Warsaw University
Faculty of Mathematics, Informatics and Mechanics

Krzysztof Ziemiański

A faithful complex representation of the 2-compact group $DI(4)$

Supervisor
prof. dr hab. Stefan Jackowski
Institute of Mathematics

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Author’s declaration

Aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means

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Abstract

p-Compact groups, introduced by Dwyer and Wilkerson [DW2], are homotopy theoretic analougues of compact Lie groups (see also [N2] and [M]). The present paper contains a construction of a faithful complex representation of 2-compact group $DI(4)$, which was provided by [DW1]. The main tool used is the homotopy decomposition of the classifying space of an arbitrary compact Lie group onto the homotopy colimit of the classifying spaces of its p-stubborn subgroups, which is due to Jackowski, McClure and Oliver [JMO1]. The paper contains a detailed discussion on a method of classification of maps $BG \to BU(n)^{\wedge}_{p}$ (called homotopy representations; G is a compact Lie group) which uses this homotopy decomposition. For $G = Spin(7), SU(2)^n$ we obtain explicit results, which are close to a full classification. We classify 2-stubborn subgroups of $Spin(7)$ and complex representations of its discrete approximations. This part depends heavily on results of [O1] and [AF] and on the representation theory of finite groups. An important (and the most difficult) part constitute calculations of obstruction groups, which appear when we extend maps from spaces at the homotopy decomposition diagrams to maps from its homotopy colimits. Methods we use are based on [JMO1] and [O2].

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Introduction

0.1 p-Compact groups

Lie groups and loop spaces

Let G be a compact Lie group and EG a contractible space equipped with a free action of G. A space of orbits $BG := EG/G$ is called *the classifying space* of the group G; the homotopy type of BG does not depend on the choice of the space EG. Unlike the group G, the space BG does not possess any differential or algebraic structure. The crucial property which relates G and BG is that there is a weak homotopy equivalence $\Omega BG \to G$. This observation motivates the following definition:

Definition 0.1.1. *A loop space* is a CW-complex X with a pointed space BX and a weak homotopy equivalence $\Omega BX \to X$. A loop space $(X, BX, \Omega BX \to X)$ is *finite* if the CW-complex X is finite.

We call BX *the classifying space* of X, what is ambiguous, since it is BX which determines X (unlike for compact Lie groups). Obviously each compact Lie group is a finite loop space. However, the class of finite loop spaces is much larger. For example, a group $SU(2) \simeq S^3$ admits uncountably many non-equivalent structures of a finite loop space [R] (i.e. there is uncountably many pairwise homotopy non-equivalent spaces BX, such that ΩBX is homotopy equivalent to S^3).

p-Compact groups

Dwyer and Wilkerson [DW2] introduced *a p-compact groups*, which are closely related to a finite loop spaces. The main idea is to consider only a part of the homotopy type of a classifying space related to a fixed prime integer p.

The definition of a p-compact group uses the construction due to Bousfield and Kan [BK], which assigns to any space Y its *p-completion* $Y_p^\wedge$, which can be interpreted as the p-part of the homotopy type of Y. If spaces Y and $Y_p^\wedge$ are weakly equivalent, then we say that the space Y is p-complete. Under mild assumptions on Y holds $\pi_i(Y_p^\wedge) \cong \pi_i(Y) \otimes \mathbb{Z}_p^\wedge$, where $\mathbb{Z}_p^\wedge$ is the ring of p-adic integers.

Definition 0.1.2. A loop space X is *a p-compact group*, if the space X is p-finite (i.e. it has the finite homology with coefficients $\mathbb{F}_p$) and the space BX is p-complete.

Examples of p-compact groups

If G is a compact Lie group such that its group of conected components is a finite p-group, then its p-completion $G_p^\wedge$ is a p-compact group and its classifying space is $(BG)_p^\wedge$. In particular, if G is a k-dimensional torus $(S^1)^k$, then its completion is *a p-compact torus*. It is an Eilenberg-McLane space $K(\mathbb{Z}_p^\wedge, 1)^k$ and its classifying space is $K(\mathbb{Z}_p^\wedge, 2)$. There are numerous examples of p-compact groups which are not a completion of any compact Lie group (such p-compact groups are called *exotic*).

Homomorphisms

Many ideas concerning compact Lie groups can be generalized onto p-compact groups. For example, any homomorphism of compact Lie groups $f : G \to H$ induces the classifying map $Bf : BG \to BH$. *A homomorphism* of p-compact groups $f : X \to Y$ is a map $Bf : BX \to BY$ (obviously it determines, up to homotopy, a map $f : X \cong \Omega BX \xrightarrow{\Omega Bf} \Omega BY \cong Y$). If maps $Bf, Bg : BX \to BY$ are homotopic, then the homomorphisms f i g are called *conjugate*. If $f : G \to H$ is a monomorphism of Lie groups, then (assuming that the models of the classifying spaces are chosen properly) Bf is a fibration with a fibre H/G. Similarly, a homomorphism of p-compact groups f is *a monomorphism*, if its homotopy fibre is p-finite (i.e. its homology with coefficients $\mathbb{F}_p$ are finite).

Centralizers

Let $f : X \to Y$ be a homomorphism of p-compact groups. *A centralizer* of X in Y is a loop space $\Omega\,\mathrm{map}(BX, BY)_{Bf}$ (it is not always a p-compact group). Unlike the definitions stated before, which are motivated by elementary properties of classifying spaces of compact Lie groups, the definition of a centralizer is motivated by the following theorem:

Theorem 0.1.3 (Dwyer-Zabrodsky)**.** *If P is a finite p-group and G a compact Lie group, then for each homomorphism $f : P \to G$ the map*

$$BC_G(P) \longrightarrow \mathrm{map}(BP, BG)_{Bf}$$

adjoint to the classifying map of the homomorphism $C_G(f(P)) \times P \to G$ induces an isomorphism on the homology with coefficients $\mathbb{F}_p$ (and hence a weak homotopy equivalence after passing to the p-completions).

Then we see that centralizers in the p-compact sense coincide with centralizers in the group sense assuming that the subgroup P is p-finite. The Dwyer-Zabrodsky Theorem can be generalized onto some larger classes on groups (for example p-toral groups and p-discrete toral groups).

Maximal torus

A torus in a p-compact group X is any monomorphism $i : T \to X$, where T is a p-compact torus. Like any compact Lie group, each p-compact group possesses *a maximal torus* $i_X : T_X \to X$, which is determined uniquely up to conjugacy by the following universal property: for any torus $i : T \to X$ in X there is a homomorphism of p-compact groups $j : T \to T_X$, such that i is conjugate to $i_X \circ j$. The group of automorphisms of T_X preserving the monomorphism i_X is called *the Weyl group* of X and denoted by W_X. The group W_X is finite and it acts on the classifying space of the maximal torus $BT_X \simeq K(\mathbb{Z}_p^\wedge, 2)^r$. This action determines a representation $W_X \to \mathrm{GL}_r(\mathbb{Z}_p^\wedge)$, which is one of the most important invariants of p-compact groups.

Complex representations of p-compact groups

Recall that a complex representation of a Lie group G is a linear action of G on a complex vector space, or equivalently, a homomorphism $\varphi : G \to \mathrm{GL}(V)$. If the group G is compact, then we can assume that the image of φ is contained in a unitary group $U(V) \to \mathrm{GL}(V)$. Similarly,

Definition 0.1.4. *A complex representation* of a p-compact group X is a homomorphism (of p-compact groups) $\varphi : X \to U(n)^{\wedge}_p$ (i.e. a map $BX \to BU(n)^{\wedge}_p$). A representation is called *faithful* if φ is a monomorphism.

The Peter-Weyl Theorem states, that each compact Lie group admits a faithful complex representation. It is interesting if p-compact groups possess the analogous property, i.e. whether or not the following conjecture is valid:

Conjecture 0.1.5. *Each p-compact group admits a faithful complex representation.*

For odd p this conjecture was proven by Castellana [C], who has constructed, for any simple p-compact group X a representation of dimension $\dim X$, which is a generalization of the adjoint representation of a compact Lie group. There is a strong evidence, that the only simple 2-compact group, which is not a completion of any compact Lie group is $DI(4)$, constructed by Dwyer and Wilkerson [DW1]. The main result of the present paper is the following:

Theorem 0.1.6. *There is a faithful complex represenation of $DI(4)$ of dimension 2^{46}.*

Since $DI(4)$ has no non-trivial normal subgroups, it is equivalent to the existence of any non-trivial map $BDI(4) \to (BU(2^{46}))^{\wedge}_2$. It seems that the methods of [C] cannot be generalized onto the case $DI(4)$, since there is no representation of $DI(4)$ of dimension $\dim DI(4) = 45$.

2-Compact group $DI(4)$

The property which determines uniquely $DI(4)$ is that the cohomology ring of $BDI(4)$ with $\mathbb{F}_2$-coefficients is a rank 4 mod 2 algebra of Dickson invariants:

$$H^*(BDI(4); \mathbb{F}_2) \cong \mathbb{F}_2[t_1, t_2, t_3, t_4]^{\mathrm{GL}_4(\mathbb{F}_2)},$$

where the generators t_i are in dimension 1. The algebra $H^*(BDI(4); \mathbb{F}_2)$ is also a polynomial algebra generated by elements c_8, c_{12}, c_{14}, c_{15} (indices indicate dimensions of generators). An action of the Steenrod algebra is determined by relations $Sq^4 c_8 = c_{12}$, $Sq^2 c_{12}$ and $Sq^1 c_{14} = c_{15}$. The Weyl group $W_{DI(4)}$ is isomorphic to $\{\pm 1\} \times \mathrm{GL}_3(\mathbb{F}_2)$ and the action of $W_{DI(4)}$ on the maximal torus determines a representation $\mathrm{GL}_3(\mathbb{F}_2) \to \mathrm{GL}_3(\mathbb{Z}^{\wedge}_2)$, which is a section of the mod 2 reduction. 2-Compact group $DI(4)$ contains a subgroup $Spin(7)^{\wedge}_2$, such that $DI(4)$ and $Spin(7)^{\wedge}_2$ have a common maximal torus. It appears that no (algebraic) representation of the Lie group $Spin(7)$ (or even its maximal torus T) extends after completion to a faithful representation of $DI(4)$. The main reason is that the homomorphism $W_{DI(4)} \to \mathrm{GL}_3(\mathbb{Z}^{\wedge}_2)$ (cf. 0.1) has no factorization through $\mathrm{GL}_3(\mathbb{Z})$. Therefore a completion of any faithful algebraic representation of T is not $W_{DI(4)}$-invariant and cannot be extended to $DI(4)$.

0.2 Homotopy decompositions

Homotopy decompositions are very useful tools for studying classifying spaces of compact Lie groups. They are also very helpful for constructing and studying p-compact groups.

Homotopy colimits

Let $\mathcal{C}$ be a small category (i.e. such that the class of its objects is a set) and F any functor from $\mathcal{C}$ into the category of topological spaces **Sp**. Obviously one can take the colimit of F, but this construction does not have good homotopy properties (i.e. homotopy limits of homotopy equivalent functors need not to be homotopy equivalent). More useful construction in the homotopy theory is *the homotopy colimit* $\mathrm{hocolim}_{\mathcal{C}} F$, which has much better properties. There is a map $\mathrm{hocolim}_{\mathcal{C}} F \to \mathrm{colim}_{\mathcal{C}} F$, which is in many cases a weak homotopy equivalence. However, the homotopy colimit is not a colimit neither in the category of spaces, nor in the homotopy category.

Obstruction theory

Fix a small category $\mathcal{C}$, a functor $F : \mathcal{C} \to \mathbf{Sp}$, a space Z and a family of maps $\{f_C : F(C) \to Z\}_{C \in \mathcal{C}}$. Assume that $\{f_C\}$ is *homotopy compatible*, i.e. for any morphism $c : C \to C'$ in $\mathcal{C}$ the maps f_C and $f_{C'} \circ F(c)$ are homotopic. Consider the following problem: when the family $\{f_C\}$ can be extended to the map $\mathrm{hocolim}_{\mathcal{C}} F \to Z$. Define, for $i \geq 1$, the following family of contravariant functors on $\mathcal{C}$ with values in the category of groups:

$$\Pi^f_i(C) := \pi_i(\mathrm{map}(F(C), Z)_{f_C}).$$

Assume that for each $C \in \mathrm{Ob}(\mathcal{C})$ the group $\Pi_1^f(C)$ is abelian. Then:

- If for $i \geq 1$ the groups $H^{i+1}(\mathcal{C}; \Pi_i^f)$ vanish, then it exists an extension of $\{f_C\}$ to a map $\mathrm{hocolim}_{\mathcal{C}} F \to Z$.
- If for $i \geq 1$ the groups $H^i(\mathcal{C}; \Pi_i^f)$ vanish, then an extension of $\{f_C\}$ to a map $\mathrm{hocolim}_{\mathcal{C}} F \to Z$ is determined uniquely up to homotopy (if it exists).

Remark. Symbols H^i denote cohomology groups of categories with coefficients in functors (it is a common generalizations of the group cohomology and the cohomology of spaces).

Homotopy decompositions

Let p be a fixed prime integer and G either a compact Lie group, or a p-compact group.

Definition 0.2.1. *A homotopy decomposition* of the classifying space BG is any functor $F : \mathcal{C} \to \mathbf{Sp}$ equipped with a map $\mathrm{hocolim}_{\mathcal{C}} F \to BG$, which induces an isomorphism of the homology with coefficients $\mathbb{F}_p$. Moreover, we assume that $\mathcal{C}$ is a finite EI-category (i.e. such that each endomorphism is actually an isomorphism) and that for each $C \in \mathcal{C}$ the space $F(C)$ is a classifying space of a Lie group (or a p-compact group).

Homotopy decompositions are presentations of a given classifying spaces in the form of a homotopy colimit of classifying spaces of simpler groups (usually subgroups), but only up to the homological equivalence. (Decompositions for which the homology equivalence is in fact a homotopy equivalence are usually not very useful). In the present paper we use two kinds of homotopy decompositions: *the centralizer decomposition*, where appear classifying spaces of centralizers of elementary abelian p-subgroups of a given p-compact group, and *the p-stubborn subgroup decomposition*, in which appear classifying spaces of some p-toral subgroups of a given group.

Construction of the 2-compact group $DI(4)$

The space $BDI(4)$ was constructed [DW1] as a homotopy colimit of some functor (which appears to be the centralizer decomposition functor of $DI(4)$ as well). Now we describe this functor. Let V be a 4-dimensional vector space over $\mathbb{F}_2$ and $\mathcal{A}$ be a category, whose objects are non-zero subspaces of V and morphisms are all monomorphisms between vector spaces. Define

$$K : \mathcal{A} \ni X \mapsto \mathbb{F}_2[V]^{\{g \in \mathrm{GL}(V): \forall_{x \in X} g(x)=x\}} \in \mathbf{GrAlg_2}, \tag{0.2.2}$$

where $\mathbf{GrAlg_2}$ is the category of graded $\mathbb{F}_2$-algebras. It appears that there is a functor $F : \mathcal{A}^{op} \to \mathbf{Sp}$, such that $H^*(F; \mathbb{F}_2) \cong K$. The space $BDI(4)$ is defined to be the 2-completion of the homotopy colimit of F. Values of the functor F are completed classifying spaces of compact Lie groups (namely $Spin(7)$, $SU(2)^3/\{\pm 1\}$, $T^3 \ltimes \{\pm 1\}$ and $\{\pm 1\}^4$). Then $BDI(4)$ is the homotopy colimit of the following diagram:

$$\overset{GL_4(\mathbb{F}_2)}{\curvearrowright} B\{\pm 1\}^4 \rightrightarrows \overset{GL_3(\mathbb{F}_2)}{\curvearrowright} B(T^3 \ltimes \{\pm 1\})^{\wedge}_2 \rightrightarrows \overset{GL_2(\mathbb{F}_2)}{\curvearrowright} B(SU(2)^3/\{\pm 1\})^{\wedge}_2 \rightrightarrows BSpin(7)^{\wedge}_2. \tag{0.2.3}$$

This diagram is very close to the centralizer decomposition diagram of the space $BSpin(7)$ — it has the same objects, but more morphisms.

0.3 Homotopy representations of compact Lie groups

In this section we describe a general method of constructing maps from BG into the 2-completion of $BU(n)$, where G is a Lie group. Such maps will be called *homotopy representations* of G. We say that two homotopy representations are isomorphic, if they are homotopic (as maps between spaces). The general properties of the p-completion imply that there is a bijection $[BG, (BU(n))^{\wedge}_p] = [(BG)^{\wedge}_p, (BU(n))^{\wedge}_p]$, then maps $(BG)^{\wedge}_p \to (BU(n))^{\wedge}_p$ will also be called homotopy representations. In the present section we use stubborn decompositions [JMO1], which seem to be the most appropriate for our purposes.

Homotopy representations of p-toral groups

A Lie group P is called *p-toral*, if it is an extension of a finite p-group by a torus. Any p-toral group has a dense subgroup P^{∞}, called *the discrete approximation*, which is an extension of a finite p-group by a p-discrete torus $(\mathbb{Z}/p^{\infty})^r$. Usefullness of discrete approximations follows from the following version of the Dwyer-Zabrodsky Theorem:

Theorem 0.3.1. *Let P be a p-toral group and G a compact Lie group. Then the map*

$$\operatorname{Rep}(P^\infty, G) \xrightarrow{B(-)^\wedge_p} [BP, (BG)^\wedge_p],$$

is a bijection, where $\operatorname{Rep}(P^\infty, G) := \operatorname{Hom}(P^\infty, G)/\operatorname{Inn}(G)$. *Moreover, for any representation $\varphi : P^\infty \to G$ the map*

$$BC_G(P) \to \operatorname{map}(BP, (BG)^\wedge_p)_{(B\varphi)^\wedge_p}$$

induces an isomorphisms on the homology with coefficients $\mathbb{F}_p$.

In particular, any homotopy complex representation of a p-toral group P is the completion of an algebraic representation of its discrete approximation P^∞. If P is not p-toral, then the set of isomorphism classes of homotopy representations has no simple description.

Decomposition onto p-stubborn subgroups

Let G be a compact connected Lie group. Let $\mathcal{O}_p(G)$ be the category of G-orbits having the form G/P, for p-toral P. p-Toral subgroup $P \subseteq G$ is called *p-stubborn*, if its Weyl group $W_G(P) := N_G(P)/P$ is finite and does not contain any non-trivial normal p-subgroup. Let $\mathcal{R}_p(G)$ be the subcategory of $\mathcal{O}_p(G)$ with objects G/P for p-stubborn P. The stubborn decomposition fuctor of BG is defined on the category $\mathcal{R}_p(G)$ and its value at the object G/P is homotopy equivalent to the space $(BP)^\wedge_p$.

$\mathcal{R}$-invariant representations

For any compact connected Lie group G the category $\mathcal{R}_p(G)$ contains exactly one *maximal* object (i.e. such that there are no morphisms from this object, except automorphisms). This object is G/N, where N is a maximal p-toral subgroup of the normalizer of the maximal torus of G. An n-dimensional complex representation φ of N^∞ is *$\mathcal{R}_p(G)$-invariant* (or, in short, *$\mathcal{R}$-invariant* if G and p are clear), if the map $(B\varphi)^\wedge_2 : (BN^\infty)^\wedge_p \to (BU(m))^\wedge_p$ determines a homotopy compatible family of maps from the decomposition diagram of BG into the space $(BU(m))^\wedge_p$. To determine whether or not a representation is $\mathcal{R}$-invariant one needs to compare characters of restrictions $\operatorname{res}^{N^\infty}_{P^\infty} \varphi$ for p-stubborn P.

Obstructions for extending $\mathcal{R}$-invariant representations

Any $\mathcal{R}$-invariant representation φ of N^∞ determines a homotopy compatible family of maps from the decomposition functor of BG. The obstructions for the existence of an extension of $(B\varphi)_p^\wedge$ to a map $(BG)_p^\wedge \to BU(n)_p^\wedge$ lie in groups $H^{i+1}(\mathcal{C}; \Pi_i^\varphi)$ (cf. 0.2). The application of Theorem 0.3.1 and the Schur Lemma allows to prove, that the functor Π_2^φ is isomorphic to a functor Ξ^φ, such that $\Xi^\varphi(G/P)$ is a free $\mathbb{Z}_p^\wedge$-module generated by isomorphism classes of irreducible subrepresentations of $\mathrm{res}_{N^\infty}^{P^\infty} \varphi$. Moreover, $\Pi_1 = \Pi_3 = 0$. By [JMO1], the cohomology of the category $\mathcal{R}_p(G)$ with coefficients in any functor having values in the category of $\mathbb{Z}_p^\wedge$-modules vanish above some dimension (which does not depend on the coefficients; the smallest possible bound will be called *a cohomological p-dimension* of the category $\mathcal{R}_p(G)$). These observations allow to calculate effectively obstruction groups (and thus to classify homotopy representations of G), assuming G is not very big.

Theorem 0.3.2. *Let G be a compact connected Lie group and φ an $\mathcal{R}$-invariant representation of N^∞. Then*

a. If $H^3(\mathcal{R}_p(G); \Xi^\varrho) = 0$ and the cohomological p-dimension of the category $\mathcal{R}_p(G)$ is not greater than 4, then $B\varrho_p^\wedge$ extends to a map $BG \to BU(m)_p^\wedge$,

b. If $H^2(\mathcal{R}_p(G); \Xi^\varrho) = 0$ and the cohomological p-dimension of the category $\mathcal{R}_p(G)$ is not greater than 4, then $B\varrho_p^\wedge$ extends uniquely (up to homotopy) to a map $BG \to BU(m)_p^\wedge$.

0.4 Application: Homotopy representations of groups $Spin(7)$ and $SU(2)^3$

Put $p = 2$.

The category $\mathcal{R}_2(Spin(7))$

As mentioned in the previous section, the classification of homotopy representations of $Spin(7)$ requires the calculation of the cohomology groups of the category $\mathcal{R}_2(Spin(7))$ (with coefficients in Ξ^ϱ). Jackowski, McClure and Oliver [JMO1] provided very effective tools for this kind of calculations, however its application requires a detailed description of the category

$\mathcal{R}_p(Spin(7))$. The proof of the following theorem uses results of the papers [O1], [AF] and [JMO1]:

Theorem 0.4.1. *(a)* $\mathcal{R}_2(Spin(7))$ *is an EI-category naturally equivalent to* $\mathcal{R}_2(O(7)$. *Moreover,*

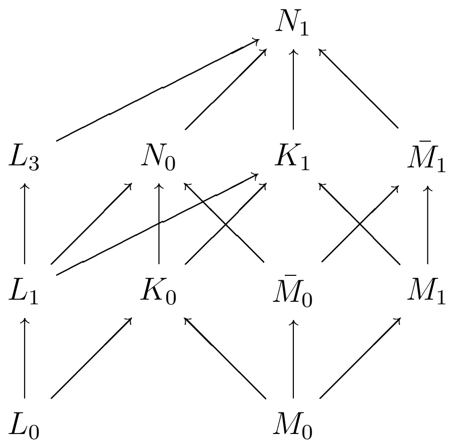
$$\mathrm{Ob}(\mathcal{R}_2(Spin(7)) = \{N_i, K_i, M_i, \bar{M}_i\}_{i=0,1} \cup \{L_i\}_{i=0,1,3}.$$

Groups of automorphisms of objects are respectively

$$\mathrm{Aut}(N_1) = 1, \qquad \mathrm{Aut}(N_0) \cong \mathrm{Aut}(L_3) \cong \mathrm{Aut}(K_1) \cong \Sigma_3$$
$$\mathrm{Aut}(L_1) \cong \Sigma_5, \qquad \mathrm{Aut}(L_0) \cong \Sigma_7, \qquad \mathrm{Aut}(K_0) \cong \mathrm{Aut}(\bar{M}_0) \cong \Sigma_3 \times \Sigma_3$$
$$\mathrm{Aut}(M_1) \cong \Sigma_3 \wr \Sigma_2, \qquad \mathrm{Aut}(M_0) \cong \Sigma_3 \wr \Sigma_2 \times \Sigma_3$$

(see Section 3.2 for definitions of these objects).

(b) The set of morphisms of $\mathcal{R}_2(Spin(7))$ *is generated by automorphisms and by inclusions presented on the following diagram:*



(c) The full subcategory of $\mathcal{R}_2(Spin(7))$ *containing objects* L_i, K_i, N_i *is naturally equivalent to the category* $\mathcal{R}_2(\Sigma_7)$. *The full subcategory of* $\mathcal{R}_2(Spin(7))$ *containing objects* M_i, $\bar{M}_i$, K_i, N_i *is naturally equivalent to the category* $\mathcal{R}_2(O(4)) \times \mathcal{R}_2(\Sigma_3)$.

Representations of discrete approximations of 2-stubborn subgroups of $Spin(7)$

The next step toward the classification of homotopy representations of $Spin(7)$ is a classification of complex representations of discrete approximations of 2-stubborn subgroups P of $Spin(7)$. Note that, in general, the groups P^∞ are not finite, only locally finite (i.e. such that any finitely generated subgroup of P^∞ is finite). However, it appears that the representation theory of locally finite groups is very close to the representation theory of finite groups. The present paper contains a complete classification of irreducible subrepresentations of discrete approximations of 2-stubborn subgroups of $Spin(7)$.

Homotopy representations of $Spin(7)$

Any irreducible representation of a locally finite subgroup of $Spin(7)$ which contains $Z := Z(Spin(7))$ is either *even* (i.e. it comes from a representation of $SO(7)$), or *odd* (i.e. its restriction to Z is a sum of isomorphic, irreducible and non-trivial representations). It implies that any $\mathcal{R}$-invariant representation φ of N_1^∞ is a direct sum of an even representation ϱ_{ev} and an odd one ϱ_{od} (and both ϱ_{ev} and ϱ_{od} are $\mathcal{R}$-invariant).

Theorem 0.4.2. *Let $\varrho \cong \varrho_{ev} \oplus \varrho_{od}$ be an $\mathcal{R}$-invariant representation N_1^∞. Then the odd part $B(\varrho_{od})_2^\wedge$ extends to a homotopy representation of $Spin(7)$. The even part $B(\varrho_{ev})_2^\wedge$ extends, if it satisfies some mild technical conditions.*

Remark. We do not know any example of an even $\mathcal{R}$-invariant representation, which does not satisfy the conditions mentioned in the theorem.

Homotopy representations of $SU(2)^n$

Let $N \subseteq SU(2)$ be the normalizer of the maximal torus. The following theorem contains a partial classification of homotopy representations of the groups L^n:

Theorem 0.4.3. *Let φ be an $\mathcal{R}$-invariant representation of $(N^\infty)^n$.*

(a) If $n \leq 3$, then $B\varphi_2^\wedge$ extends to $(BSU(2)^n)_2^\wedge$.

(b) If $n \leq 2$, then $B\varphi_2^\wedge$ extends uniquely to $(BSU(2)^n)_2^\wedge$.

This theorem allows for constructing of interesting examples of homotopy represenations of $SU(2)^n$, which illustrate deep differences between homotopy representations and algebraic representations. For example,

- Every irreducible homotopy representation has dimension ≤ 4 (there are irreducible algebraic representations having arbitrarily large dimensions).
- Every homotopy representation of $SU(2)$ admits a decomposition onto a sum of irreducible subrepresentations, but the decomposition it is not uniquely determined.
- An irreducible homotopy representation of $SU(2)^2$ does not need to be a tensor product of irreducible representations of $SU(2)$.

For us the most interesting is the group $SU(2)^3$, since the classifying space of its quotient $SU(2)^3/\{\pm 1\}$ appears in the decomposition diagram of $BDI(4)$. Theorem 0.4.3 implies only, that any $\mathcal{R}^3$-invariant representation can be extended to a homotopy representation of $SU(2)^3$. However, we need some information about the set of extensions of a given $\mathcal{R}$-invariant representation of $(N^\infty)^3$.

Theorem 0.4.4. *Let φ be an $\mathcal{R}_2(SU(2)^3)$-invariant representation of $(N^\infty)^3$. The set of homotopy classes of extensions of $B\varphi_2^\wedge$ to $(BSU(2)^3)_2^\wedge$ admits a structure of a free and transitive $H^2(\mathcal{R}_2(SU(2)^3); \Xi^\varrho)$-set, which is preserved by homotopy self-equivalences of $(BSU(2)^3)_2^\wedge$.*

A proof of the theorem requires the following construction:

Adams operations on $BSU(2)_2^\wedge$

An Adams operation on $SU(2)$ is any map $\psi_k : BSU(2)_2^\wedge \to BSU(2)_2^\wedge$, where k is an odd 2-adic integer, such that its restriction to the completion of the classifying space of the maximal torus (which is homotopy equivalent to $K(\mathbb{Z}_2^\wedge, 2)$) is induced by multiplication by k. It turns out [JMO2], that for each $k \in (\mathbb{Z}_2^\wedge)^*$ it exists an Adams operation ψ_k. Moreover, every homotopy self-equivalence of $(BSU(2))_2^\wedge$ is an Adams operation and ψ_k is homotopic to ψ_l if and only if $k = \pm l$. Therefore, the group $\mathrm{HAut}(BSU(2)_2^\wedge)$ of homotopy self-equivalences of $(BSU(2))_2^\wedge$ is isomorphic to $(\mathbb{Z}_2^\wedge)^*/\{\pm 1\}$. In the present paper we prove that the action of $\mathrm{HAut}(BSU(2)_2^\wedge)$ can be realized at the level of the stubborn homotopy decomposition of $BSU(2)_2^\wedge$.

Theorem 0.4.5. *It exists a functor F on the category $\mathcal{R}_2(SU(2))$ with values in the category of* $\mathrm{HAut}(BSU(2)^\wedge_2)$*-spaces, which is homotopy equivalent to the stubborn decoposition functor of $BSU(2)$. Moreover, the induced action of the group* $\mathrm{HAut}(BSU(2)^\wedge_2)$ *on* $(\mathrm{hocolim}\, F)^\wedge_2$ *coincides (up to homotopy) with the action of* $\mathrm{HAut}(BSU(2)^\wedge_2)$ *on* $BSU(2)^\wedge_2$.

0.5 Proof of Main Theorem

Homotopy representation of a 2-normalizer of a maximal torus of $Spin(7)$

Let T be a maximal torus of $Spin(7)$ and let N_1 be a maximal 2-toral subgroup of the normalizer of T in $Spin(7)$. Note that the 2-completion of T is also a maximal torus of $DI(4)$. Let W be a Weyl group of $DI(4)$. Since W acts on the 2-completed torus $(BT)^\wedge_2$, it also acts on its discrete approximation T^∞ (although, this action cannot be realized on T). Let ϱ be a representation of T^∞ determined by the root $(1, 0, 0)$ and θ a one-dimensional trivial representation. Define

$$\varphi := \mathrm{ind}_{T^\infty}^{N_1^\infty} \bigotimes_{w \in W/W_\varrho} (\theta \oplus w^* \varrho).$$

The dimension of φ is $m := 2^{46}$. In the next steps we extend it to a representation of the 2-compact group $DI(4)$.

Homotopy representation of $Spin(7)$

We prove that the representation φ is $\mathcal{R}_2(Spin(7))$-invariant (by calculating its character); it satisfies also the conditions of Theorem 0.4.2. Hence it extends to a homotopy representation $f : BSpin(7)^\wedge_2 \to BU(m)^\wedge_2$ of G.

$\mathcal{A}$-Invariance of the map f

Let $F : \mathcal{A} \to \mathbf{Sp}$ be the centralizer decomposition diagram of $BDI(4)$ (cf. 0.2). We need to prove now, that the map f is $\mathcal{A}$-invariant, (i.e. it defines a homotopy compactible family of maps from the diagram F to the space $(BU(m)^\wedge_2)$. The hard part is to check that f restricted to the space

$B(SU(2)^3/\{\pm 1\})^\wedge_2$ is $GL_2(\mathbb{F}_2)$-invariant (por. 0.2). It is obtained as a consequence of Theorem 0.4.4.

A faithful complex representation of $DI(4)$

The obstruction groups to the existence of an extension of f to a map from $BDI(4)$ can be expressed in terms of Steinberg modules [O2], assuming that homotopy groups of the suitable mapping spaces are abelian. Hence the last serious difficulty is to prove that the group

$$\Pi_1^\varrho := \pi_1(\mathrm{map}((BSU(2)^3/\{\pm 1\})^\wedge_2, BU(m)^\wedge_2)_{B\varrho_1})$$

is abelian. The stubborn homotopy decomposition of $SU(2)^3$ allows to present the mapping space as a homotopy limit of some functor. Now the homotopy groups of the mapping space can be computed using the Bousfield spectral sequence [BK].

0.6 Notation

The letter p will always stand for a fixed prime number and $\mathbb{F}_p$ is a field with p elements. Symbols C_n and Σ_n stand for the cyclic group with n-elements and the symmetric group on n letters respectively. All homology and cohomology groups are taken with $\mathbb{F}_p$-coefficients unless stated otherwise. Symbols **Sp**, **Gr**, **Ab**, **Cat**, **Poset**, $\mathbf{Mod}_R$ denote respectively the categories of topological spaces, groups, abelian groups, small categories, partially ordered sets and modules over a commutative ring R. $\mathbb{Z}_p^\wedge$ is the ring of p-adic integers.

Chapter 1

p-Compact groups and homotopy decompositions

1.1 Completions and localizations

Throughout this paper we use intensively the localization with respect to homology with coefficients $\mathbb{F}_p$, which was constructed by Bousfield [B], called in short *the p-localization*. For nilpotent spaces it coincides with the p-completion of Bousfield and Kan [BK]. Recall basic definitions and properties of localizations.

Definition 1.1.1. A space X is *p-local* if for each map $f : A \to B$ which induces an isomorphism on the homology with coefficients $\mathbb{F}_p$, the map

$$f^* : \mathrm{map}(B, X) \longrightarrow \mathrm{map}(A, X)$$

is a weak homotopy equivalence.

Definition 1.1.2. A functor $F : \mathbf{Sp} \to \mathbf{Sp}$ is a p-localization functor if it is

- *coaugmented*, i.e. it is equipped with a natural transformation $\eta : Id_{\mathbf{Sp}} \to F$,
- *idempotent*, i.e. for each space X the maps $\eta(F(X))$ and $F(\eta(X))$ are weak homotopy equivalences and they are homotopic to each other,
- *universal with respect to p-local spaces*, i.e. every map $f : X \to T$ into any p-local space T factors uniquely up to homotopy through $F(X)$.

Bousfield [B] proved that for each prime integer p there is a p-localization functor and it is unique up to homotopy. We denote it by $(-)^\wedge_p$. On the other hand Bousfield and Kan [BK] constructed the completion functor $\mathbb{F}_p^\infty : \mathbf{Sp} \to \mathbf{Sp}$, which coincides with the p-localization for a large class of spaces. Such spaces are called p-good. All nilpotent spaces are p-good. Let us state some properties of completions and localizations:

Proposition 1.1.3.

(a) If X is p-good, then $\pi_i(X^\wedge_p) = \mathbb{Z}^\wedge_p \otimes \pi_i(X)$.

(b) If P is a finite p-group, then BP is p-local.

(c) If X is p-local, then for each space A the mapping space $\mathrm{map}(A, X)$ *is p-local.*

Proof. For (a) and (b) see [DW2, 11.4], and (c) follows immediately from definition. □

Proposition 1.1.4. *Let P be a finite p-group and X an $\mathbb{F}_p$-good space. Assume that the homotopy groups of X are abelian and finitely generated. Then the map*

$$\eta_* : [BP, X] \longrightarrow [BP, X^\wedge_p]$$

is a bijection.

Proof. Fix $f : BP \to X^\wedge_p$. Let Y be the homotopy fibre of the completion $X \to X^\wedge_p$. The obstructions to the existence (resp. the uniqueness) of a lift $\tilde{f} : BP \to X$ lie in the groups $H^{n+1}(P; \pi_n Y)$ (resp. $H^n(P; \pi_n Y)$). For each n the group $\pi_n Y$ lies in the exact sequence

$$\ldots \longrightarrow \pi_{n+1} X \longrightarrow \pi_{n+1} X \otimes \mathbb{Z}^\wedge_p \longrightarrow \pi_n Y \longrightarrow \pi_n X \longrightarrow \pi_n X \otimes \mathbb{Z}^\wedge_p \longrightarrow \ldots$$

Thus for each n the group $\pi_n Y$ is an extension of a finite group abelian group of order prime to p by $(\mathbb{Z}^\wedge_p/\mathbb{Z})^s$. Therefore multiplication by p is an isomorphism on $\pi_n Y$ and then $H^i(P; \pi_n Y) = 0$ for $i > 0$. □

1.2 p-Compact groups

In this section we recall the definition of p-compact groups (cf. [DW2]) and we state its basic properties. Fix a prime integer p. We say that a space is *p-finite* if its homology with coefficients $\mathbb{F}_p$ are finite.

Definition 1.2.1. A *p-compact group* X is a triple (X, BX, ε) where:

(a) X is a p-finite space,

(b) BX is a pointed p-complete space,

(c) $\varepsilon : \Omega BX \longrightarrow X$ is a weak homotopy equivalence.

Note that it is a compilation of definitions 0.1.1 and 0.1.2. Usually we use a single symbol X for the triple (X, BX, ε). The space BX, by analogy to groups, is called a classifying space of a p-compact group X.

Example 1.2.2. If P is a finite p-group then (P, BP, ε) is a p-compact group. More generally, if G is a compact Lie group such that $\pi_0 G$ is a finite p-group then $(G^\wedge_p, BG^\wedge_p, \varepsilon^\wedge_p)$ is a p-compact group. In both cases ε is an obvious weak equivalence $\Omega BP \to P$ (resp. $\Omega BG \to G$).

Definition 1.2.3. A p-compact group $(K(\mathbb{Z}^\wedge_p, 1)^n, K(\mathbb{Z}^\wedge_p, 2)^n, \varepsilon)$ is called *an n-dimensional p-compact torus*. Note that it is a special case of the previous example for $G = (S^1)^n$.

Definition 1.2.4. A *homomorphism* of p-compact groups $X \to Y$ is any pointed map $Bf : BX \to BY$. Two homomorphisms are *conjugate* if they are freely homotopic (homotopy does not need to preserve basepoints). A homomorphism $f : X \to Y$ is *a monomorphism* if the homotopy fibre of Bf is p-finite.

Definition 1.2.5. A p-compact group P is *toral* if it sits in the exact sequence

$$1 \longrightarrow T \longrightarrow P \longrightarrow W \longrightarrow 1,$$

where T is a p-compact torus and W is a finite p-group (the exact sequence means that $BT \to BP \to BW$ is a fibration).

Definition 1.2.6. A monomorphism $i : T \to X$ is *a maximal torus* of X if T is a p-compact torus and i admits the following universal property: each homomorphism from a p-compact torus to X factors through i.

Proposition 1.2.7 ([DW2, Theorem 8.13])**.** *Each p-compact group admits a maximal torus, which is determined uniquely up to conjugacy.*

Definition 1.2.8 ([DW2, 9.2, 9.6])**.** Let $Bi : BT^k \to BX$ be a fibration homotopy equivalent to the classifying map of the maximal torus of X. *The Weyl group* W_X of X is the group $\pi_0(\mathrm{Aut}(BT^k)_{Bi})$, where $\mathrm{Aut}(BT^k)_{Bi}$ is the space of self-maps of BT^k which commute with Bi. Note that there is a natural action of W_X on $\pi_2(BT^k) \simeq (\mathbb{Z}_2^\wedge)^k$.

Definition 1.2.9. Let $f : X \to Y$ be a homomorphism of p-compact groups. *The centralizer* $C_Y(f(X))$ is the loop space obtained as $\Omega\,\mathrm{map}(BX, BY)_{Bf}$. If X is a finite p-group, then the centralizer is a p-compact group [DW2, 5.1]. If X is a finite p-group and Y is the completion of a compact Lie group, then, due to the Dwyer-Zabrodsky theorem (2.2.1), the centralizer $C_Y(f(X))$ in the p-compact sense coincides with the algebraic centralizer.

We conclude with a convenient characterization of monomorphisms of p-compact groups:

Proposition 1.2.10. *A homomorphism $f : X \to Y$ of p-compact groups is a monomorphism if one of the following equivalent conditions hold:*

(a) the homotopy fibre of Bf is p-finite,

(b) $H^*(BX; \mathbb{F}_p)$ is a finitely generated $H^*(BY; \mathbb{F}_p)$-module,

(c) for each monomorphism $j : \mathbb{Z}/p \to X$ the homomorphism $B(f \circ j)$ is non-trivial.

Proof. An equivalence of (a) and (b) is proved in [DW2, 9.11]. An equivalence of (a) and (c) follows from [DW4, 3.2]. □

Proposition 1.2.11. *Let X, Y be p-compact groups and let $i_X : T_X \subseteq X$ be a maximal torus of X. A homomorphism $f : X \to Y$ is a monomorphism if and only if $f \circ i_X : T_X \to Y$ is a monomorphism.*

Proof. Assume that $f \circ i_X$ is a monomorphism. By [DW2, 5.6] and [DW2, 6.8] any monomorphism $j : \mathbb{Z}/p \to X$ extends to a map $S \to X$, where S is a p-compact torus. Then by universal property of the maximal torus j factors through T_X (i.e. $j \sim i_X \circ j'$). Therefore $f \circ j \sim (f \circ i_X) \circ j'$ is a composition of monomorphisms and hence it is a monomorphism. Now the conclusion follows from 1.2.10.(c). □

1.3 Homotopy decompositions

A very useful tool in studying p-compact groups are homotopy decompositions, i.e. presentations of classifying spaces of p-compact groups (or classifying spaces of Lie groups completed at p) as homotopy colimits of simpler spaces, usually classifying spaces of its subgroups. In this section we define simplicial sets, nerves of categorties and homotopy colimits. Then we give two examples of homotopy decompositions of p-compact groups, namely the centralizer decomposition and the decomposition on p-stubborn subgroups (called also the p-stubborn decomposition).

Definition 1.3.1 ([Ma],[BK, VIII.2.1]). *A simplicial set* $X_\bullet$ is a sequence of sets X_n for $n \geq 0$ (called n-simplices), equipped with maps $d_i^n : X_n \to X_{n-1}$ and $s_i^n : X_n \to X_{n+1}$ for $0 \leq i \leq n$ (called *face maps* and *degeneracy maps* respectively), which satisfy the following relations:

$$d_i^{n-1} d_j^n = d_{j-1}^{n-1} d_i^n \quad \text{for } i < j$$

$$d_i^{n+1} s_j^n = \begin{cases} s_{j-1}^{n-1} d_i^n & \text{for } i < j \\ id & \text{for } i = j, j+1 \\ s_j^{n-1} d_{i-1}^n & \text{for } i > j+1 \end{cases}$$

$$s_i^{n+1} s_j^n = s_j^{n+1} s_i^n \quad \text{for } i > j.$$

Given simplicial sets $X_\bullet$ and $Y_\bullet$, *a simplicial map* $f : X_\bullet \to Y_\bullet$ is a sequence of maps $f_n : X_n \to Y_n$ which commute with the face and degeneracy maps.

Let Δ be a category of finite ordered sets. Note that each object of Δ is isomorphic to $\mathbf{n} := \{0 < 1 < \ldots < n\}$.

Definition 1.3.2. Let $\mathcal{C}$ be a small category. *A nerve* of $\mathcal{C}$ is a simplicial set $N(\mathcal{C})$ such that $N(\mathcal{C})_n$ is a set of all natural transformations $\mathbf{n} \to \mathcal{C}$ (or, equivalently, a set of all sequences of morphisms of $\mathcal{C}$ having length n), and face and degeneracy maps are as follows:

$$d_0^n(C_0 \xrightarrow{c} C_1 \to \ldots \to C_n) = (C_1 \to \ldots \to C_n)$$

$$d_n^n(C_0 \to \ldots \to C_{n-1} \xrightarrow{c} C_n) = (C_0 \to \ldots \to C_{n-1})$$

$$d_i^n(C_0 \to \ldots \to C_{i-1} \xrightarrow{c} C_i \xrightarrow{c'} C_{i+1} \to \ldots \to C_n)$$
$$= (C_0 \to \ldots \to C_{i-1} \xrightarrow{c'c} C_{i+1} \to \ldots \to C_n) \quad \text{for } i > 0$$

$$s_i^n(C_0 \to \ldots \to C_n) = (C_0 \to \ldots \to C_i \xrightarrow{id} C_i \to \ldots \to C_n)$$

Let $\Delta^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^n : x_0 + \cdots + x_n = 0,\ x_i \geq 0\}$ be a standard n-dimensional topological simplex. For $0 \leq i \leq n$ define maps

$$\mathfrak{d}_i^n : \Delta^{n-1} \ni (t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}) \in \Delta^n$$
$$\mathfrak{s}_i^n : \Delta^{n+1} \ni (t_0, \ldots, t_{n+1}) \mapsto (t_0, \ldots t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{n+1}) \in \Delta^n.$$

Definition 1.3.3. Let $\mathcal{C}$ be a small category and let $F : \mathcal{C} \to \mathbf{Sp}$ be any functor. *A homotopy colimit* of F is a space

$$\operatorname{hocolim}_{\mathcal{C}} F := \coprod_n \coprod_{\sigma \in N(\mathcal{C})_n} \Delta^n \times F(\sigma(0))/\sim,$$

where $\sim$ is generated by

$$(\sigma,\ \mathfrak{d}_i^n(x),\ y) \sim \begin{cases} (d_i^n(\sigma),\ x,\ F(\sigma(0 \to 1))(y)) & \text{for } i = 0 \\ (d_i^n(\sigma),\ x,\ y) & \text{for } i > 0 \end{cases}$$
$$(\sigma,\ \mathfrak{s}_i^n(x),\ y) \sim (s_i^n(\sigma),\ x,\ y)$$

for all $0 \leq i \leq n$, $\sigma \in N(\mathcal{C})_n$, $x \in \Delta^n$, $y \in F(C_0)$. A *k-th skeleton* of $\operatorname{hocolim}_{\mathcal{C}} F$, denoted by $\operatorname{hocolim}_{\mathcal{C}}^{(k)} F$, is a space

$$\operatorname{hocolim}_{\mathcal{C}}^{(k)} F := \left(\coprod_{n \leq k} \coprod_{\sigma \in N(\mathcal{C})_n} \Delta^n \times F(\sigma(0))/\sim \right) \subseteq \operatorname{hocolim}_{\mathcal{C}} F.$$

Definition 1.3.4. Let X be a p-compact group. A functor $F : \mathcal{C} \to \mathbf{Sp}$ is *a homotopy decomposition* of X if there is a family of maps $f_C : F(C) \to BX$ which extends to a weak homotopy equivalence

$$(\operatorname{hocolim}_{\mathcal{C}} F)_p^\wedge \xrightarrow{\simeq} BX$$

and each value of F is the classifying space of a p-compact group. A homotopy decomposition F is *toral* if for each $C \in \operatorname{Ob}(\mathcal{C})$ the space $F(C)$ is the classifying space of a p-compact toral group.

Centralizer decomposition

Definition 1.3.5. Assume that X is a p-compact group. Let $\mathcal{A}_p(X)$ be the category whose objects are classifying maps of monomorphisms of the form

$Bf : B(\mathbb{Z}/p)^n \to BX$ and morphisms are commutative diagrams

$$\begin{array}{ccc} BC_p^m & \xrightarrow{Bf} & BX \\ \big\downarrow & \nearrow_{Bg} & \\ BC_p^n & & \end{array}$$

Theorem 1.3.6 ([DW3])**.** *If X is a p-compact group, then the evaluation maps*

$$\mathrm{map}(B(\mathbb{Z}/p)^n, BX)_{Bf} \to BX$$

induce a homotopy equivalence

$$(\mathrm{hocolim}_{Bf\in\mathcal{A}_p(X)}\, \mathrm{map}(B(\mathbb{Z}/p)^n, BX)_{Bf})^\wedge_p \xrightarrow{\simeq} BX.$$

Stubborn decomposition

The stubborn homotopy decomposition, originally due to Jackowski, McClure and Oliver [JMO1], works only for Lie groups (or for p-compact groups which are its completions). Recently it was extended to the general case by Notbohm [N1].

Definition 1.3.7. A closed subgroup P of a compact Lie group G is *p-stubborn* if:

- P is p-toral i.e. it is an extension of a finite p-group by a torus $(S^1)^n$,
- $W_G(P) := N_G(P)/P$ is finite,
- $W_G(P)$ has no normal p-subgroups (excluding the trivial subgroup).

Definition 1.3.8. Let $\mathcal{O}(G)$ denote the category of G-orbits and G-maps (by a G-orbit we mean a transitive G-set). Let $\mathcal{R}_p(G)$ denote the full subcategory of $\mathcal{O}(G)$ containing objects G/P for p-stubborn P.

Theorem 1.3.9 ([JMO1])**.** *The map*

$$(\mathrm{hocolim}_{G/P\in\mathcal{R}_p(G)}\, EG \times_G G/P)^\wedge_p \simeq BG^\wedge_p$$

is a weak homotopy equivalence. In other words, the functor $EG \times_G (-)$ is a homotopy decomposition of G. Since $EG \times_G G/P \simeq BP$, this decomposition is toral.

1.4 Extending maps to homotopy colimits

Homotopy decompositions are very useful tools for constructing maps from classifying spaces of p-compact groups. However, homotopy colimits are not colimits even in the homotopy category. In this section we present the obstruction theory for homotopy colimits.

Fix a small category $\mathcal{C}$, a functor $F : \mathcal{C} \to \mathbf{Sp}$ and a space Z. Moreover, fix a homotopy compatible family of maps $f = \{f_C : F(C) \to Z\} \in \lim_{\mathcal{C}}[F, Z]$. Assume that for each $C \in \mathcal{C}$ the mapping space $map(F(C), Z)_{f_C}$ is 1-connected. Define the functor

$$\Pi_i^f : \mathcal{C}^{op} \ni C \mapsto \pi_i(\mathrm{map}(F(C), Z)_{f_C}) \in \mathbf{Ab}. \tag{1.4.1}$$

A theorem provided by [W] states that if $H^{i+1}(\mathcal{C}; \Pi_i^f) = 0$ for all $i > 1$, then there is a map $\tilde{f} : \mathrm{hocolim}_{\mathcal{C}} F \to Z$ which extends f (i.e. $\tilde{f}|_{F(C)} = f_C$). If additionally $H^i(\mathcal{C}; \Pi_i^f) = 0$ for all $i > 1$, then $\tilde{f}$ is determined uniquely up to homotopy. We reprove this result, and then we construct some action on the set of homotopy classes of extensions of f if one of groups $H^i(\mathcal{C}; \Pi_i^f)$ does not vanish. We also prove that this action is functorial in some sense.

Denote for short $X := \mathrm{hocolim}_{\mathcal{C}}\ F$ and $X_n := \mathrm{hocolim}_{\mathcal{C}}^{(n)} F$. Let

$$C_i^j = \prod_{\sigma \in N(\mathcal{C})_j} \Pi_i^f(\sigma(0))$$

and let for $u \in C_i^j$

$$\delta_i^j(u)(\sigma) = F(\sigma(0 \to 1))^* u(d_0(\sigma)) + \sum_{k=1}^{j+1} (-1)^k u(d_k(\sigma)) \in C_i^{j+1}.$$

By [O2, Lemma 2], $H^*(C_i^*, \delta_i^*) = H^*(\mathcal{C}; \Pi_i^f)$. For each i let Z_i^j, B_i^j, H_i^j denote the cocycles, the coboundaries and the cohomology of the cochain complex C_i^* (note that $H_i^j = H^j(\mathcal{C}; \Pi_i^f)$).

Let $g : X_i \to Z$ be any map extending f (i.e. such that $g|_{F(C)} = f_C$) and let $\mathrm{sk}_i\, \Delta^j$ denote i-skeleton of Δ^j. For any $\sigma \in N(\mathcal{C})_j$ let

$$Ad_\sigma(g) : \mathrm{sk}_i\, \Delta^j \to \mathrm{map}(F(\sigma(0)), Z)_{f_{\sigma(0)}} \tag{1.4.2}$$

be an adjoint map to a composition

$$\mathrm{sk}_i\, \Delta^j \times F(\sigma(0)) \cong \{\sigma\} \times \mathrm{sk}_i\, \Delta^j \times F(\sigma(0)) \longrightarrow X_n \xrightarrow{g} Z.$$

Since the image of a map $\{\sigma\} \times \mathrm{sk}_i \Delta^j \times F(\sigma(0)) \to X_{i+1}$ lies in X_i, then $Ad_\sigma(g)$ is well-defined.

Remark. By the definition of a homotopy colimit (1.3.3), any map $g : X_n \to Z$ (resp. $g : X \to Z$) is determined uniquely by a collection of maps $Ad_\sigma(g)$, where $\sigma \in N(\mathcal{C})_i$, $0 \le i \le n$ (resp. $i \ge 0$) which satisfies conditions

$$\begin{aligned}
(d^i_k : \Delta^{i-1} \to \Delta^i) \circ Ad_\sigma(g) &= Ad_{d^i_k(\sigma)}(g), \qquad \text{if } k > 0,\\
(d^i_0 : \Delta^{i-1} \to \Delta^i) \circ Ad_\sigma(g) &= Ad_{d^i_0(\sigma)}(g) \circ F(\sigma(0 \to 1))^*\\
(s^k_i : \Delta^{i+1} \to \Delta^i) \circ Ad_\sigma(g) &= Ad_{s^i_k(\sigma)}(g)
\end{aligned}$$

for any i, $\sigma \in N(\mathcal{C})_i$.

Now define a cochain $o_i(g) \in C^{i+1}_i$ by

$$o_i(g)(\sigma) = Ad_\sigma(g)_*[\partial\Delta^{i+1}] \in \pi_{i+1} \operatorname{map}(F(\sigma(0)), Z)_{f_{\sigma(0)}}. \tag{1.4.3}$$

(Note that we do not need to care about basepoints since $\operatorname{map}(F(\sigma(0)), Z)_{f_{\sigma(0)}}$ is supposed to be 1-connected).

Now let $h : X_{i-1} \to Z$ be a map extending f, and let E^i_h be a set of extensions of h to X_i modulo homotopy constant on X_{i-1}. Our goal is to define a free and transitive action of H^i_i on E^i_h. It will be a generalization of the following very elementary construction. Let Y be a 1-connected space and let $a, b : \Delta^i \to Y$ be any maps such that $a|_{\partial\Delta^i} = b_{\partial\Delta^i}$. Then a map

$$S^i \simeq \Delta^i \cup_{\partial\Delta^i} \Delta^i \xrightarrow{a \cup b} Y$$

determines a class $a - b \in \pi_i(Y)$. If b is fixed, then for any class $s \in \pi_i(Y)$ there exists a map a satisfying $a - b = s$ (and such a is determined uniquely up to homotopy mod $\partial\Delta^i$). As a consequence we obtain the following

Proposition 1.4.4. *The set of extensions of a given $\partial\Delta^i \to Y$ to Δ^i modulo homotopy constant on $\partial\Delta^i$ carries a natural action of $\pi_n(Y)$ which is free and transitive.* □

For any $u \in C^i_i$ and any $[g] \in E^i_h$ let $g + c : X_i \to Z$ be a map such that $g + c|_{X_{i-1}} = h$ and $Ad_\sigma(g + u) = Ad_\sigma(g) + u(\sigma)$ for each $\sigma \in N(\mathcal{C})_i$.

Proposition 1.4.5. *The action of C^i_i on E^i_h is free an transitive.*

Proof. Maps $g : X_i \to Z$ which extend h are in 1-1 correspondence with collections of maps $Ad_\sigma(g)$, $\sigma \in N(\mathcal{C})_i$ such that $Ad_\sigma(g)|_{\partial\Delta^i} = Ad_\sigma(h)$. Now the conclusion follows from 1.4.4. □

The following proposition enlists some properties of concepts introduced above:

Proposition 1.4.6. *Fix $g : X_i \to Z$ such that $g|_{F(C)} = f_C$ for each $C \in \mathcal{C}$. Then*

(a) g extends to X_{i+1} if and only if $o_i(g) = 0$.

(b) $o_i(g) \in Z_i^{i+1}$.

(c) $o_i(g+u) = o_i(g) + \delta_i^i(u)$ for each $u \in C_i^i$

(d) $g|_{X_{i-1}}$ extends to X_{i+1} if and only if $o_i(g) \in B_i^{i+1}$

(e) Fix $u \in C_i^i$. Then g is homotopic to $g+u$ mod X_{i-2} if and only if $u \in B_i^i$.

Proof. Ad (a): Follows immediately from the definition of $o_i(g)$.

Ad (b): Let $\partial_k \Delta^j$ denotes a k-th face of Δ^j. For any $\sigma \in N(\mathcal{C})_{i+2}$ we have

$$\delta_i^{i+1}(o_i(g))(\sigma) = F(\sigma(0 \to 1))^* u(d_0(\sigma)) + \sum_{k=1}^{i+2} (-1)^k o_i(g)(d_k(\sigma))$$
$$= F(\sigma(0 \to 1))^* Ad_{d_0(\sigma)}(g)_*[\partial\Delta^{i+1}] + \sum_{k=1}^{i+2} (-1)^k Ad_{d_k(\sigma)})(g)_*[\partial\Delta^{i+1}]$$
$$= \sum_{k=0}^{i+2} (-1)^k Ad_\sigma(g)_*[\partial\partial_k\Delta^{i+2}] = Ad_\sigma(g)_* \left(\sum_{k=0}^{i+2} (-1)^k [\partial\partial_k\Delta^{i+2}] \right) = 0.$$

Ad (c): For any $\sigma \in N(\mathcal{C})_{i+1}$ we have

$$o_i(g+u)(\sigma) = Ad_\sigma(g+u)_*[\partial\Delta^{i+1}] =$$
$$(Ad_\sigma(g)_*[\partial\Delta^{i+1}]) + F(\sigma(0 \to 1))^* u(d_0(\sigma)) + \sum_{k=1}^{i+1} u(d_k(\sigma)) = o_i(g) + \delta_i^i(u)$$

Ad (d): Assume that $o_i(g) = \delta_i^i(u)$ for some $u \in C_i^i$. By (c) we have $o_i(g+(-u)) = o_i(g) - \delta_i^i(u) = 0$. Thus $g+(-u)$ extends to X_{i+1} and so does $g|_{X_{i-1}} = (g + (-u))|_{X_{i-1}}$. Now assume that $g|_{X_{i-1}}$ extends to a map $h : X_{i+1} \to Z$. By 1.4.5 there is a cochain $u \in C_i^i$ such that $g = h|_{X_i} + u$. Then by (c) $\delta_i^{i+1}(o_i(g)) = \delta_i^{i+1}(o_i(h|_{X_i}) + \delta_i^i(u)) = \delta_i^{i+1}(\delta_i^i(u)) = 0$.

Ad (e): Let $h : [0,1] \times X_i \to Z$ be a homotopy between g and $g + u$ constant on X_{i-2} and for any $\sigma \in N(\mathcal{C})_j$, $j \leq i$ let

$$Ad_\sigma(h) : [0,1] \times \Delta^j \to \mathrm{map}(F(\sigma(0)), Z)_{f_{\sigma(0)}}$$

be a map defined analogously to 1.4.2. For any $\sigma \in N(\mathcal{C})_{i-1}$ let $v(\sigma) \in \Pi_i^f$ be a homotopy class of the map

$$S^i \simeq [0,1]\times\Delta^{i-1}\cup_{\partial([0,1]\times\Delta^{i-1})}[0,1]\times\Delta^{i-1} \xrightarrow{Ad_\sigma(g)\circ p \cup Ad_\sigma(h)} \mathrm{map}(F(\sigma(0)), Z)_{f_{\sigma(0)}}$$

where $p : [0,1] \times \Delta^{i-1} \to \Delta^{i-1}$ is a projection. Then it is easy to check that $u = \delta_i^{i-1}(v)$. Given $v \in C_i^{i-1}$ such that $u = \delta_i^{i-1}(v)$, a homotopy between g and $g + u$ can be defined in a similar way. □

Proposition 1.4.7. *If $H^{i+1}(\mathcal{C}; \Pi_i^f) = 0$ for $i > 1$, then f extends to a map $X \to Z$.*

Proof. Since all mapping spaces $\mathrm{map}(F(C), Z)_{f_C}$ are assumed to be 1-connected, then f extends to X_1. If f extends to a map $g : X_i \to Z$ (where $i > 1$), then $o_i(g) \in Z_i^{i+1}$ (by 1.4.6.(b)) and then $o_i(g) \in B_i^{i+1}$ (by assumption). Hence $g|_{X_{i-1}}$ extends to X_{i+1} (by 1.4.6.(d)). By induction, f extends to X. □

For the rest of the section we fix $n > 1$, and assume that $H^{i+1}(\mathcal{C}; \Pi_i^f) = 0$ for all $i > 1$ and that $H^i(\mathcal{C}; \Pi_i^f) = 0$ for all $1 < i \neq n$. Under these assumptions holds

Proposition 1.4.8. *Let $g, g' : X \to Z$ be any extensions of f. Then*

(a) $g|_{X_{n-1}} \sim g'|_{X_{n-1}}$.

(b) If $g|_{X_n} \sim g'|_{X_n}$, *then* $g \sim g'$.

(c) If $u \in C_n^n$, *then* $g|_{X_n} + u$ *extends to* X *if and only if* $u \in Z_n^n$.

Proof. By 1-connectivity of the suitable mapping spaces $g|_{X_1} \sim g'|_{X_1}$. Let $n \neq i > 1$ and assume $g|_{X_{i-1}} \sim g'|_{X_{i-1}}$. We can replace g' by a homotopic map such that $g'|_{X_{i-1}} = g|_{X_{i-1}}$. By 1.4.5 there is $u \in C_i^i$ such that $g' = g + u$.

Moreover, $0 = o_i(g'|_{X_i}) = o_i(g|_{X_i}) + \delta_i^i(u) = \delta_i^i$. Thus $u \in Z_i^i$ and by assumptions we have $u \in B_i^i$. Now an application of 1.4.6.(e) shows that $g|_{X_i} \simeq g'|_{X_i}$. By induction on i starting from 2 we obtain (a) and by induction starting from $n+1$ we obtain (b). To prove (c) note that the condition $u \in Z_n^n$ is necessary to extensibility of $g|_{X_n} + u$ to X_{n+1} (by 1.4.6.(a) and 1.4.6.(c)). On the other hand, if $g|_{X_n} + u$ extends to X_{n+1} it extends also to X by an inductive argument similar to the one used in the proof of 1.4.7. □

Let $g : X \to Z$ be any map extending f, and let $[u] \in H_n^n$. Let $(g + [u]) : X \to Z$ be a map such that $(g + [u])|_{X_{n-1}} = g|_{X_{n-1}}$, $(g + [u])|_{X_n} = g|_{X_n}$. By 1.4.8.(c) such a map exists, by 1.4.8.(b) it is unique up to homotopy and by 1.4.6.(e) it does not depend on a choice of a cocycle u. Then, by 1.4.5, it is an action of H_n^n on the set E_f^∞ of homotopy classes of maps extending f.

Proposition 1.4.9. *The action of H_n^n on E_f^∞ is transitive. If $n = 2$, then it is also free.*

Proof. The first part follows form 1.4.8. Since for each $C \in \mathcal{C}$ the mapping spaces $\mathrm{map}(F(C), Z)_{f_C}$ are 1-connected, then any two homotopic maps $g \sim g' : X_2 \to Z$ are homotopic modulo X_0. Now the second part follows from 1.4.6.(e). □

The main result of the section is that the action of H_n^n on E_f^∞ is functorial, in some sense. To make a precise statement we need the following

Definition 1.4.10. *The category of diagrams on a category $\mathcal{A}$*, denoted by $\mathbf{Diag}_{\mathcal{A}}$, is the category whose objects are functors $A : \mathcal{C}_A \to \mathcal{A}$ from a small category $\mathcal{C}_A$. A morphism φ from $A : \mathcal{C}_A \to \mathcal{A}$ to $B : \mathcal{C}_B \to \mathcal{A}$ is a pair (T_φ, t_φ), where $T_\varphi : \mathcal{C}_A \to \mathcal{C}_B$ is a functor and $t_\varphi : A \to B \circ T_\varphi$ is a natural transformation.

Remark. Every morphism $\varphi : A \to B$ in the category $\mathbf{Diag}_{\mathbf{Sp}}$ induces a map $\varphi_* : \mathrm{hocolim}_{\mathcal{C}_A} A \to \mathrm{hocolim}_{\mathcal{C}_B} B$.

Obviously, F is an object in the category $\mathbf{Diag}_{\mathbf{Sp}}$. Let $\varphi : F \to F$ be an automorphism in the category $\mathbf{Diag}_{\mathbf{Sp}}$ such that for each $C \in \mathcal{C}$ maps f_C and $(\varphi^* f)_C := f_{T_\varphi(C)} \circ t_\varphi(C)$ are homotopic. As before E_f^∞ (resp. $E_{\varphi^* f^\infty}$) denotes a set of homotopy classes of extensions of f (resp. $\varphi^* f$). Obviously there is a natural bijection $E_f^\infty \cong E_{\varphi^* f}^\infty$.

The morphism φ induces a bijection $\varphi* : E_f^\infty \to E_f^\infty$ by

$$E_f^\infty \ni [g] \mapsto [X \xrightarrow{\varphi_*} X \xrightarrow{g} Z] \in E_{\varphi^* f}^\infty \cong E_f^\infty \qquad (1.4.11)$$

and automorphisms of chain complexes C^*_k by

$$(\varphi^*(u))(\sigma) = t^*_\varphi(u(T^*_\varphi(\sigma))), \qquad \text{for } u \in C^l_k,\ \sigma \in N(\mathcal{C})_l \tag{1.4.12}$$

which obviously induces automorphisms $\varphi^* : H^n_n \to H^n_n$.

Now we are ready to prove the main result of this section:

Theorem 1.4.13. *The diagram*

$$\begin{array}{ccc} H^n_n \times E^\infty_f & \xrightarrow{+} & E^\infty_f \\ \Big\downarrow{\scriptstyle \varphi^*\times\varphi^*} & & \Big\downarrow{\scriptstyle \varphi^*} \\ H^n_n \times E^\infty_f & \xrightarrow{+} & E^\infty_f \end{array}$$

commutes.

Proof. Let $g : X \to Z$ be a map extending f, and let $g' : X \to Z$ be a map homotopic to $\varphi^* g$ such that $g'|_{X_{n-1}} = g|_{X_{n-1}}$.

Note that for any simplex σ we have

$$Ad_\sigma(\varphi^* g) = t^*_\varphi(\sigma(0)) \circ Ad_{T_\varphi \sigma}(g)$$

Then for any $u \in Z^n_n$, $\sigma \in N(\mathcal{C})_n$

$$\begin{aligned} Ad_\sigma(\varphi^*(g+u)) &= t^*_\varphi u(\sigma(0)) \circ Ad_{T_\varphi\sigma}(g+u) \\ &= t^*_\varphi u(\sigma(0)) \circ [Ad_{T_\varphi\sigma}(g) + u(T_\varphi(g))] \\ &= t^*_\varphi u(\sigma(0)) \circ Ad_{T_\varphi\sigma}(g) + t^*_\varphi(u(T_\varphi(g))) = Ad_\sigma(\varphi^* g) + \varphi^*(u)(\sigma) \end{aligned}$$

Then $\varphi^*(g+u)|_{X_n} = \varphi^*(g)|_{X_n} + \varphi^* u$. Now the conclusion follows by 1.4.8.(b). □

1.5 2-Compact group $DI(4)$

In this section we recall the construction and some properties of the 2-compact group $DI(4)$ constructed by Dwyer and Wilkerson in [DW1].

The main result is

Theorem 1.5.1. ([DW1, 1.1]) *There exists a 2-compact group* $DI(4)$ *such that* $H^*(BDI(4);\mathbb{F}_2)$ *is isomorphic as an algebra over Steenrod algebra to the ring of rank 4 mod 2 Dickson invariants* $\mathbb{F}_2[x_1,x_2,x_3,x_4]^{\mathrm{GL}_4(\mathbb{F}_2)}$.

Now let us give state properties of $DI(4)$. All of them are proved in [DW1]:

- $H^*(BDI(4);\mathbb{F}_2)$ is a polynomial algebra on classes c_8, c_{12}, c_{14}, c_{15} (where $c_i \in H^i(BDI(4);\mathbb{F}_2)$),
- The Weyl group $W_{DI(4)}$ is isomorphic to $\{\pm 1\} \times \mathrm{GL}_3(\mathbb{F}_2)$.
- $\dim DI(4) = 45$.

Recall a sketch of the proof of 1.5.1. The first step is to find a suitable action of $W := W_{DI(4)}$ on the maximal torus, or equivalently the representation $W \to \mathrm{GL}(3,\mathbb{Z}_2^\wedge)$. Let $a : W_{Spin(7)} \to \mathrm{GL}_3(\mathbb{Z})$ be a natural homomorphism. Let V be a subgroup of $W_{Spin(7)}$ of index 2 such that the composition

$$i : V \subset W_{Spin(7)} \xrightarrow{a} \mathrm{GL}_3(\mathbb{Z}) \xrightarrow{\text{mod } 2} \mathrm{GL}_3(\mathbb{F}_2)$$

is a monomorphism (one can easily see that V is isomorphic to $C_2 \wr \Sigma_3$).

Proposition 1.5.2. ([DW3, 4.1]) *There exists a section* j *of the mod 2 reduction* $\mathrm{GL}_3(\mathbb{Z}_2^\wedge) \to \mathrm{GL}_3(\mathbb{F}_2)$ *such that compositions*

$$V \xrightarrow{\subset} W_{Spin(7)} \xrightarrow{a} \mathrm{GL}_3(\mathbb{Z}) \xrightarrow{\subset} \mathrm{GL}_3(\mathbb{Z}_2^\wedge) \tag{1.5.3}$$

and

$$V \xrightarrow{i} \mathrm{GL}_3(\mathbb{F}_2) \xrightarrow{j} \mathrm{GL}_3(\mathbb{Z}_2^\wedge) \tag{1.5.4}$$

are equal. □

Remark. In Section 5.2 we give an implicit description of the homomorphism $j : \mathrm{GL}_3(\mathbb{F}_2) \to \mathrm{GL}_3(\mathbb{Z}_2^\wedge)$.

Let $\mathcal{A}$ be the inverse category of the category of vector subspaces of $\mathbb{F}_2^4$ and monomorphisms. It has (up to isomorphism) four objects A_i, where $1 \le i \le 4$ and $\dim A_i = i$. We will define a functor $F : \mathcal{A} \to \mathbf{Sp}$ such that $\operatorname{hocolim} F \simeq BDI(4)$. As mentioned before we require that $H^*(F;\mathbb{F}_2) \cong K$ (cf. 0.2.2). Let

- $G_1 = SO(7)$
- $G_2 = SO(4) \times SO(3) \subseteq G_1$
- $G_3 = (SO(2) \times SO(2)) \rtimes C_2 \times SO(2) \times SO(1) \subseteq G_2$
- $G_4 = \langle -I_2 \times I_5, I_2 \times -I_2 \times I_3, I_4 \times -I_2 \times I_1 \rangle \subseteq G_3$,

where $I_k \in O(k)$ is an identity matrix. Next, put $F(A_i) := (B\tilde{G}_i)^\wedge_2$, where $\tilde{G}$ is the counterimage of $G \subseteq SO(7)$ in $Spin(7)$. Now we need to define the action of $\mathrm{Aut}(A_i) \simeq \mathrm{GL}_i(\mathbb{F}_2)$ on $F(A_i)$. The action on $F(A_1)$ is trivial (since $\mathrm{Aut}(A_1) = 1$) and the action of $\mathrm{Aut}(A_4) \simeq \mathrm{GL}_4(\mathbb{F}_2)$ on $F(A_4)$ is induced by the canonical action on $\tilde{G}_4 \simeq C_2^4$. The action on the remaining two spaces is more complicated. The action of $\mathrm{GL}_3(\mathbb{F}_2)$ on $F(A_3)$ comes from 1.5.2. The construction of the action of $\mathrm{GL}_2(\mathbb{F}_2)$ on $F(A_2)$ follows from the following proposition:

Proposition 1.5.5. *If $H \simeq SU(2)^n$ or $H \simeq SU(2)^n/\{\pm 1\}$, then the group of homotopy classes of self-equivalences of $BH^\wedge_2$ is isomorphic to the group $N_{\mathrm{GL}_n(\mathbb{Z}^\wedge_2)}(W_H)/W_H$.*

Proof. This is a direct corollary from [DW1, 5.5]. □

Note that the group $\tilde{G}_2 = \widetilde{SO(4) \times SO}(3)$ is isomorphic $SU(2)^3/\{\pm 1\}$ and that $\mathrm{GL}_2(\mathbb{F}_2) \simeq N_{C_2 \times \mathrm{GL}_2(\mathbb{F}_3)}(W_{\tilde{G}_2}) \subseteq N_{\mathrm{GL}_3(\mathbb{Z}^\wedge_2)}(W_{\tilde{G}_2})$. Hence 1.5.5 gives the suitable action on $F(A_2)$. The detailed checking that this data gives the required functor into the homotopy category is presented in [DW1, Section 6]. Moreover, the functor F can be lifted to the functor which commutes on the nose ([DW1, 7.7]). Finally, $DI(4)$ is a 2-compact group defined by

$$BDI(4) := (\mathrm{hocolim}_{\mathcal{A}} F)^\wedge_2.$$

Chapter 2

Homotopy representations of compact Lie groups

Let G be a compact Lie group and let p be a prime integer. *A homotopy complex representation* of G at p is a map $BG \to BU(m)^\wedge_p$. In this chapter we describe a method of constructing homotopy complex representations of compact Lie groups using the stubborn homotopy decompositions (1.3.9). For any complex representation $\varphi : G \to U(m)$ the p-completion of the classifying map $B\varphi^\wedge_p : BG^\wedge_p \to BU(m)^\wedge_p$ is obviously a homotopy complex representation. Moreover, isomorphic representations produce isomorphic (i.e. homotopic) homotopy representations. However, many homotopy representations are *not* completions of "algebraic" ones.

Consider the following diagram

$$\begin{array}{ccc}
\mathrm{Rep}(N^\infty, U(m)) & \overset{B(-)^\wedge_p}{=\!=\!=\!=\!=} & [BN^\infty, BU(m)^\wedge_p] \\
\Big\uparrow & & \Big\uparrow \\
\lim\limits_{G/P \in \mathcal{R}} \mathrm{Rep}(P^\infty, U(m)) & \overset{B(-)^\wedge_p}{=\!=\!=} & \lim\limits_{G/P \in \mathcal{R}} [EG \times G/P, BU(m)^\wedge_p] \\
\Big\uparrow & & \Big\uparrow \\
\mathrm{Rep}(G, U(m)) & \xrightarrow{B(-)^\wedge_p} & [BG, BU(m)^\wedge_p]
\end{array} \tag{2.0.1}$$

where N is the p-normalizer of the maximal torus of G.

Both the upper and the middle horizontal arrow are bijections due to a version of the Dwyer-Zabrodsky theorem (2.2.4), which relates homotopy representations of a p-toral group P with algebraic representations of its approximation $P^\infty \subseteq P$ (2.1.4). Obviously the lower horizontal arrow is neither an injection nor a surjection — in fact it is rather difficult to understand. To find a map $BG \to BU(m)^\wedge_p$ we need to choose some representation $\varrho \in \mathrm{Rep}(N^\infty, U(m))$ first. Since p-discrete approximations of p-toral groups are not finite (unless the group itself is finite), then the classical representation theory cannot be used directly to classify representations of N^∞. However, discrete approximations are locally finite and its complex (algebraic) representations admit, as it will be shown in Section 2.3, many properties similar to the properties of finite groups.

The problem of descending of $\varrho \in \mathrm{Rep}(N^\infty, U(m))$ at the middle level is purely algebraic: if ϱ extends to an element of $\lim_{G/P \in \mathcal{R}} \mathrm{Rep}(P^\infty, U(m))$ it is called *$\mathcal{R}$-invariant*. The next step is to check if an $\mathcal{R}$-invariant representation ϱ extends (or extends uniquely) to a map $BG \to BU(m)^\wedge_p$. The answer is stated in terms of some obstruction groups (cf. 1.4).

2.1 Discrete approximations of p-toral groups

We consider three kinds of toral groups: p-toral groups, p-compact toral groups and p-discrete toral groups. Let us state the definitions of these objects and explain relations between them.

Definition 2.1.1. A group T is

- *a torus* if it is isomorphic to $(S^1)^r \cong (SO(2))^r$,
- *a p-discrete torus* if it is isomorphic to $(\mathbb{Z}/p^\infty)^r$ (with discrete topology!),
- *a p-compact torus* if it is a p-compact group isomorphic to $K(\mathbb{Z}^\wedge_p, 1)^r$ with classifying space $K(\mathbb{Z}^\wedge_p, 2)$.

Definition 2.1.2. A group P is *a p-toral group* (resp. *a p-discrete toral*, *a p-compact toral*) if it sits in the exact sequence

$$1 \longrightarrow T(P) \longrightarrow P \longrightarrow W(P) \longrightarrow 1,$$

where $W(P)$ is a finite p-group and $T(P)$ is a torus (resp. a p-discrete torus, a p-compact torus). Note that in the last case "the exact sequence" means that that there is the suitable fibration of the classifying spaces (cf. 1.2.5).

Remark. If P is a p-toral group (resp. a p-discrete toral, a p-compact toral), then the group $T(P)$ is uniquely determined as the connected component of unity (resp. the subgroup of all infinitely p-divisible elements, the universal covering of the classifying space).

There are two operations between these classes of groups. The first one is the completion producing a p-compact toral group out of any of the other classes. The second one is taking a discrete approximation producing a p-discrete toral group.

Proposition 2.1.3. *Let P be either a toral group or a p-discrete toral group. Then $P^\wedge_p = (P^\wedge_p, BP^\wedge_p, \epsilon_P)$ is a p-compact toral group.*

Proof. The first part is clear and the second is [DW2, 6.10]. □

Definition 2.1.4. Let P be a p-toral group and $P^\wedge$ be a p-compact toral group.

- A p-discrete toral subgroup $P^\infty \subseteq P$ is *a discrete approximation* if it is dense in P.
- A p-discrete toral group P^∞ equipped with an $\mathbb{F}_p$-equivalence $BP^\infty \to BP^\wedge$ is called *a discrete approximation* of $P^\wedge$.

Proposition 2.1.5 ([DW2, 6.9])**.** *Each p-compact toral group admits a p-discrete approximation.*

Proposition 2.1.6. *Each p-toral group P admits a unique up to conjugacy p-discrete approximation.*

Proof. Each discrete approximation P^∞ of P fits into the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & (\mathbb{Z}/p^\infty)^s & \longrightarrow & P^\infty & \longrightarrow & W(P) & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle i} & & \Big\downarrow & & \Big\| & & \\
1 & \longrightarrow & (S^1)^s & \longrightarrow & P & \longrightarrow & W(P) & \longrightarrow & 1
\end{array}$$

Let $c^\infty \in H^2(W(P); (\mathbb{Z}/p^\infty)^s)$ and $c \in H^2(W(P); (S^1)^s)$ be the cohomology classes of these extensions. Obviously $i_* c^\infty = c$. Consider the exact sequences

$$1 \longrightarrow \mathbb{Z}^s \longrightarrow \mathbb{Z}[\tfrac{1}{p}]^s \longrightarrow (\mathbb{Z}/p^\infty)^s \longrightarrow 1$$

$$1 \longrightarrow \mathbb{Z}^s \longrightarrow \mathbb{R}^s \longrightarrow (S^1)^s \longrightarrow 1.$$

Since $H^i(W(P); \mathbb{R}^s) = H^i(W(P); \mathbb{Z}[\frac{1}{p}]^s) = 0$, then for each $i > 0$ we have isomorphisms

$$H^2(W(P); (\mathbb{Z}/p^\infty)^s) \cong H^3(W(P); \mathbb{Z}^s) \cong H^2(W(P); (S^1)^s).$$

Therefore there is a bijection between the set of extensions of $W(P)$ by $(\mathbb{Z}/p^\infty)^s$ and the set of extensions of $W(P)$ by $(S^1)^s$. □

Proposition 2.1.7. ([DW2, 6.20]) *For each p-discrete toral group P^∞ there exists a sequence of finite p-subgroups*

$$P^{(i)} \subseteq P^{(i+1)} \subseteq P^{(i+2)} \subseteq \ldots \subseteq P^\infty$$

fitting into the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T^{(i)}(P) & \longrightarrow & P^{(i)} & \longrightarrow & W(P) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & T(P) & \longrightarrow & P & \longrightarrow & W(P) & \longrightarrow & 1 \end{array}$$

where $T^{(i)}(P) \simeq (\mathbb{Z}/p^i)^n \subseteq T(P)$ is the subgroup of elements of order p^k, $k \leq i$. □

Definition 2.1.8. The group $P^{(i)}$ will be called *an i-th discrete approximation* of P.

Remark. The sequence $P^{(i)}$ is not unique.

2.2 The Dwyer-Zabrodsky theorem

Throughout the present section P is a p-discrete toral group and G is a compact connected Lie group. Recall the classical version of the Dwyer-Zabrodsky theorem:

Theorem 2.2.1 ([DZ]). *If P is finite, then $B : \mathrm{Rep}(P, G) \longrightarrow [BP, BG]$ is a bijection and for each $\varrho \in \mathrm{Rep}(P, G)$ the map*

$$BC_G(\varrho(P)) \longrightarrow \mathrm{map}(BP, BG)_{B\varrho}$$

adjoint to the pairing $C_G(\varrho(P)) \times P \ni (g, p) \mapsto (\varrho(g)p) \in G$ induces an isomorphism on $\mathbb{F}_p$-homology.

In this section we give a slightly different version of this theorem. We compute the p-homotopy type of the mapping space $\mathrm{map}(BP, BG^\wedge_p)$, where P is any p-discrete toral group.

Proposition 2.2.2. *If P is a finite p-group, then the map*

$$[BP, BG] \longrightarrow [BP, BG^\wedge_p]$$

induced by the p-completion $\varepsilon_{BG} : BG \to BG^\wedge_p$ is a bijection.

Proof. Replace the map $BG \to BG^\wedge_p$ by a homotopy equivalent fibration

$$F \longrightarrow BG \xrightarrow{e} BG^\wedge_p$$

Fix $f : BP \to BG^\wedge_p$. The obstructions for the existence (resp. the uniqueness) of a lift $\widetilde{f} : BP \to BG$ lie in groups $H^{i+1}(P, \pi_i(F))$ (resp. $H^i(P, \pi_i(F))$). But groups $\pi_i(F)$ are abelian and multiplication by p on them is an isomorphism. □

Proposition 2.2.3. *The restrictions $\mathrm{Rep}(P, G) \to \mathrm{Rep}(P^{(r)}, G)$ induce the bijection*

$$\mathrm{Rep}(P, G) \longrightarrow \lim_{r\to\infty} \mathrm{Rep}(P^{(r)}, G)$$

Proof. Let $\varphi, \psi \in \mathrm{Hom}(P, G)$ be non-conjugate homomorphisms and put

$$G_r = \{g \in G : \forall_{p\in P^{(r)}}\ g^{-1}\varphi(p)g = \psi(p)\}.$$

G_r is a non-increasing sequence of closed subsets of G. If $\bigcap_r G_r = \emptyset$ then by compactness of G there exists r such that $G_r = \emptyset$. Then $\varphi|_{P^r}$ is not conjugate to $\psi|_{P^r}$. This shows injectivity of the map $\operatorname{Rep}(P, G) \to \lim_r \operatorname{Rep}(P^{(r)}, G)$.

Now let $\varphi_r \in \operatorname{Hom}(P^{(r)}, G)$ for $r \geq i$ be a sequence such that φ_r is conjugate to $\varphi_s|_{P^{(r)}}$ for $r \leq s$. Define homomorphisms $\psi_r \in \operatorname{Hom}(P^{(r)}, G)$ by induction. For $r = i$ put $\psi_i = \varphi_i$ and for $r > i$ let $\psi_r(p) = g^{-1}\varphi_r(p)g$ where g is any element such that $\psi_{r-1} = g^{-1}\varphi_r|_{P^{(r-1)}}g$. Then $\psi = \bigcup_r \psi_r$ is a well-defined homomorphism $P \to G$ such that $\psi|_{P^r}$ is conjugate to φ_r for each r. This implies the surjectivity. □

Theorem 2.2.4 ([JMO2, Thm. 1.1])**.** *The map*

$$\varepsilon_{BG} \circ B : \operatorname{Rep}(P, G) \longrightarrow [BP, BG^\wedge_p]$$

is a bijection and for each $\varrho \in \operatorname{Rep}(P, G)$ the map

$$BC_G(\varrho(P)) \longrightarrow \operatorname{map}(BP, BG^\wedge_p)_{B\varrho}$$

is a mod p homology equivalence.

Proof. Consider the diagram

$$\begin{array}{ccccc}
\operatorname{Rep}(P, G) & & \xrightarrow[\ (2.2.3)\ ]{\cong} & & \lim\limits_{r\to\infty} \operatorname{Rep}(P^{(r)}, G) \\
\big\downarrow & & & & \cong \big\downarrow (2.2.1) \\
[BP, BG^\wedge_p] & = = = & [\operatorname{hocolim}\limits_{r\to\infty} BP^{(r)}, BG^\wedge_p] & \xrightarrow{R} & \lim\limits_{r\to\infty} [BP^{(r)}, BG^\wedge_p]
\end{array}$$

By 2.2.1 each element of $\lim_{r\to\infty}[BP^{(r)}, BG^\wedge_p]$ has the form $\{\varepsilon_{BG} \circ B\varrho_r\}$, where $\varrho_r \in \operatorname{Rep}(P^{(r)}, G)$. By 2.2.3 we can assume that $\varrho_r = \operatorname{res}^P_{P^{(r)}} \varrho$ for some $\varrho \in \operatorname{Rep}(P, G)$. The collection of maps $\{\varepsilon_{BG} \circ B\varrho_r\}$ always extends to a map $\operatorname{hocolim}_{r\to\infty} BP^{(r)} \to BG^\wedge_p$ but the uniqueness of the extension is determined by vanishing obstruction lying in

$$H^1(\mathbb{N}; \pi_1 \operatorname{map}(BP^{(r)}, BG^\wedge_p)_{B\varrho_r}) = H^1(\mathbb{N}; \pi_1 BC_G(\varrho(P^{(r)}))^\wedge_p), \qquad \text{(by 2.2.1)}$$

where $\mathbb{N}$ is the category of positive integers with a single morphism $r \to r'$ for $r \leq r'$. Since $C_G(\varrho(P))$ is a descending sequence of closed subgroups of G it eventually stabilizes. Therefore the group H^1 vanishes and R is a bijection. □

2.3 Complex representations of locally finite groups

In the present section we generalize numerous theorems concerning complex representations of finite groups to locally finite groups. These facts allow to classify irreducible representations of p-discrete toral groups, especially the ones which appear as discrete approximations of stubborn subgroups of $Spin(7)$. Finally, we prove that any complex representation of a locally finite group has a unique unitary structure (up to isomorphism).

Let us start with the definition.

Definition 2.3.1. A group Γ is *locally finite* if it is countable and every finitely generated subgroup of Γ is finite.

The following proposition states the crucial property of countably generated locally finite groups:

Proposition 2.3.2. *For each locally finite group* Γ *there is an ascending sequence of finite subgroups*

$$\{e\} = \Gamma^{(0)} \subseteq \Gamma^{(1)} \subseteq \Gamma^{(2)} \subseteq \cdots \subseteq \Gamma$$

such that $\bigcup_{r \geq 0} \Gamma^r = \Gamma$.

Proof. Let $\{g_s\}_{s=1}^{\infty}$ be a set of generators of Γ. Put $\Gamma^{(r)} = \langle g_s \rangle_{s=0}^{r}$. □

Definition 2.3.3. *A complex representation* of a group Γ is a left action of Γ on a complex vector space V which preserves. In other words it is a homomorphism $\varrho : \Gamma \to \mathrm{GL}(V)$. Representations $\varrho : \Gamma \to GL(V)$, $\varrho' : \Gamma \to GL(V')$ are *isomorphic* if and only if there is an isomorphism $f : V \to V'$, such that for each $g \in \Gamma$ the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{\varrho(g)} & V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ V' & \xrightarrow{\varrho'(g)} & V' \end{array}$$

We denote the set of isomorphism classes of representations of Γ by $\mathrm{Rep}(\Gamma)$. The subset containing only m-dimensional representations is denoted by $\mathrm{Rep}_m(\Gamma)$. A representation $\varrho : \Gamma \to \mathrm{GL}(V)$ is said to be *irreducible*, if V does not contain any non-trivial subrepresentation $0 \neq W \subsetneq V$. The set of isomorphism classes of irreducible representations of Γ is denoted by $\mathrm{IR}(\Gamma)$. Fix once for all representatives $\sigma : \Gamma \to \mathrm{GL}(W_\sigma)$ for each $\sigma \in \mathrm{IR}(\Gamma)$. Let $R(\Gamma)$ be the representation ring, i. e. the Grothendieck construction on $\mathrm{Rep}(\Gamma)$.

Definition 2.3.4. Let $\varrho : \Gamma \to \mathrm{GL}(V)$ and $\varrho' : \Gamma \to \mathrm{GL}(V')$ be complex representations. We can form *the direct sum* $\varrho \oplus \varrho' : \Gamma \to \mathrm{GL}(V \oplus V')$ by

$$(\varrho \oplus \varrho')(g)(v, v') := (\varrho(g)(v), \varrho'(g)(v')) \in V \oplus V'$$

and the tensor product $\varrho \otimes \varrho' : \Gamma \to \mathrm{GL}(V \otimes V')$ by

$$(\varrho \otimes \varrho')(g)(v \otimes v') := (\varrho(g)(v) \otimes \varrho'(g)(v')) \in V \otimes V'.$$

These operations give a structure of an abelian monoid with multiplication on $\mathrm{Rep}(\Gamma)$ and a structure of a commutative ring on $R(\Gamma)$.

Definition 2.3.5. Let $a : \Gamma \to \Delta$ be a homomorphism and $\varrho : \Delta \to \mathrm{GL}(V)$ a representation. *The restriction of* ϱ *along* a is a composition:

$$a^*\varrho : \Gamma \xrightarrow{a} \Delta \xrightarrow{\varrho} \mathrm{GL}(V).$$

If a is clear, for example if Γ is a subgroup of Δ, we denote the restricted representation by $\mathrm{res}_\Gamma^\Delta\, \varrho$.

For any group Γ let $\mathbb{C}[\Gamma]$ denote a group ring of Γ over $\mathbb{C}$.

Definition 2.3.6. Fix groups $\Gamma \subseteq \Delta$ and a representation $\varrho : \Gamma \to \mathrm{GL}(V)$. *The induced representation* $\mathrm{ind}_\Gamma^\Delta\, \varrho$ is a vector space $\mathbb{C}[\Delta] \times_{\mathbb{C}[\Gamma]} V$ with the obvious left action of Δ.

Our first goal is to prove that the Schur lemma holds for locally finite groups. Let us recall the classical version of this lemma and some of its consequences.

Proposition 2.3.7 (Schur lemma)**.** ([S, Prop. 6]) *Let Γ be a finite group.*

(a) If $\varrho : \Gamma \to \mathrm{GL}(V)$ is an irreducible representation, then

$$\mathrm{End}_\Gamma(V) \cong \mathbb{C}.$$

(b) If $\varrho : \Gamma \to \mathrm{GL}(V)$*,* $\sigma : P \to \mathrm{GL}(W)$ *are non-isomorphic irreducible representations, then*

$$\mathrm{Hom}_\Gamma(V, W) = 0.$$

Definition 2.3.8. Let $\varrho : \Gamma \to \mathrm{GL}(V)$ be any representation. A presentation of V as a direct sum of its subrepresentations

$$V \simeq \bigoplus_{\sigma \in \mathrm{IR}(\Gamma)} V_\sigma$$

is *a canonical decomposition* if and only if for each $\sigma \in \mathrm{IR}(\Gamma)$ the representation V_σ is isomorphic to a direct sum of representations isomorphic to σ.

Proposition 2.3.9 ([S, 2.6])**.** *Each representation of a finite group admits a unique canonical decomposition.*

Now fix a locally finite group Γ and choose a filtration $\Gamma^{(s)}$ (cf. Prop. 2.3.2). The following elementary observation allows to prove many results of this section:

Proposition 2.3.10. *Let* $\varrho : \Gamma \to \mathrm{GL}(V)$ *and* $\varrho' : \Gamma \to \mathrm{GL}(V')$ *be any representations. For large enough* s *we have* $\mathrm{Hom}_\Gamma(V, V') = \mathrm{Hom}_\Gamma^{(s)}(V, V')$.

Proof. A sequence

$$\mathrm{Hom}_{\Gamma^{(0)}}(V, V') \supseteq \mathrm{Hom}_{\Gamma^{(1)}}(V, V') \supseteq \mathrm{Hom}_{\Gamma^{(2)}}(V, V') \supseteq \dots$$

eventually stabilizes since the dimensions of these spaces do. Moreover

$$\mathrm{Hom}_\Gamma(V, V') = \bigcap_{r=0}^{\infty} \mathrm{Hom}_{\Gamma^{(r)}}(V).$$

Thus, for large enough s we have $\mathrm{Hom}_\Gamma(V, V') = \mathrm{Hom}_{\Gamma^{(s)}}(V, V')$. □

The following proposition plays a crucial role in the further considerations. It allows to relate representations of a locally finite group with representations of its finite subgroups. For each locally finite group Γ

Proposition 2.3.11. *Let* Γ *be a locally finite group.*

(a) *If* $\varrho : \Gamma \to \mathrm{GL}(V)$ *is a representation of* Γ*, then* ϱ *is irreducible of and only if for large enough* s *a restriction* $\mathrm{res}^{\Gamma}_{\Gamma^{(s)}}\ \varrho$ *is irreducible.*

(b) *If* $\varrho : \Gamma \to \mathrm{GL}(V)$*,* $\varrho' : \Gamma \to \mathrm{GL}(V')$ *be non-isomorphic irreducible representations, then for large enough* s *restrictions* $\mathrm{res}^{\Gamma}_{\Gamma^{(s)}}\ \varrho$ *and* $\mathrm{res}^{\Gamma}_{\Gamma^{(s)}}\ \varrho'$ *are non-isomorphic and irreducible.*

Proof. By 2.3.10 there is an integer s such that $\mathrm{End}_{\Gamma^{(s)}}(V) = \mathrm{End}_{\Gamma}(V)$. If the space $\mathrm{End}_{\Gamma^{(s)}}(V)$ is 1-dimensional then $\mathrm{res}^{\Gamma}_{\Gamma^{(s)}}\ \varrho$ is irreducible (2.3.7) and so is ϱ. If $\dim \mathrm{End}_{\Gamma^{(s)}}(V) > 1$ then $\mathrm{res}^{\Gamma}_{\Gamma^{(s)}}\ \varrho$ is not irreducible and there exists $f \in \mathrm{End}_{\Gamma^{(s)}}(V)$ having non-trivial kernel (for example projection onto one of the irreducible summands). But $f \in \mathrm{End}_{\Gamma}(V)$ and $\ker f$ is a non-trivial subrepresentation of ϱ.

To prove (b) set $f \in \mathrm{Hom}_{\Gamma}(V, V')$. Both $\ker f$ and $\mathrm{im}\, f$ must be either trivial or the whole space; otherwise it would deny the irreducibility of either ϱ or ϱ'. Since ϱ and ϱ' are not isomorphic then f is trivial and $\mathrm{Hom}_{\Gamma}(V, V') = 0$. By 2.3.10 for s large enough we have $\mathrm{Hom}_{\Gamma^{(s)}}(V, V') = 0$. Then $\mathrm{res}^{\Gamma}_{\Gamma^{(s)}}\ \varrho$ and $\mathrm{res}^{\Gamma}_{\Gamma^{(s)}}\ \varrho'$ are not isomorphic. By (a) they are also irreducible (after increasing s if necessary). ☐

As a corollary we obtain

Proposition 2.3.12. *The Schur lemma holds for locally finite groups.*

Proof. Let Γ be a locally finite group. If $\varrho : \Gamma \to \mathrm{GL}(V)$ is irreducible, then for some s $\mathrm{res}^{\Gamma}_{\Gamma^{(s)}}\ \varrho$ is also irreducible and $\mathrm{End}_{\Gamma}(V) \subseteq \mathrm{End}_{\Gamma^{(s)}}(V) \simeq \mathbb{C}$. But multiplication by any scalar is a Γ-homomorphism. Hence $\mathrm{End}_{\Gamma}(V) \simeq \mathbb{C}$. If $\varrho : \Gamma \to \mathrm{GL}(V)$ and $\varrho' : \Gamma \to \mathrm{GL}(V')$ are non-isomorphic and irreducible, then for some s representations $\mathrm{res}^{\Gamma}_{\Gamma^{(s)}}\ \varrho$ and $\mathrm{res}^{\Gamma}_{\Gamma^{(s)}}\ \varrho'$ are irreducible and non-isomorphic. Finally, $\mathrm{End}_{\Gamma}(V, V') \subseteq \mathrm{End}_{\Gamma^{(s)}}(V, V') = 0$. ☐

Proposition 2.3.13. *Each representation* $\varrho : \Gamma \to \mathrm{GL}(V)$ *of a locally finite group* Γ *is isomorphic to a direct sum of irreducible representations. The presentation as direct sum is unique up to permutation of irreducible summands.*

Proof. Choose s such that $\mathrm{End}_{\Gamma}(V) = \mathrm{End}_{\Gamma^{(s)}}(V)$. Any $\Gamma^{(s)}$-subrepresentation $W \subseteq V$ is also a Γ-subrepresentation since any $\Gamma^{(s)}$-projection $f : V \to W$ onto W is also a Γ-homomorphism. Thus a decomposition of V onto irreducible $\Gamma^{(s)}$-subrepresentations is also a decomposition onto irreducible Γ-subrepresentations. The uniqueness follows from the Schur lemma. ☐

Corollary 2.3.14. If Γ is a locally finite group, then $\operatorname{Rep}(\Gamma)$ is a free monoid with basis $\operatorname{IR}(\Gamma)$.

Proposition 2.3.15. *Let $\varrho : \Gamma \to \operatorname{GL}(V)$ be a representation of a locally finite group. The evaluation*

$$\bigoplus_{\sigma\in\operatorname{IR}(\Gamma)} W_\sigma \otimes \operatorname{Hom}_\Gamma(W_\sigma, V) \xrightarrow{ev} V$$

is an isomorphism.

Proof. It is sufficient to consider only the case $V = \bigoplus_{\sigma\in\operatorname{IR}(\Gamma)} W_\sigma^{\oplus l_\sigma}$. The evaluation $ev_\sigma : W_\sigma \otimes \operatorname{Hom}_\Gamma(W_\sigma, W_\sigma^{\oplus l_\sigma}) \longrightarrow W_\sigma^{\oplus l_\sigma}$ is a composition

$$W_\sigma \otimes \operatorname{Hom}_\Gamma(W_\sigma, V) \overset{(2.3.12)}{=} W_\sigma \otimes \operatorname{Hom}_\Gamma(W_\sigma, W_\sigma^{\oplus l_\sigma}) = W_\sigma \otimes \mathbb{C}^{l_\sigma} \cong W_\sigma^{\oplus l_\sigma}$$

Since $ev = \bigoplus ev_\sigma$, then the conclusion follows. □

Proposition 2.3.16. *For any representation $\varrho : \Gamma \to \operatorname{GL}(V)$ of a locally finite group the homomorphism*

$$\bigoplus_{\sigma\in\operatorname{IR}(\Gamma)} \operatorname{End}(\operatorname{Hom}_\Gamma(W_\sigma, V)) \ni \bigoplus_\sigma f_\sigma \mapsto$$
$$\mapsto ev \circ \Big(\bigoplus_\sigma Id_{W_\sigma} \otimes f_\sigma\Big) \circ (ev)^{-1} \in \operatorname{End}_\Gamma(V)$$

is an isomorphism.

Proof. By the Schur lemma we have

$$\operatorname{End}_\Gamma(V) \simeq \bigoplus_{\sigma\in\operatorname{IR}(\Gamma)} \operatorname{End}_\Gamma(W_\sigma \otimes \operatorname{Hom}_\Gamma(W_\sigma, V)).$$

Moreover,

$$\begin{aligned}
&\operatorname{End}_\Gamma(W_\sigma \otimes \operatorname{Hom}_\Gamma(W_\sigma, V)) \\
&\quad = \operatorname{Hom}_\Gamma(W_\sigma \otimes \operatorname{Hom}_\Gamma(W_\sigma, V), W_\sigma \otimes \operatorname{Hom}_\Gamma(W_\sigma, V)) \\
&\quad = \operatorname{Hom}_\Gamma(\operatorname{Hom}_\Gamma(W_\sigma, V), \operatorname{Hom}_\Gamma(W_\sigma, W_\sigma \otimes \operatorname{Hom}_\Gamma(W_\sigma, V))) \\
&\quad = \operatorname{Hom}_\Gamma(\operatorname{Hom}_\Gamma(W_\sigma, V), \operatorname{Hom}_\Gamma(W_\sigma, W_\sigma) \otimes \operatorname{Hom}_\Gamma(W_\sigma, V)) \\
&\quad = \operatorname{End}_\Gamma(\operatorname{Hom}_\Gamma(W_\sigma, V)).
\end{aligned}$$
□

Corollary 2.3.17. Under the assumptions above there is an isomorphism

$$\bigoplus_{\sigma\in \mathrm{IR}(\Gamma)} \mathrm{GL}(\mathrm{Hom}(W_\sigma, V)) \longrightarrow \mathrm{GL}_\Gamma(V).$$

Useful tools in the representation theory of finite groups are characters.

Definition 2.3.18. *The character* of a representation $\varrho : \Gamma \to \mathrm{GL}(V)$ is a function

$$\chi_\varrho : \Gamma \ni g \mapsto \mathrm{Tr}(\varrho(g)) \in \mathbb{C}$$

Let $Ch(\Gamma) \in \mathbb{C}^\Gamma$ be a vector subspace spanned by all characters of representations. If Γ is finite, then there is the hermitian product on $Ch(\Gamma)$:

$$(\chi|\chi') = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} \chi(g)\overline{\chi'(g)}.$$

Proposition 2.3.19. ([S, Th.6]) *Let* Γ *be a finite group.*

(a) $Ch(\Gamma)$ *is a subspace of all* $\mathbb{C}$*-valued functions which are constant on conjugacy classes.*

(b) Characters of irreducible representations form an orthonormal basis of $Ch(\Gamma)$*.*

Corollary 2.3.20 ([S, Th. 7])**.** The number if isomorphism classes of irreducible representations of a finite group Γ is equal to the number of conjugacy classes of elements of Γ.

Proposition 2.3.21. *Fix a locally finite group* Γ *and characters* $\chi, \chi' \in Ch(\Gamma)$*. Then the sequence* $(\chi|\chi')_r := (\chi|_{\Gamma^r}|\chi'|_{\Gamma^r})$ *stabilizes. In particular,*

$$(\chi|\chi') := \lim_{r\to\infty} (\chi|\chi')_r$$

is a hermitian product on $Ch(\Gamma)$ *and characters of irreducible representations form an orthonormal basis of* $Ch(\Gamma)$*.*

Proof. It is sufficient to prove that the sequnce $(\chi|\chi')_r$ stabilizes for $\chi = \chi_\varrho$, $\chi' = \chi_{\varrho'}$, where $\varrho, \varrho' \in \mathrm{Rep}(\Gamma)$. Let

$$\varrho \simeq \bigoplus_{\sigma\in \mathrm{IR}(\Gamma)} \sigma^{\oplus l_\sigma}, \qquad \varrho' \simeq \bigoplus_{\sigma\in \mathrm{IR}(\Gamma)} \sigma^{\oplus l'_\sigma}$$

be the decompositions onto irreducible summands. There exists r such that each irreducible representation of Γ which appears in these decompositions is irreducible when restricted to $\Gamma^{(r)}$ and non-isomorphic summands restrict to non-isomorphic summands (cf. 2.3.11). Then

$$(\chi_\varrho|\chi_{\varrho'})_s = \sum_{\sigma\in \mathrm{IR}(\Gamma)} l_\sigma l'_\sigma$$

for all $s \geq r$. Moreover, by 2.3.11 irreducible representations are pairwise orthogonal. □

Corollary 2.3.22. Let Γ be a locally finite group. Then representations $\varrho, \sigma \in \mathrm{Rep}(\Gamma)$ are isomorphic iff its characters χ_ϱ and χ_σ are equal.

Proof. Representations are isomorphic iff they have isomorphic decompositions into irreducible summands (2.3.13). By 2.3.21 it is equivalent to the equality of characters. □

Throughout the rest of this section we will list lemmas which will be used later in computations of sets of irreducible representations. Here follows the classical application of the character theory:

Definition 2.3.23. *The regular representation* ψ_Γ of a group Γ is the vector space $\mathbb{C}[\Gamma]$ with the obvious left Γ-action.

Proposition 2.3.24. *Each irreducible representation of a finite group* Γ *is contained in the regular representation with multiplicity equal to its dimension.*

Proof. The character of the regular representation is

$$\chi_{\psi_\Gamma}(g) = \begin{cases} |\Gamma| & \text{for } g = e \\ 0 & \text{for } g \neq e. \end{cases}$$

Then for $\varrho \in \mathrm{IR}(\Gamma)$ we have

$$(\chi_{\psi_\Gamma}|\chi_\varrho) = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} \chi_{\psi_\Gamma}(g)\overline{\chi_\varrho(g)} = \frac{1}{|\Gamma|}\chi_{\psi_\Gamma}(e)\overline{\chi_\varrho(e)} = \dim\varrho.$$ □

Corollary 2.3.25. If Γ is a finite group, then

$$|\Gamma| = \sum_{\varrho\in \mathrm{IR}(\Gamma)} (\dim\varrho)^2$$

Proposition 2.3.26. *Each irreducible representation of a locally finite abelian group Γ is one-dimensional.*

Proof. If Γ is finite, we obtain the conclusion combining 2.3.25 with 2.3.20. In the general case it reduces to the finite case by Proposition 2.3.11. $\square$

Definition 2.3.27. Let p be a prime integer and let k be a p-adic integer. Define the representation

$$\varrho_k : \mathbb{Z}/p^\infty \ni \frac{a}{p^n} \mapsto \exp\left(\frac{2\pi i a k}{p^n}\right) \in \mathrm{GL}_1(\mathbb{C}).$$

Proposition 2.3.28. *Each irreducible representation of $\mathbb{Z}/p^\infty$ is isomorphic to ϱ_k for some $k \in \mathbb{Z}_p^\wedge$, and ϱ_k is isomorphic to ϱ_l if and only if $k = l$. In short*

$$\mathrm{IR}(\mathbb{Z}/p^\infty) = \{\varrho_k\}_{k\in\mathbb{Z}_p^\wedge}.$$

Proof. Since each irreducible representation of $\mathbb{Z}/p^\infty$ is 1-dimensional we have

$$\mathrm{IR}(\mathbb{Z}/p^\infty) \simeq \mathrm{Hom}(\mathbb{Z}/p^\infty, \mathrm{GL}_1(\mathbb{C})) \simeq \mathrm{Hom}(\mathbb{Z}/p^\infty, \mathbb{Z}/p^\infty) \simeq \mathbb{Z}_p^\wedge. \qquad \square$$

Proposition 2.3.29. *Let Γ and Δ be locally finite groups. The map*

$$\bar{\otimes} : \mathrm{IR}(\Gamma) \times \mathrm{IR}(\Delta) \ni (\varrho, \sigma) \mapsto \mathrm{res}^{\Gamma}_{\Gamma\times\Delta}\, \varrho \otimes \mathrm{res}^{\Delta}_{\Gamma\times\Delta}\, \sigma \in \mathrm{IR}(\Gamma \times \Delta)$$

is a bijection.

Proof. If $\varrho \in \mathrm{IR}(\Gamma)$, $\sigma \in \mathrm{IR}(\Delta)$ then for some s both representations $\mathrm{res}^{\Gamma}_{\Gamma^{(s)}}\, \varrho$ and $\mathrm{res}^{\Delta}_{\Delta^{(s)}}\, \sigma$ are irreducible. Hence $\mathrm{res}^{\Gamma\times\Delta}_{\Gamma^{(s)}\times\Delta^{(s)}}\, \varrho\bar{\otimes}\sigma$ is irreducible and so is $\varrho\bar{\otimes}\sigma$. Then the map $\bar{\otimes}$ is well-defined.

Now let $\varrho \in \mathrm{IR}(\Gamma \times \Delta)$. For s large enough $\mathrm{res}^{\Gamma}_{\Gamma^{(s)}}\, \varrho$ is irreducible and then it is isomorphic to a tensor product of irreducible representations of factors. Using 2.3.11 we obtain

$$\mathrm{res}^{\Gamma\times\Delta}_{\Gamma\times\{1\}}\, \varrho \simeq \sigma^{\oplus \dim \tau}, \qquad \mathrm{res}^{\Gamma\times\Delta}_{\{1\}\times\Delta}\, \varrho \simeq \tau^{\oplus \dim \sigma}$$

for some $\sigma \in \mathrm{IR}(\Gamma)$, $\tau \in \mathrm{IR}(\Delta)$. For each $i \geq s$ the characters of ϱ and $\sigma\bar{\otimes}\tau$ are equal on $\Gamma^{(s)} \times \Delta^{(s)}$. Hence the characters are equal on $\Gamma \times \Delta$ and by 2.3.22 we have $\varrho \simeq \sigma\bar{\otimes}\tau$. $\square$

Corollary 2.3.30. Define $\varrho_{k_1,\ldots,k_n} := \bar{\bigotimes}_{i=1}^{n} \varrho_{k_i} \in \mathrm{IR}((\mathbb{Z}/p^\infty)^n)$. Then

$$\mathrm{IR}((\mathbb{Z}/p^\infty)^n) = \{\varrho_{k_1,\ldots,k_n}\}_{k_i\in\mathbb{Z}_p^\wedge}.$$

We conclude with three easy technical lemmas used intensively in Chapter 3:

Proposition 2.3.31. *Let Γ be a locally finite group, Δ a normal subgroup of Γ having the finite index and $\sigma : \Gamma \to \mathrm{GL}(V)$ an irreducible representation. Then for every irreducible subrepresentation $\varrho : \Delta \to \mathrm{GL}(V')$ of $\mathrm{res}^\Gamma_\Delta \sigma$ there exists a subrepresentation of $\mathrm{ind}^\Gamma_\Delta \varrho$ which is isomorphic to σ.*

Proof. The homomorphism

$$\mathbb{C}[\Gamma] \times_{\mathbb{C}[\Delta]} V' \ni (g, x) \mapsto \sigma(g)(x) \in V$$

is an epimorphism. Obviously it splits. □

Remark. Let Γ be a locally finite group and Δ a normal subgroup of Γ having the finite index. The quotient group Γ/Δ acts on $\mathrm{IR}(\Delta)$ by conjugation.

Proposition 2.3.32. *Let Γ be a locally finite group, Δ a normal subgroup of Γ having the finite index, σ an irreducible representation of Δ and ϱ an irreducible subrepresentation of $\mathrm{ind}^\Gamma_\Delta \sigma$. Then the dimension of ϱ is divisible by $|(\Gamma/\Delta)\sigma| \cdot \dim \sigma$. In particular, if the isotropy group $(\Gamma/\Delta)_\sigma$ is trivial, then $\mathrm{ind}^\Gamma_\Delta \sigma$ is irreducible.*

Proof. For each $g \in \Gamma$ let $c_g \in \mathrm{Aut}(\Gamma)$ be the conjugation by g. We have

$$\mathrm{res}^\Gamma_\Delta \, \mathrm{ind}^\Gamma_\Delta \, \sigma \simeq \bigoplus_{g\Delta \in \Gamma/\Delta} c_g^* \sigma.$$

Since ϱ is invariant under the conjugation by any $g \in \Gamma$, then

$$\mathrm{res}^\Gamma_\Delta \, \varrho \simeq \left(\bigoplus_{\sigma' \in (\Gamma/\Delta)\sigma} \sigma' \right)^{\oplus l}$$

for some integer l. □

Proposition 2.3.33. *Let Γ be a locally finite group, Δ a normal subgroup of Γ having the finite index and σ an irreducible representation of Δ. Then if the isotropy group $(\Gamma/\Delta)_\sigma$ has order 2, then $\mathrm{ind}^\Gamma_\Delta \sigma$ is a direct sum of two irreducible and non-isomorphic representations, both having dimension $\frac{1}{2}(\Gamma : \Delta) \cdot \dim \sigma$*

Proof. Let $\Delta' = \{g \in \Gamma : c_g^*\sigma \simeq \sigma\}$. The group Δ is a normal subgroup of Δ' (since it has index 2). Since

$$\chi_{\mathrm{ind}_\Delta^{\Delta'}\sigma}(g) = \begin{cases} 2\cdot\chi_\sigma(g) & \text{for } g \in \Delta \\ 0 & \text{for } g \notin \Delta \end{cases},$$

then the length of the character $\chi_{\mathrm{ind}_\Delta^{\Delta'}\sigma}$ is equal $\sqrt{2}$. Therefore $\mathrm{ind}_\Delta^{\Delta'}\sigma$ is a sum of two irreducible non-isomorphic representations σ_1, σ_2. Then

$$\mathrm{ind}_\Delta^\Gamma \sigma \simeq \mathrm{ind}_{\Delta'}^\Gamma \sigma_1 \oplus \mathrm{ind}_{\Delta'}^\Gamma \sigma_2.$$

The dimensions of the summands and their irreducibility follows from 2.3.32. ☐

Till now we have considered complex representations which do not carry any unitary structure. However, the Dwyer-Zabrodsky theorem states that homotopy representations of locally finite groups are in bijection with unitary algebraic representations (not just complex ones). The following proposition proves that in fact any complex representation is unitary, and that a unitary structure is unique.

Proposition 2.3.34. *Let V be an n-dimensional unitary space, and let Γ be a locally finite group. Then map*

$$\mathrm{Rep}(\Gamma, U(V)) \longrightarrow \mathrm{Rep}(\Gamma, \mathrm{GL}(V)) \simeq \mathrm{Rep}_n(\Gamma) \tag{2.3.35}$$

is a bijection.

Proof. The proposition is clear for finite groups. Then $\mathrm{Rep}(\Gamma^{(s)}, U(V)) \cong \mathrm{Rep}(\Gamma^{(s)}, \mathrm{GL}(V))$ for any s. By 2.2.3 we have

$$\mathrm{Rep}(\Gamma, U(V)) \cong \lim_{s\to\infty} \mathrm{Rep}(\Gamma^{(s)}, U(V)) \cong \lim_{s\to\infty} \mathrm{Rep}(\Gamma^{(s)}, \mathrm{GL}(V)).$$

The composition

$$\mathrm{Rep}(\Gamma, \mathrm{GL}(V)) \longrightarrow \lim_{s\to\infty} \mathrm{Rep}(\Gamma^{(s)}, \mathrm{GL}(V)) \xleftarrow{\cong} \xleftarrow{\cong} \lim_{s\to\infty} \mathrm{Rep}(\Gamma^{(s)}, U(V)) \xleftarrow{\cong} \mathrm{Rep}(\Gamma, U(V))$$

is an inverse of the map $\mathrm{Rep}(\Gamma, U(V)) \longrightarrow \mathrm{Rep}(\Gamma, \mathrm{GL}(V))$. ☐

2.4 Obstruction functors of $\mathcal{R}$-invariant representations

Throughout this section G is a compact connected Lie group. Let N be a p-normalizer of a maximal torus of G and let $\mathcal{R} := \mathcal{R}_p(G)$. For any $\mathcal{R}$-invariant representation $\varrho : N^\infty \to U(V)$ let

$$\Pi_i^\varrho(P) := \pi_i \operatorname{map}((EG/P)_p^\wedge, BU(m)_P^\wedge)_{B\varrho_p^\wedge|_{(EG/P)_p^\wedge}}$$

Obstructions to the existence and (resp. uniqueness) of a topological realization of ϱ depend on the obstructions lying in groups $H^{i+1}(\mathcal{R};\Pi_i^\varrho)$ (resp. $H^i(\mathcal{R};\Pi_i^\varrho)$, cf. 1.4). The main goal of this section is to describe the isomorphism of functors Π_i^ϱ with some functors which are much easier to calculate.

Definition 2.4.1. For any representation $\varrho : \Gamma \to U(V)$ let $\mathrm{IR}(\Gamma,\varrho)$ be the subset of $\mathrm{IR}(\Gamma)$ consisting of all irreducible subrepresentations of ϱ. Similarly let $\mathrm{Rep}(\Gamma,\varrho)$ denotes the set of isomorphisms classes of subrepresentations of $\varrho^{\oplus s}$ for some s. (In other words it contains all representations with irreducible factors contained in ϱ). Note that $\mathrm{Rep}(\Gamma,\varrho)$ is the free abelian monoid with basis $\mathrm{IR}(\Gamma,\varrho)$. Let $R(\Gamma,\varrho)$ be the Grothendieck construction on $\mathrm{Rep}(\Gamma,\varrho)$.

Remark. Given a representation $\varrho : P \to U(V)$ and a homomophism $a : Q \to P$, there are obvious restrictions $a^* : \mathrm{Rep}(P,\varrho) \to \mathrm{Rep}(Q,a^*\varrho)$ and $a^* : R(P,\varrho) \to R(Q,a^*\varrho)$. If a and a' are conjugate then $a^* = (a')^*$. Therefore for any $\mathcal{R}$-invariant representation $\varrho : N^\infty \to U(V)$ there exist functors

$$\mathrm{Rep}(-,\varrho) : \mathcal{R}^{op} \longrightarrow \mathbf{Set}, \quad \text{and} \quad R(-,\varrho) : \mathcal{R}^{op} \longrightarrow \mathbf{Ab}$$

We begin with some easy observations. If $i : W \hookrightarrow V$ is an inclusion of unitary spaces then there exists the canonical inclusion of unitary groups

$$i_* : U(W) \ni f \mapsto (f, id_{W^\perp}) \in U(W \oplus W^\perp) \cong U(V).$$

For a unitary space V the loop

$$[0,1] \ni t \mapsto e^{2\pi i t} \cdot Id_V \in U(V)$$

represents the canonical generator $l_V \in \pi_1 U(V)$.

Fix a representation $\varrho : P \to U(V)$. Let $\sigma : P \to U(W)$ be a representative of an element of $\mathrm{Rep}(P, \varrho)$. Choose any monomorphism $i : W \to V^{\oplus s}$. Define the map

$$J_{P,\varrho} : R(P, \varrho) \ni [\sigma] \mapsto i_*(l_W) \in \pi_1 U_P(V^{\oplus s}) \tag{2.4.2}$$

The $J_{P,\varrho}$ does not depend on the choice of i since every inclusion $W \to V^{\oplus s}$ determines the conjugate homomorphism $U_P(W) \to U_P(V^{\oplus s})$. It also does not depend on the choice of s since for $s < s'$ the map $U_P(V^{\oplus s}) \to U_P(V^{\oplus s'})$ induces an isomorphism on π_1.

Proposition 2.4.3. *For any $\mathcal{R}$-invariant representation $\varrho : N^\infty \to U(V)$ the collection of maps $J_{P,\varrho|_P}$ defines a natural equivalence*

$$J_\varrho : R(-, \varrho) \to \pi_1 \operatorname{Aut}_{(-)}(V).$$

of functors on $\mathcal{R}$.

Proof. Let $a : Q \to P$ be a morphism of $\mathcal{R}$. The commutativity of a diagram

$$\begin{array}{ccc} R(P, \varrho|_P) & \xrightarrow{J_{P,\varrho|_P}} & \pi_1 \operatorname{Aut}_P(V) \\ \downarrow{\scriptstyle a^*} & & \downarrow \\ R(Q, \varrho|_Q) & \xrightarrow{J_{Q,\varrho|_Q}} & \pi_1 \operatorname{Aut}_Q(V) \end{array}$$

is clear. Thus J_ϱ is a natural transformation. Both groups $R(P, \varrho|_P)$ and $\pi_1 \operatorname{Aut}_P(V)$ are free and abelian, and the generator set of them both is $\mathrm{IR}(P, \varrho|_P)$ (cf. 2.3.17) and the generators are mapped to the corresponding generators. □

Introduce the following notation:

$$\Xi^\varrho := \mathbb{Z}_p^\wedge \otimes R(-, \varrho) : \mathcal{R}^{op} \longrightarrow \mathbf{Mod}_{\mathbb{Z}_p^\wedge} \tag{2.4.4}$$

Theorem 2.4.5. *Functors Ξ^ϱ and Π_2^ϱ are naturally equivalent.*

Proof. The transformation is a composition of the natural equivalences

$$\Xi^\varrho = \mathbb{Z}_p^\wedge \otimes R(-, \varrho) \xrightarrow{Id \otimes J_\varrho} \mathbb{Z}_p^\wedge \otimes \pi_1 \operatorname{Aut}_{(-)}(V) \cong$$
$$\cong \mathbb{Z}_p^\wedge \otimes \pi_1(C_{U(V)}(\varrho(-))) \xrightarrow{\cong} \mathbb{Z}_p^\wedge \otimes \pi_2(BC_{U(V)}(\varrho(-))) \longrightarrow$$
$$\longrightarrow \pi_2(BC_{U(V)}(\varrho(-)))_p^\wedge \xrightarrow[(2.2.4)]{\cong} \pi_2(\mathrm{map}(EG/(-), BU(V)_p^\wedge)_{B\varrho|_P})_p^\wedge. \quad \square$$

Definition 2.4.6. Let $\mathcal{C}$ be a small category. *The cohomological dimension* $cd_p(\mathcal{C})$ of $\mathcal{C}$ at the prime p is a greatest number i such that $H^i(\mathcal{C}; M) \neq 0$ for some functor $M : \mathcal{C} \to \mathbf{Mod}_{\mathbb{Z}_p^\wedge}$. If no such number exists we put $cd_p(\mathcal{C}) = +\infty$.

Theorem 2.4.7 ([JMO1]). *$cd_p(\mathcal{R}) < +\infty$.*

Proposition 2.4.8. *For each $\mathcal{R}$-invariant representation $\varrho : N^\infty \to U(V)$ the functors Π_1^ϱ and Π_3^ϱ vanish.*

Proof. By the Dwyer-Zabrodsky theorem, for each stubborn P the map

$$BU_P(V)_p^\wedge \simeq BC_{U(V)}(\varrho(P))_p^\wedge \longrightarrow (\mathrm{map}(EG/P, BU(V)_p^\wedge)_{B\varrho|_P})_p^\wedge$$

is a homotopy equivalence. But $U_P(V)$ is a product of unitary groups by 2.3.17 and then $\pi_0(U_P(V)) = \pi_2(U_P(V)) = 0$. □

Theorem 2.4.9. *Let $\varrho : N^\infty \to U(V)$ be an $\mathcal{R}$-invariant representation. Then*

(a) If $H^3(\mathcal{R}; \Xi^\varrho) = 0$ and $cd_p(\mathcal{R}) \leq 4$ then the map $B\varrho_2^\wedge$ extends to a map $BG \to BU(m)_p^\wedge$.

(b) If $H^2(\mathcal{R}; \Xi^\varrho) = 0$ and $cd_p(\mathcal{R}) \leq 3$ then there exists at most one extension of $B\varrho_2^\wedge$ to a map $BG \to BU(m)_p^\wedge$.

(c) If $H^i(\mathcal{R}; \Xi^\varrho) = 0$ for all $i \geq 2$ then there exists r such that $(B\varrho^{\oplus r})_2^\wedge$ extends to map $BG \to BU(rm)_p^\wedge$

Proof. (a) By 2.4.8 $H^2(\mathcal{R}; \Pi_1^\varrho) = H^4(\mathcal{R}; \Pi_3^\varrho) = 0$. Other obstruction groups vanish by the assumptions.

(b) Similarly by 2.4.8 we have $H^1(\mathcal{R}; \Pi_1^\varrho) = H^3(\mathcal{R}; \Pi_3^\varrho) = 0$ and the assumptions imply that $H^2(\mathcal{R}; \Pi_2^\varrho) = 0$ and $H^i(\mathcal{R}; \Pi_i^\varrho) = 0$ for $i > 3$.

(c) Fix s such that $2s > cd_p(\mathcal{R})$ (it is possible by 2.4.7). It is enough to prove that Π_i^ϱ is either 0 or Ξ^ϱ for $i \leq s$. We have

$$\Pi_i^\varrho(P) \cong \pi_{i-1}(U_P(V^{\oplus s})_p^\wedge) \xrightarrow[(2.3.17)]{\cong} \bigoplus_{\sigma \in \mathrm{IR}(P)} \pi_{i-1}(U(\mathrm{Hom}_P(W_\sigma, V^{\oplus s}))_p^\wedge)$$

Since $\dim(\mathrm{Hom}_P(W_\sigma, V^{\oplus s}))$ is at least s or 0 then for each $i < 2s$ by Bott periodicity

$$\pi_{i-1}(U(\mathrm{Hom}_P(W_\sigma, V^{\oplus s}))_p^\wedge) = \begin{cases} \pi_1(U(\mathrm{Hom}_P(W_\sigma, V^{\oplus s}))_p^\wedge) & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd} \end{cases}$$

Thus for each $i \leq cd_p(\mathcal{R}) \leq 2s$ we have $\Pi_i^\varrho = \Xi^\varrho$ for even i and $\Pi_i^\varrho = 0$ for odd i.

□

Chapter 3

Representations of stubborn subgroups of $Spin(7)$

3.1 Stubborn subgroups of symmetric groups and orthogonal groups

In this section we classify 2-stubborn subgroups of symmetric groups (after [AF]) and orthogonal groups (after [O1]). Next, we describe the categories $\mathcal{R}_2(O(n))$ and $\mathcal{R}_2(\Sigma_n)$. Finally we construct a functor $\mathcal{R}_2(\Sigma_n) \to \mathcal{R}_2(O(n))$ which is an inclusion onto a full subcategory.

2-Stubborn subgroups of orthogonal groups

Each 2-stubborn subgroup of an orthogonal group is built up out of some number of "pieces" by taking products and wreath products. (Recall that the wreath product $G \wr H$ is the semi-direct product $G^H \ltimes H$, where H acts on G^H by shifting coordinates). Let $I_n \in GL_n(\mathbb{C})$ denote the identity matrix and let C_2^t be the elementary abelian group of rank t. If $M \in GL_m(\mathbb{R})$ and $N \in GL_n(\mathbb{R})$, then $M \otimes N$ is the matrix with entries

$$(M \otimes N)_{am+b-m,cm+d-m} = M_{b,d}N_{a,c}.$$

for $a, c = 1, \dots, n$, $b, d = 1, \dots, m$.

Definition 3.1.1. Let

$$A := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For $i < n$ define the matrices $A_i^n, B_i^n \in O(2^n)$ by

$$A_i^n := I_{2^i} \otimes A \otimes I_{2^{n-i-1}}, \qquad B_i^n := I_{2^i} \otimes B \otimes I_{2^{n-i-1}}.$$

These matrices satisfy the following relations

$$(A_i^n)^2 = (B_i^n)^2 = I_{2^n}, \qquad A_i^n A_j^n = A_j^n A_i^n, \qquad B_i^n B_j^n = B_j^n B_i^n$$

$$A_i^n B_j^n = \begin{cases} B_j^n A_i^n & \text{for } i \neq j \\ -B_j^n A_i^n & \text{for } i = j \end{cases}$$

and for any $X \in SO(2)$

$$A_i^n (X \otimes I_{2^{n-1}}) = \begin{cases} (X \otimes I_{2^{n-1}})^{-1} A_0^n & \text{for } i = 0 \\ (X \otimes I_{2^{n-1}}) A_i^n & \text{for } i > 0 \end{cases}$$

$$B_i^n (X \otimes I_{2^{n-1}}) = \begin{cases} (X \otimes I_{2^{n-1}})^{-1} B_0^n & \text{for } i = 0 \\ (X \otimes I_{2^{n-1}}) B_i^n & \text{for } i > 0. \end{cases}$$

Definition 3.1.2. Let

$$\Gamma_{2^n} := \langle -I_{2^n}, A_0^n, \ldots, A_{n-1}^n, B_0^n, \ldots, B_{n-1}^n \rangle \subseteq O(2^n)$$

$$\bar{\Gamma}_{2^n} := \langle \{X \otimes I_{2^{n-1}}\}_{X \in SO(2)}, A_0^n, \ldots, A_{n-1}^n, B_0^n, \ldots, B_{n-1}^n \rangle \subseteq O(2^n)$$

Let V_n and X_n be the elementary abelian 2-groups with bases $\{a_i, b_i\}_{i=1}^{n-1}$ and $\{a_i, b_i\}_{i=0}^{n-1}$ respectively and let $W_n := X_n / \langle a_0 - b_0 \rangle$. The ranks of V_n, W_n and X_n are respectively $2n-2$, $2n-1$ and $2n$. The groups Γ_{2^n} and $\bar{\Gamma}_{2^n}$ lie in the exact sequences

$$1 \longrightarrow \bar{\Gamma}_2 \longrightarrow \bar{\Gamma}_{2^n} \xrightarrow{p_V} V_n \longrightarrow 1 \tag{3.1.3}$$

$$1 \longrightarrow SO(2) \longrightarrow \bar{\Gamma}_{2^n} \xrightarrow{p_W} W_n \longrightarrow 1 \tag{3.1.4}$$

$$1 \longrightarrow \{\pm I_{2^n}\} \longrightarrow \Gamma_{2^n} \xrightarrow{p_X} X_n \longrightarrow 1 \tag{3.1.5}$$

$$1 \longrightarrow \{\pm I_{2^n}\} \longrightarrow \bar{\Gamma}_{2^n} \xrightarrow{p'_V} \bar{\Gamma}_2 / \{\pm I\} \times V_n \longrightarrow 1, \tag{3.1.6}$$

where $p_V(A_i^n) = p_W(A_i^n) = p_X(A_i^n) = a_i$, $p_V(B_i^n) = p_W(B_i^n) = p_X(B_i^n) = b_i$, $p'_V(X \otimes I_{2^{n-1}}) = (X, 0)$ and

$$p'_V(A_i^n) = \begin{cases} (A, 0) & \text{for } i = 0 \\ (I, a_i) & \text{for } i > 0 \end{cases}, \qquad p'_V(B_i^n) = \begin{cases} (B, 0) & \text{for } i = 0 \\ (I, b_i) & \text{for } i > 0 \end{cases}.$$

The group Γ_{2^n} is an extra-special group and it has order 2^{2n+1}. $\bar{\Gamma}_{2^n}$ is a 2-toral group with 1-dimensional torus. Note that $\Gamma_1 \cong \{\pm 1\} = O(1)$ and $\bar{\Gamma}_2 \cong O(2)$.

Definition 3.1.7. ([O1, Def. 2]) Let

$$\mathcal{T}_\Gamma(k) \subseteq \mathcal{T}_{irr}(k) \subseteq \mathcal{T}_{prod}(k)$$

be the sets of 2-toral subgroups of $O(k)$ defined as follows:

- $\mathcal{T}_\Gamma(1) = \{\Gamma_1 = O(1)\}$,
- $\mathcal{T}_\Gamma(2) = \{\bar{\Gamma}_2 = O(2)\}$,
- $\mathcal{T}_\Gamma(2^n) = \{\Gamma_{2^n}, \bar{\Gamma}_{2^n}\}$ for $n > 1$,
- $\mathcal{T}_\Gamma(k) = \emptyset$ for $k \neq 2^n$,
- $\mathcal{T}_{irr}(2^n)$ is the set of those wreath products in $O(2^n)$ of the form

  $$\Gamma \wr C_2^{t_1} \wr \ldots \wr C_2^{t_s},$$

  where $\Gamma \in \mathcal{T}_\Gamma(2^l)$, $n = l + t_1 + \ldots + t_s$ and $t_1 > 1$ if $\Gamma = \Gamma_1$,
- $\mathcal{T}_{prod}(k)$ is the set of all products in $O(k)$ of the form

  $$P = P_1 \times \ldots \times P_s \subseteq O(k_1) \times O(k_2) \times O(k_s) \subseteq O(k),$$

  where $P_i \in \mathcal{T}_{irr}(k_i)$ and $k = k_1 + \ldots + k_s$.

Remark. If $\Gamma \subseteq O(2^l)$, then the embedding $\Gamma \wr C_2^t \subseteq O(2^{l+t})$ is chosen as follows:

$$\Gamma \wr C_2^t \cong \langle \Gamma^{2^t}, B_l^{l+t}, B_{l+1}^{l+t}, \ldots, B_{l+t-1}^{l+t} \rangle \subseteq O(2^{l+t}).$$

By [O1, Theorem 8] each 2-stubborn subgroup of $O(k)$ is conjugate to a subgroup in $\mathcal{T}_{prod}(O(k))$. Possibly not all elements $\Gamma \in \mathcal{T}_{prod}(O(k))$ are 2-stubborn — it depends on the Weyl group of Γ. The following proposition follows immediately from [O1, Theorem 6]:

Proposition 3.1.8. *Let $W_G(P) := N_G(P)/P$. Then*

- $W_{O(2^n)}(\Gamma_{2^n}) \simeq O^+(X_n)$,
- $W_{O(2^n)}(\bar{\Gamma}_{2^n}) \simeq Sp(V_n)$,

- $W_{O(2^n)}(\Gamma \wr C_2^{t_1} \wr \ldots \wr C_2^{t_s}) \simeq W_{O(2^m)}(\Gamma) \times \mathrm{GL}_{t_1}(\mathbb{F}_2) \times \ldots \times \mathrm{GL}_{t_s}(\mathbb{F}_2)$, $\Gamma \in \mathcal{T}(2, O(2^m))$.

Remark. $O^+(X_n)$ is the group of all automorphisms of X_n which preserve the quadratic form

$$q\left(\sum x_i a_i + \sum y_i b_i\right) = \sum x_i y_i \tag{3.1.9}$$

and $Sp(V_n)$ is the group of all automorphisms of V_n which preserve the non-singular form

$$\begin{pmatrix} 0 & I_{n-1} \\ I_{n-1} & 0 \end{pmatrix} \tag{3.1.10}$$

with basis of V_n ordered $a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}$.

The following theorem determines which elements of $\mathcal{T}_{prod}(k)$ are in fact 2-stubborn subgroups of $O(k)$:

Theorem 3.1.11. ([O1, Theorem 6]) *Let $P \in O(k)$ be a 2-stubborn subgroup. Then P is conjugate to an element of $\mathcal{T}_{prod}(k)$. If $P \in \mathcal{T}_{prod}(k)$ then P is 2-stubborn if and only if when written as a product*

$$P = P_1 \times \ldots \times P_s, \qquad (P_i \in \mathcal{T}_{irr}(k_i))$$

there is no factor P_i with $W_{O(k_i)}(P_i) = 1$ which occurs with multiplicity exactly 2 or 4.

Theorem 3.1.12. ([O1, Proposition 9]) *Let $P, P' \in \mathcal{T}_{prod}(k)$. If P' is conjugate to a subgroup of P, then $x^{-1}P'x \subseteq P$ for some permutation matrix x which permutates irreducible factors of P'. If $P' \subseteq P$ then the inclusion is a composite of products of the following types:*

(a) $O(1) \times O(1) \subseteq O(2)$,

(b) $O(1) \wr C_2^{2t_1} \wr C_2^{t_2} \wr \cdots \wr C_2^{t_s} \subseteq O(2) \wr C_2^{t_1} \wr C_2^{t_2} \wr \cdots \wr C_2^{t_s}$,

(c) $(\Gamma \wr C_2^{t_1} \wr \cdots \wr C_2^{t_s})^{t_{s+1}} \subseteq \Gamma \wr C_2^{t_1} \wr \cdots \wr C_2^{t_s} \wr C_2^{t_{s+1}}$,

(d) $\Gamma_{2^k} \wr C_2^{t_1} \wr \cdots \wr C_2^{t_s} \subseteq \bar{\Gamma}_{2^k} \wr C_2^{t_1} \wr \cdots \wr C_2^{t_s}$,

(e) $\Gamma \wr \cdots \wr C_2^{t_i} \wr C_2^{t_{i+1}} \wr \cdots \wr C_2^{t_s} \subseteq \Gamma \wr \cdots \wr C_2^{t_i + t_{i+1}} \wr \cdots \wr C_2^{t_s}$,

(f) $\Gamma_{2^{k+t_1}} \wr C_2^{t_2} \wr \cdots \wr C_2^{t_s} \subseteq \Gamma_{2^k} \wr C_2^{t_1} \wr C_2^{t_2} \wr \cdots \wr C_2^{t_s}$,

(g) $\bar{\Gamma}_{2^{k+t_1}} \wr C_2^{t_2} \wr \cdots \wr C_2^{t_s} \subseteq \bar{\Gamma}_{2^k} \wr C_2^{t_1} \wr C_2^{t_2} \wr \cdots \wr C_2^{t_s}$,

where Γ *stands for either* Γ_{2^k} *or* $\bar{\Gamma}_{2^k}$*.*

Corollary 3.1.13. Each morphism in $\mathcal{R}_2(O(n))$ is a composition of automorphisms and inclusions enlisted in 3.1.12.

2-Stubborn subgroups of symmetric groups

Now we describe the category $\mathcal{R}_2(\Sigma_k)$. The classification of 2-stubborn subgroups of symmetric groups is due to Alperin and Fong [AF] (they call them 2-*radical* subgroups). Note that if $G \subseteq \Sigma_m$, $H \subseteq \Sigma_n$, then the product $G \times H$ is a subgroup of Σ_{m+n} and the wreath product $G \wr H$ is a subgroup of Σ_{mn}.

Definition 3.1.14. For any sequence $t_1, \ldots, t_s$ of positive integers let

$$B(t_1, \ldots, t_s) := 1 \wr C_2^{t_1} \wr \cdots \wr C_2^{t_s} \subseteq \Sigma_{2^t},$$

where $t = t_1 + \cdots + t_s$ (we treat 1 as a subgroup of Σ_1). The groups $B(t_1, \ldots, t_s)$ will be called *basic* subgroups of Σ_{2^t}. The set of all basic subgroups of Σ_{2^t} will be denoted by $\mathcal{B}_{irr}(2^t)$.

Remark. Note that each basic subgroup of Σ_{2^t} acts transitively on the set $\{1, \ldots, 2^t\}$.

Definition 3.1.15. Let $\mathcal{B}_{prod}(n)$ denotes the family of all products of basic subgroups in Σ_n, i.e.

$$\mathcal{B}_{prod}(n) = \{G_1 \times \cdots \times G_r : G_i \in \mathcal{B}_{irr}(2^{t_i}), n = 2^{t_1} + \cdots + 2^{t_r}\}.$$

Here follow two propositions which are consequences of [AF, (2B)]:

Proposition 3.1.16. *If* $t = t_1 + \cdots + t_s$*, then*

$$W_{\Sigma_{2^t}}(B(t_1, \ldots, t_s)) \simeq GL_{t_1}(\mathbb{F}_2) \times \cdots \times GL_{t_s}(\mathbb{F}_2).$$

Proposition 3.1.17. *Let* $G_i \subseteq \Sigma_{2^{t_i}}$*, for* $i = 1, \ldots, r$*, be a collection of pairwise non-isomorphic basic subgroups, and let* $k = \sum_{i=1}^r 2^{t_i} l_i$*. Then*

$$W_{\Sigma_k}\left(\prod_{i=1}^r G_i^{l_i}\right) = \prod_{i=1}^r W_{\Sigma_{2^{t_i}}}(G_i) \wr \Sigma_{l_i}.$$

Theorem 3.1.18. *Each 2-stubborn subgroup $G \subseteq \Sigma_k$ is, up to conjugacy, a product of basic subgroups (i.e. $G \in \mathcal{B}_{prod}(k)$). A group $G \in \mathcal{B}_{prod}(k)$ is stubborn if and only if written as a product of basic subgroups $G = G_1 \times \cdots \times G_r$ there is no factor isomorphic to $B(1, \ldots, 1)$ which occurs with multiplicity exactly 2 or 4.*

Proof. The first statement is a consequence of [AF, (2A)]. Since both groups $GL_n(\mathbb{F}_2)$ and Σ_n have a non-trivial normal 2-subgroup if and only if $n = 2$ or $n = 4$, then the second statement follows immediately from Propositions 3.1.16 and 3.1.17. □

Proposition 3.1.19. *If groups $G, H \in \mathcal{B}_{prod}(k)$ are conjugate, then there exists a conjugacy between them which permutes its basic factors.*

Proof. Since each basic subgroup of a symmetric group acts transitively on the set of letters, basic factors of G (and, similarly, H) are in bijection with the set of G-orbits (H-orbits). The conjugacy between G and H permutes the orbits and therefore it also permutes its basic factors. □

Proposition 3.1.20. *Fix collections of subgroups $G_i \subseteq \Sigma_{k_i}$ for $i = 1, \ldots, r$ and $H_j \subseteq \Sigma_{l_j}$ for $j = 1, \ldots, s$. Assume that for each i the group G_i acts transitively on the set of letters, and that $n := k_1 + \cdots + k_r = l_1 + \cdots + l_s$. If*

$$H := H_1 \times \cdots \times H_s \subseteq G := G_1 \times \cdots \times G_r \subseteq \Sigma_n,$$

then $H = (H \cap G_1) \times \cdots \times (H \cap G_r)$.

Proof. For each $i = 1, \ldots, r$ let O_i^G be an orbit of $G_i \subseteq G$. Note that $\{1, \ldots, k\} = \bigcup_{i=1}^r O_i^G$ is a decomposition onto G-orbits. Similarly define H-sets O_j^H, for $j = 1, \ldots, s$. Since $H \subseteq G$, then for each j there exists i such that $O_j^H \subseteq O_i^G$. Hence $H_j \subseteq G \cap \Sigma_{O_i^G} = G_i$ and the conclusion follows. □

Proposition 3.1.21. *For any subgroup $G \subseteq \Sigma_k$ let $\delta(G) \subseteq G$ denotes the subgroup generated by all elements $g \in G$ which have a fixed point. The following holds:*

- *if $G \in \mathcal{B}_{prod}(k)$ is a non-trivial product of basic subgroups, then $\delta(G) = G$.*
- *$\delta(B(t_1, \ldots, t_r)) = B(t_1, \ldots, t_{r-1})^{2^{t_r}}$,*

Proof. The product $H \times H'$ is generated by $H \times \{1\} \cup \{1\} \times H'$. It implies that the first statement holds, and that $B(t_1, \dots, t_{r-1})^{2^{t_r}} \subseteq \delta(B(t_1, \dots, t_r))$. On the other hand, for $C_2^{t_r} \subseteq \Sigma_{2^{t_r}}$ we have $\delta(C_2^{t_r}) = \{1\}$. Then each element $g \in B(t_1, \dots, t_r) \setminus B(t_1, \dots, t_{r-1})^{2^{t_r}}$ acts freely on the set $\{1, \dots, 2^t\}$ (where $t = t_1 + \dots + t_r$). As a consequence we obtain the second statement. □

Theorem 3.1.22. *Let $G, H \in \mathcal{B}_{prod}(k)$. Assume that $H \subseteq G$. Then the inclusion $H \subseteq G$ is a composite of products of inclusions of the following types:*

(a) $B(t_1, \dots, t_{r-1})^{2^{t_r}} \subseteq B(t_1, \dots, t_{r-1}, t_r)$,

(b) $B(t_1, \dots, t_j + t_{j+1}, \dots, t_r) \subseteq B(t_1, \dots, t_j, t_{j+1}, \dots, t_r)$.

Proof. If G is redicible, then by Proposition 3.1.20 the inclusion is the product of inclusions $(H \cap G_i) \subseteq G_i$, where G_i's are irreducible, so assume that $G = B(t_1, \dots, t_r)$. If H is reducible, then by Proposition 3.1.21 we obtain

$$H = \delta(H) \subseteq \delta(G) = B(t_1, \dots, t_{r-1})^{2^{t_r}} \subseteq B(t_1, \dots, t_{r-1}, t_r) = G.$$

In this case the inclusion is the composition of an inclusion of type (a) with $H \subseteq \delta(G)$. Finally, if H is irreducible then $H = B(t'_1, \dots, t'_s)$. We have

$$\delta(H) = B(t'_1, \dots, t'_{s-1})^{2^{t'_s}} \subseteq B(t_1, \dots, t_{r-1})^{2^{t_r}} = \delta(G).$$

Since H-orbits are contained in G-orbits, then $t'_s \geq t_r$. If $t'_s = t_r$, then we are reduced to the case of smaller inclusion of irreducible subgroups. If $t'_s > t_r$, then the inclusion

$$B(t'_1, \dots, t'_{s-1})^{2^{t'_s - t_r}} \subseteq B(t_1, \dots, t_{r-1})$$

factors through $B(t_1, \dots, t_{r-2})^{2^{t_{r-1}}}$. Hence $t'_s - t_r \geq t_{r-1}$ and then $t'_s \geq t_{r-1} + t_r$. Finally, we obtain the factorization

$$H \subseteq B(t_1, \dots, t_{r-2}, t_{r-1} + t_r) \subseteq B(t_1, \dots, t_{r-2}, t_{r-1}, t_r) = G. \qquad \square$$

Definition 3.1.23. Let $\mathcal{C}$ be a small category and let $\mathrm{Mor}_{\mathcal{C}}$ be the set of all morphisms of $\mathcal{C}$. We say that a subset $X \subseteq \mathrm{Mor}_{\mathcal{C}}$ *generates* the category $\mathcal{C}$ if there exists a full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ which is naturally equivalent to $\mathcal{C}$ such that each morphism of $\mathcal{C}'$ is a composition of morphisms which belong to X.

Theorem 3.1.24. *The category $\mathcal{R}_2(\Sigma_k)$ is generated by automorphisms of objects and inclusions enlisted in Theorem 3.1.22.*

Proof. Fix $G, H \in \mathcal{B}_{prod}(k)$ and choose a morphism $H \to G$ represented by $g \in \Sigma_k$. By Proposition 3.1.19, H and $g^{-1}Hg$ are isomorphic objects of $\mathcal{R}_2(\Sigma_k)$. Therefore we can assume that $g^{-1}Hg = H$. Then $g : H \to G$ is the composition of the conjugation by g on H and the inclusion $H \subseteq G$. □

A full inclusion $\mathcal{R}_2(\Sigma_k) \to \mathcal{R}_2(O(k))$

Definition 3.1.25. For any $G \in \mathcal{B}_{prod}(k)$ let $\bar{G} \in \mathcal{T}_{prod}(k)$ be given by

$$\bar{G} = \{\pm 1\} \wr C_2^{t_1} \wr \cdots \wr C_2^{t_r} \quad \text{for } G = B(t_1, \ldots, t_r),\ t_1 > 1$$

$$\bar{G} = O(2) \wr C_2^{t_2} \wr \cdots \wr C_2^{t_r} \quad \text{for } G = B(1, t_2, \ldots, t_r)$$

$$\bar{G} = \bar{G}_1 \times \cdots \times \bar{G}_r \quad \text{for } G_i \in \mathcal{B}_{irr}(k_i)$$

Remark. For each $G \in \mathcal{B}_{prod}(k)$ holds

$$\bar{G} \cap \{\pm 1\} \wr \Sigma_k = \{\pm 1\} \wr G. \tag{3.1.26}$$

Theorem 3.1.27. *The formulae*

$$\mathcal{R}_2(\Sigma_k) \ni G \mapsto \bar{G} \in \mathcal{R}_2(O(k))$$

$$\mathrm{Mor}_{\mathcal{R}_2(\Sigma_k)}(H, G) \ni gG \mapsto g\bar{G} \in \mathrm{Mor}_{\mathcal{R}_2(O(k))}(\bar{H}, \bar{G})$$

define the functor $I : \mathcal{R}_2(\Sigma_k) \to \mathcal{R}_2(O(k))$ which is an inclusion onto the full subcategory.

Proof. The functor I is well-defined.
It is sufficient to check that for each generating morphism $gG : H \to G$ holds $g^{-1}\bar{H}g \subseteq \bar{G}$. It is clear for automorphisms (cf. Propositions 3.1.16 and 3.1.17), so assume that $g = 1$ and $H \to G$ is a product of the inclusions enlisted in Theorem 3.1.22. If the inclusion $H \subseteq G$ is a non-trivial product of inclusions $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$, then $\bar{H} \subseteq \bar{G}$ if and only if $\bar{H}_1 \subseteq \bar{G}_1$ and $\bar{H}_2 \subseteq \bar{G}_2$. Hence we are reduced to the case when the inclusion is of type (a) or type (b) (cf. 3.1.22). If

$$H = B(t_1, \ldots, t_{r-1})^{2^{t_r}} \subseteq B(t_1, \ldots, t_{r-1}, t_r) = G,$$

then for $t_1 > 1$ we obtain

$$\bar{H} = (\{\pm 1\} \wr C_2^{t_1} \wr \cdots \wr C_2^{t_{r-1}})^{2^{t_r}} \subseteq \{\pm 1\} \wr C_2^{t_1} \wr \cdots \wr C_2^{t_r} = \bar{G}$$

and for $t_1 = 1$

$$\bar{H} = (O(2) \wr C_2^{t_2} \wr \cdots \wr C_2^{t_{r-1}})^{2^{t_r}} \subseteq O(2) \wr C_2^{t_2} \wr \cdots \wr C_2^{t_r} = \bar{G}.$$

If

$$H = B(t_1, \ldots, t_j + t_{j+1}, \ldots, t_r) \subseteq B(t_1, \ldots, t_j, t_{j+1}, \ldots, t_r) = G,$$

then for $j > 1$ we obtain the inclusion

$$\bar{H} = K \wr C_2^{t_2} \wr \cdots \wr C_2^{t_j + t_{j+1}} \wr \cdots \wr C_2^{t_r} \subseteq K \wr C_2^{t_2} \wr \cdots \wr C_2^{t_j} \wr C_2^{t_{j+1}} \wr \cdots \wr C_2^{t_r} = \bar{G},$$

where $K = O(2)$ if $t_1 = 1$ and $K = \{\pm 1\} \wr C_2^{t_1}$ otherwise. Similarly if $j = 1$ and $t_1 > 1$, then the inclusion $\bar{H} \subseteq \bar{G}$ is straightforward. The only non-trivial case appears when $j = t_1 = 1$. Then

$$\bar{H} = \{\pm 1\} \wr C_2^{1+t_2} \wr \cdots \wr C_2^{t_r} \subseteq O(2) \wr C_2^{t_2} \wr \cdots \wr C_2^{t_r} = \bar{G},$$

since $\{\pm 1\} \wr C_2^{1+t_2} \subseteq \{\pm 1\} \wr C_2 \wr C_2^{t_2} \subseteq O(2) \wr C_2^{t_2}$. Hence I is well-defined.

The functor I is faithful.
By combining Propositions 3.1.8, 3.1.16 and 3.1.17 we see that for each subgroup $G \in \mathcal{B}_{prod}(k)$ the homomorphism $I : \mathrm{Aut}_{\mathcal{R}_2(\Sigma_k)}(G) \to \mathrm{Aut}_{\mathcal{R}_2(O(k))}(\bar{G})$ is actually an isomorphism. Now fix $H \neq G \in \mathcal{B}_{prod}(k)$ and choose morphisms $g_1 G, g_2 G : H \to G$ in the category of Σ_k-orbits. Let us consider the compositions

$$H \xrightarrow{g_i G} g_i^{-1} H g_i \xrightarrow{1 \cdot G} G$$

for $i = 1, 2$. By Proposition 3.1.19 $g_1^{-1} H g_1$ and $g_2^{-1} H g_2$ differ by conjugation by an element h which permutes irreducible factors. Hence the conjugation by $i(h)$, where $i : \Sigma_k \to O(k)$ is an obvious inclusion, sends the group $g_1^{-1} \bar{H} g_1$ onto $g_2^{-1} \bar{H} g_2$ and also permutes irreducible factors. By (3.1.26) $h \in G$ if and only if $i(h) \in \bar{G}$. It shows that $g_1 G$ and $g_2 G$ represent the same morphism $H \to G$ in $\mathcal{R}_2(\Sigma_k)$ if and only if they represent the same morphism in $\mathcal{R}_2(\Sigma_k)$. As a consequence we get that I is an isomorphism on sets of morphisms. □

3.2 The category $\mathcal{R}_2(Spin(7))$.

The main goal of the present section is to give a detailed description of the category of 2-stubborn $Spin(7)$-orbits. First, we note that there is a natural equivalence between categories $\mathcal{R}_2(Spin(7))$ and $\mathcal{R}(O(7))$. Next, we apply results of Section 3.1.

Proposition 3.2.1. *For each $n \not\equiv 0 \mod 4$, if $P \subseteq O(n)$ is a 2-stubborn subgroup, then $P \cap SO(n) \subseteq SO(n)$ is also 2-stubborn. The functor*

$$\mathcal{R}_2(O(n)) \ni P \mapsto P \cap SO(n) \in \mathcal{R}_2(SO(n))$$

is an equivalence of categories.

Proof. It is a consequence of [O1, Prop. 11] and [O1, Th. 12]. □

Proposition 3.2.2. ([JMO1, Prop. 1.6.(i)]) *If $f : G \to H$ is a covering of compact connected Lie groups, then the functor*

$$\mathcal{R}_2(H) \ni P \mapsto f^{-1}(P) \in \mathcal{R}_2(G).$$

is an equivalence of categories. In particular, for each n the categories $\mathcal{R}_2(Spin(n))$ and $\mathcal{R}_2(SO(n))$ are equivalent.

Let $\pi : Spin(7) \to SO(7)$ be an obvious projection.

Corollary 3.2.3. If $n \not\equiv 0 \mod 4$, then the functor

$$\mathcal{R}_2(O(n)) \ni P \mapsto \pi^{-1}(P \cap SO(n)) \in \mathcal{R}_2(Spin(n))$$

is an equivalence of categories. In particular, the categories $\mathcal{R}_2(Spin(7))$ and $\mathcal{R}(O(7))$ are naturally equivalent.

According to Theorem 3.1.11, each 2-stubborn subgroup of $O(7)$ is conjugate to an element of $\mathcal{T}_{prod}(7)$ (see 3.1.7). By definition, we have

$$\mathcal{T}_{irr}(1) = \{\{\pm 1\}\}, \qquad \mathcal{T}_{irr}(2) = \{O(2)\},$$

$$\mathcal{T}_{irr}(4) = \{O(2) \wr C_2,\ \{\pm 1\} \wr C_2^2,\ \bar{\Gamma}_4,\ \Gamma_4\}.$$

Hence

$$\mathcal{T}_{prod}(7) = \{J \times O(2) \times \{\pm 1\},\ J \times \{\pm 1\}^3\}_{J \in \mathcal{T}_{irr}(4)} \cup \{O(2)^i \times \{\pm 1\}^{7-2i}\}_{i=0,\ldots,3}.$$

By 3.1.11 all elements of $\mathcal{T}_{prod}(7)$ except $O(2)^2 \times \{\pm 1\}^3$ are 2-stubborn in $O(7)$.

We will frequently use 2-stubborn subgroups of $Spin(7)$, so we will introduce a simplier notation for them. We denote elements of $\mathcal{T}_{irr}(4)$ by

$$\begin{aligned} N &:= O(2) \wr C_2 \\ K &:= \{\pm 1\} \wr C_2^2 \\ \bar{M} &:= \bar{\Gamma}_4 \\ M &:= \Gamma_4 \end{aligned}$$

and for each $J \in \mathcal{T}_{irr}(4)$ let

$$J_1 := \pi^{-1}(J \times O(2) \times \{\pm 1\}) \subseteq \pi^{-1}(O(4) \times O(2) \times O(1)) \subset Spin(7) \tag{3.2.4}$$

$$J_0 := \pi^{-1}(J \times \{\pm 1\}^3) \subseteq \pi^{-1}(O(4) \times O(3)) \subset Spin(7), \tag{3.2.5}$$

and for $i = 0, 1, 3$ let

$$L_i := \pi^{-1}(O(2)^i \times \{\pm 1\}^{7-2i}) \subseteq \pi^{-1}(O(2i) \times O(7-2i)) \subseteq Spin(7). \tag{3.2.6}$$

Symbols defined above will be also used to denote objects of the category $\mathcal{R}_2(Spin(7))$ (and $\mathcal{R}_2(O(7))$).

As a consequence of 3.1.8 and 3.2.3 we obtain

Proposition 3.2.7. *We have*

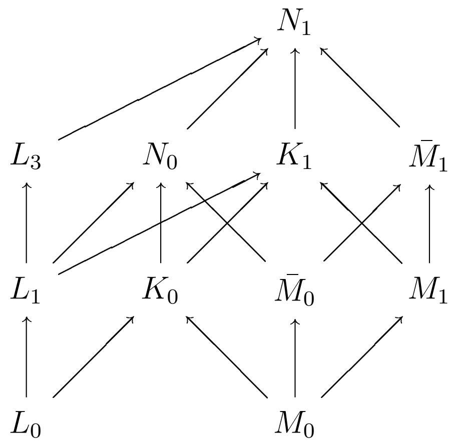
- $W_{O(4)}(N) = 1$,
- $W_{O(4)}(K) \simeq W_{O(4)}(\bar{M}) \simeq \Sigma_3$,
- $W_{O(4)}(M) \simeq O^+(X_2) \simeq \Sigma_3 \wr \Sigma_2$.

For any $J \in \mathcal{T}_{irr}(4)$ *we have*

- $\mathrm{Aut}_{\mathcal{R}_2(Spin(7))}(J_1) = W_{Spin(7)}(J_1) \cong W_{O(4)}(J)$,
- $\mathrm{Aut}_{\mathcal{R}_2(Spin(7))}(J_0) = W_{Spin(7)}(J_0) \cong W_{O(4)}(J) \times \Sigma_3$,
- $\mathrm{Aut}_{\mathcal{R}_2(Spin(7))}(L_i) = W_{Spin(7)}(L_i) \cong \Sigma_i \times \Sigma_{7-2i}$. □

As a consequence, we obtain the following theorem:

Theorem 3.2.8. *Every 2-stubborn subgroup of $Spin(7)$ is conjugate to one of the following subgroups: N_1, N_0, K_1, K_0, $\bar{M}_1$, $\bar{M}_0$, M_1, M_0, L_3, L_1, L_0. Every morphism of the category $\mathcal{R}_2(Spin(7))$ is a composition of automorphisms and morphisms presented on the following diagram:*



The groups of automorphisms of objects are respectively

$$\mathrm{Aut}(N_1) = 1, \qquad \mathrm{Aut}(N_0) \cong \mathrm{Aut}(L_3) \cong \mathrm{Aut}(K_1) \cong \mathrm{Aut}(\bar{M}_1) \cong \Sigma_3$$
$$\mathrm{Aut}(L_1) \cong \Sigma_5, \qquad \mathrm{Aut}(L_0) \cong \Sigma_7, \qquad \mathrm{Aut}(K_0) \cong \mathrm{Aut}(\bar{M}_0) \cong \Sigma_3 \times \Sigma_3$$
$$\mathrm{Aut}(M_1) \cong \Sigma_3 \wr \Sigma_2, \qquad \mathrm{Aut}(M_0) \cong \Sigma_3 \wr \Sigma_2 \times \Sigma_3$$

Proof. It is a consequence of 3.1.11 and 3.1.12. □

We conclude with a presentation of $\mathcal{R}_2(Spin(7))$ as a sum of two smaller subcategories. We say, that a small category $\mathcal{C}$ is *a sum* of its subcategories $\mathcal{C}'$ and $\mathcal{C}''$ if every morphism of $\mathcal{C}$ is a morphism of $\mathcal{C}'$ or $\mathcal{C}''$. Recall (3.1.27) that there is an inclusion $I : \mathcal{R}_2(\Sigma_7) \to \mathcal{R}_2(O(7))$ onto a full subcategory.

To construct the other subcategory we need

Proposition 3.2.9. *A group $J \in \mathcal{T}_{prod}(4)$ is stubborn in $O(4)$ if and only if $J \in \mathcal{T}_{irr}(4)$.*

Proof. If $J \in T_{prod}(4) \backslash \mathcal{T}_{irr}(4)$, then $P \cong O(2)^r \times \{\pm 1\}^{4-2r}$ for some $r = 0, 1, 2$. In this case it fails the conditions of 3.1.11, since both groups $W_{O(2)}(O(2))$ and $W_{O(1)}(\{\pm 1\})$ are trivial. □

If P is a stubborn subgroup of $O(4) \times O(3)$, then it is conjugate to a product $P' \times P''$ (by [JMO1, 1.6.(ii)]), such that P' is a stubborn subgroup of $O(4)$ and P'' is a stubborn subgroup of $O(3)$. Hence (by 3.2.9) P is conjugate to J_i for some $J \in \mathcal{T}_{irr}(4)$ and $i = 0, 1$. Therefore the inclusion $O(4) \times O(3) \subset O(7)$ induces a functor

$$\mathcal{R}_2(O(4)) \times \mathcal{R}_2(O(3)) \cong \mathcal{R}_2(O(4) \times O(3)) \longrightarrow \mathcal{R}_2(O(7)),$$

whose image is the full subcategory of $\mathcal{R}_2(O(7))$. Since $\mathcal{R}_2(Spin(7)) \cong \mathcal{R}_2(O(7))$, we obtain

Proposition 3.2.10. *The category $\mathcal{R}_2(Spin(7))$ is a sum of the subcategories $I(\mathcal{R}_2(\Sigma_7))$ and $\mathcal{R}_2(O(4)) \times \mathcal{R}_2(O(3))$. The first subcategory contains objects L_j, N_i and K_i (where $j = 0, 1, 3$, $i = 0, 1$) and the second one contains objects $N_i, K_i, M_i, \bar{M}_i$.* □

3.3 Even representations of 2-stubborn subgroups of $Spin(7)$

Assume that P is a locally finite subgroup of $G := Spin(n)$ containing the non-trivial lift of unity u. Each irreducible representation ϱ of P is either a restriction of a representation of $P/\langle u \rangle$ (and $\chi_\varrho(u) = \dim \varrho$) or it is not (and $\chi_\varrho(u) = -\dim \varrho$). In the first case we say that ϱ is *even*; otherwise it is *odd*. The present section contains the classification of even irreducible representations of discrete approximations of stubborn subgroups of G, or, equivalently representations of the corresponding subgroups of $SO(7)$ (cf. 3.2.1). Odd representations must be considered separately; the similar classification is presented in the next section.

Before we begin, let us introduce some notation. The letter θ denotes the one-dimensional trivial representation and τ denotes the non-trivial irreducible representation of the group having two elements. There is a bijection between the set of irreducible representations of an $\mathbb{F}_2$-vector space V and its dual $V^* = \mathrm{Hom}(V, \mathbb{F}_2)$, and for $\mathbf{v} \in V^*$ we have $\chi_{\mathbf{v}}(v) = (-1)^{\mathbf{v}(v)}$. Let $\mathbf{a_i}$, $\mathbf{b_i}$ denote elements of the basis of X_n^* which is dual to the basis $\{a_i\} \cup \{b_i\}$ of X_n (cf. 3.1.2).

Representations of groups Γ_{2^n}

Fix a positive integer n. Recall (3.1.5) the exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \Gamma_{2^n} \xrightarrow{p_X} X_n \longrightarrow 1.$$

If $g, g' \in \Gamma_{2^n}$ are conjugate, then obviously $p_X(g) = p_X(g')$. On the other hand, $Z(\Gamma_{2^n}) = \{\pm I\}$. Thus Γ_{2^n} has $2^{2n}+1$ conjugacy classes of elements (and the same number of isomorphism classes of irreducible representations).

For any $\mathbf{x} \in X_n^*$ define

$$\tau_{\mathbf{x}} := (p_X)^* \mathbf{x} \in \mathrm{IR}(\Gamma_{2^n}). \tag{3.3.1}$$

Proposition 3.3.2.

$$\mathrm{IR}(\Gamma_{2^n}) = \{\tau_{\mathbf{x}}\}_{\mathbf{x}\in X_n^*} \cup \{\alpha\},$$

where α is the 2^n-dimensional representation with its character given by the formula

$$\chi_\alpha(g) = \begin{cases} 2^n & \text{for } g = I \\ -2^n & \text{for } g = -I \\ 0 & \text{for } g \neq \pm I \end{cases}$$

Proof. Since there exist 2^{2n} pairwise non-isomorphic representations of Γ_{2^n} which are restrictions of representations of X_n, then there exists exactly one which is not, say α. The regular representation ψ of Γ_{2^n} is a direct sum

$$\psi \simeq \left(\bigoplus_{\mathbf{x}\in X_n^*} \tau_{\mathbf{x}} \right) \oplus \alpha^{\oplus \dim \alpha}.$$

Then $\dim \alpha = 2^n$. Since the first summand is isomorphic to $\mathrm{ind}_{\{\pm I\}}^{\Gamma_{2^n}} \theta$, then $\mathrm{ind}_{\{\pm I\}}^{\Gamma_{2^n}} \tau = \alpha^{\oplus 2^n}$. As a consequence we obtain the formula for the character of α. □

Proposition 3.3.3. *If $a \in W_{O(2^n)}(\Gamma_{2^n}) \simeq O^+(X_n)$, then $a^*\tau_{\mathbf{x}} \cong \tau_{a^*\mathbf{x}}$ and $a^*\alpha \cong \alpha$. In particular, for $n = 2$ the orbits of the action of $W_{O(4)}(\Gamma_4)$ on $\mathrm{IR}(\Gamma_4)$ are as follows:*

$$\{\tau_{\mathbf{0}}\}, \quad \{\alpha\}, \quad \{\tau_{\mathbf{a_0}+\mathbf{b_0}}, \tau_{\mathbf{a_1}+\mathbf{b_1}}, \tau_{\mathbf{a_0}+\mathbf{b_0}+\mathbf{a_1}}, \tau_{\mathbf{a_0}+\mathbf{b_0}+\mathbf{b_1}}, \tau_{\mathbf{a_0}+\mathbf{a_1}+\mathbf{b_1}}, \tau_{\mathbf{b_0}+\mathbf{a_1}+\mathbf{b_1}}\}$$
$$\{\tau_{\mathbf{a_0}}, \tau_{\mathbf{b_0}}, \tau_{\mathbf{a_1}}, \tau_{\mathbf{b_1}}, \tau_{\mathbf{a_0}+\mathbf{a_1}}, \tau_{\mathbf{a_0}+\mathbf{b_1}}, \tau_{\mathbf{b_0}+\mathbf{a_1}}, \tau_{\mathbf{b_0}+\mathbf{b_1}}, \tau_{\mathbf{a_0}+\mathbf{b_0}+\mathbf{a_1}+\mathbf{b_1}}\}.$$

Proof. The first statement follows immediately from the definition of $\tau_{\mathbf{x}}$. Fix $n = 2$. Recall that $O^+(X_2)$ is the subgroup of $\mathrm{GL}(X_2)$ containing matrices which preserve the quadratic form q (3.1.9). The set $Z := \{x \in X_2 : q(x) = 1\}$ has six elements, and the relation on Z

$$x \sim x' \Leftrightarrow x + x' \in Z \cup \{0\}$$

is an equivalence relation and has two equivalence classes: $\{a_0 + b_0, a_0 + a_1 + b_1, b_0 + a_1 + b_1\}$ and $\{a_1 + b_1, a_1 + a_0 + b_0, b_1 + a_0 + b_0\}$, which are preserved by the elements of $O^+(X_2)$. Moreover, Z spans X_2. Hence $O^+(X_2) \subseteq \Sigma_3 \wr \Sigma_2$ and one can check immediately that in fact $O^+(X_2) \cong \Sigma_3 \wr \Sigma_2$. Now elementary calculations prove the conclusion. □

Representations of groups $\bar{\Gamma}^{\infty}_{2^n}$

Define

$$c(t) := \cos 2\pi t, \quad s(t) := \sin 2\pi t, \quad e(t) = \begin{pmatrix} c(t) & s(t) \\ -s(t) & c(t) \end{pmatrix} \in SO(2) \quad (3.3.4)$$

Let $T^{\infty} := \{e(\frac{l}{2^s})\}_{l,s\in\mathbb{Z}} \otimes I_{2^{n-1}}$. The group

$$\bar{\Gamma}^{\infty}_{2^n} := \langle T^{\infty}, \Gamma_{2^n} \rangle \subseteq O(2^n), \quad (3.3.5)$$

is a discrete approximation of $\bar{\Gamma}_{2^n}$.

We begin with the case $n = 2$. We have $\mathrm{IR}(T^{\infty}) = \{\varrho_k\}_{k\in\mathbb{Z}_2^{\wedge}}$ (cf. 2.3.28), and by 2.3.32 $\mathrm{ind}_{T^{\infty}}^{\bar{\Gamma}^{\infty}_2} \varrho_k$ is irreducible iff $k \neq 0$; otherwise it splits into the sum of a trivial representation θ and a non-trivial τ (since $\bar{\Gamma}_2/T^{\infty} = W_1 \simeq C_2$). Denote

$$\alpha_{2k+1} := \mathrm{ind}_{T^{\infty}}^{\bar{\Gamma}^{\infty}_2} \varrho_{2k+1}, \qquad \beta_{2k} := \mathrm{ind}_{T^{\infty}}^{\bar{\Gamma}^{\infty}_2} \varrho_{2k}. \quad (3.3.6)$$

As a consequence we obtain

Corollary 3.3.7.

$$\mathrm{IR}(\bar{\Gamma}^{\infty}_2) = \{\alpha_{2k+1}\}_{k\in\mathbb{Z}_2^{\wedge}} \cup \{\beta_{2k}\}_{k\in\mathbb{Z}_2^{\wedge}} \cup \{\theta, \tau\}.$$

Now assume $n > 2$. Recall the exact sequences (3.1.4) and (3.1.6)

$$1 \longrightarrow T^{\infty} \longrightarrow \bar{\Gamma}^{\infty}_{2^n} \xrightarrow{p'_W} W_n \longrightarrow 1.$$

$$1 \longrightarrow \{\pm I\} \longrightarrow \bar\Gamma^\infty_{2^n} \xrightarrow{p'_V} \bar\Gamma^\infty_2/\{\pm I\} \times V_n \longrightarrow 1.$$

For each $\mathbf{w} \in W_n^*$, $\mathbf{v} \in V_n^*$ and $k \in \mathbb{Z}_2^\wedge$ define the following irreducible representations of $\bar\Gamma^\infty_2$:

$$\tau_{\mathbf{w}} := (p'_W)^* \mathbf{w}$$

$$\beta_{2k,\mathbf{v}} := (p'_V)^* \Big((\mathrm{ind}_{T^\infty/\{\pm I\}}^{\bar\Gamma^\infty_2/\{\pm I\}} \varrho_k) \bar\otimes \mathbf{v} \Big),$$

where ϱ_k is defined in 2.3.28 (note that $T^\infty/\{\pm I\} \simeq \mathbb{Z}/2^\infty$).

By 2.3.31 each irreducible representation of $\bar\Gamma^\infty_{2^n}$ is isomorphic to a subrepresentation of $\mathrm{ind}_{T^\infty}^{\bar\Gamma^\infty_{2^n}} \varrho_k$ for some $k \in \mathbb{Z}_2^\wedge$. We have the following decompositions:

$$\mathrm{ind}_{T^\infty}^{\bar\Gamma^\infty_{2^n}} \varrho_0 = \mathrm{ind}_{T^\infty}^{\bar\Gamma^\infty_{2^n}} \theta = \bigoplus_{\mathbf{w} \in W_n^*} \tau_{\mathbf{w}} \tag{3.3.8}$$

$$\mathrm{ind}_{T^\infty}^{\bar\Gamma^\infty_{2^n}} \varrho_{2k} = \bigoplus_{\mathbf{v} \in V_n^*} \beta_{2k,\mathbf{v}}. \tag{3.3.9}$$

Proposition 3.3.10. *For any $k \in \mathbb{Z}_2^\wedge$ the representation $\mathrm{ind}_{T^\infty}^{\bar\Gamma^\infty_{2^n}} \varrho_{2k+1}$ is a direct sum of 2^{n-1} pairwise isomorphic irreducible representations (which will be denoted by α_{2k+1}).*

Proof. For each s let $T^{(s)} := \{e(\frac{l}{2^s})_{l\in\mathbb{Z}}\} \otimes I_{2^{n-1}}$, $\bar\Gamma^{(s)}_{2^n} := \langle T^{(s)}, \Gamma_{2^n}\rangle$ and let ϱ_m, for $m \in \mathbb{Z}/2^s$, be the representation of $T^{(s)}$ defined by

$$\varrho_m \left(e\left(\tfrac{l}{2^s}\right)\right) = \exp\left(\tfrac{ml}{2^s}\right) \in U(1)$$

Let $\sigma := \mathrm{ind}_{T^{(s)}}^{\bar\Gamma^{(s)}_{2^n}} \varrho_{(2k+1 \mod 2^s)}$. Since $\mathrm{res}_{\Gamma_{2^n}}^{\bar\Gamma^{(s)}_{2^n}} \sigma \simeq \alpha^{\oplus 2^n}$ (cf. 3.3.2), then σ decomposes into the sum of irreducible representations of dimensions which are divisible by 2^n. The regular representation of $\bar\Gamma^{(s)}_{2^n}$ decomposes as $\omega = \bigoplus_{m\in\mathbb{Z}/2^s} \mathrm{ind}_{T^{(s)}}^{\bar\Gamma^{(s)}_{2^n}} \varrho_m$. If φ is an irreducible representation of σ, then it appears in the regular representation only in the summands $\mathrm{ind}_{T^{(s)}}^{\bar\Gamma^{(s)}_{2^n}} \varrho_{2k+1}$ and $\mathrm{ind}_{T^{(s)}}^{\bar\Gamma^{(s)}_{2^n}} \varrho_{-2k-1}$ (which are isomorphic). Since the dimensions of both these summands is 2^{2n-1} and $\dim \varphi \geq 2^n$, then

$$\mathrm{ind}_{T^{(s)}}^{\bar\Gamma^{(s)}_{2^n}} \varrho_{2k+1} \simeq \mathrm{ind}_{T^{(s)}}^{\bar\Gamma^{(s)}_{2^n}} \varrho_{-2k-1} \simeq \varphi^{\oplus 2^{n-1}}.$$

Therefore, by 2.3.11, $\mathrm{ind}_{T^\infty}^{\bar\Gamma^\infty_{2^n}} \varrho_{2k+1}$ is the direct sum of 2^{n-1} copies of pairwise isomorphic representations of dimension 2^n. ◻

Denote irreducible summand of $\mathrm{ind}_{T^\infty}^{\bar{\Gamma}_{2^n}^\infty} \varrho_{2k+1}$ from the proof above by α_{2k+1}. The character of α_{2k+1} is

$$\chi_{\alpha_{2k+1}}(g) = \begin{cases} 2^{n-1} c\left(\frac{l}{2^s}\right) & \text{for } g = e\left(\frac{l}{2^s}\right) \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.3.11. *For each n*

$$\mathrm{IR}(\bar{\Gamma}_{2^n}^\infty) = \{\tau_{\mathbf{w}}\}_{\mathbf{w}\in W_n^*} \cup \{\beta_{2k,\mathbf{v}}\}_{\mathbf{v}\in V_n^*}^{k\in\mathbb{Z}_2^\wedge\setminus\{0\}} \cup \{\alpha_{2k+1}\}^{k\in\mathbb{Z}_2^\wedge}$$

and $\beta_{2k,\mathbf{v}} \simeq \beta_{-2k,\mathbf{v}}$, $\alpha_{2k+1} \simeq \alpha_{-2k-1}$.

Proof. Each irreducible representation of $\bar{\Gamma}_{2^n}^\infty$ is, for some $k \in \mathbb{Z}_2^\wedge$, a subrepresentation of $\mathrm{ind}_{T^\infty}^{\bar{\Gamma}_{2^n}^\infty} \varrho_k$. All the cases have been considered in 3.3.8, 3.3.9 and 3.3.10. □

The following proposition shows how irreducible representations of $\bar{\Gamma}_{2^n}^\infty$ restrict to Γ_{2^n}:

Proposition 3.3.12. *Let $p : X_n \to W_n$ and $q : X_n \to V_n$ denote the obvious projections. Then*

$$\mathrm{res}_{\Gamma_{2^n}}^{\bar{\Gamma}_{2^n}^\infty} \tau_{\mathbf{w}} = \tau_{p^*\mathbf{w}}$$

$$\mathrm{res}_{\Gamma_{2^n}}^{\bar{\Gamma}_{2^n}^\infty} \beta_{2k,\mathbf{v}} = \begin{cases} \tau_{q^*\mathbf{v}} \oplus \tau_{q^*\mathbf{v}+\mathbf{a_0}+\mathbf{b_0}} & \textit{for } k \textit{ even} \\ \tau_{q^*\mathbf{v}+\mathbf{a_0}} \oplus \tau_{q^*\mathbf{v}+\mathbf{b_0}} & \textit{for } k \textit{ odd} \end{cases}$$

$$\mathrm{res}_{\Gamma_{2^n}}^{\bar{\Gamma}_{2^n}^\infty} \alpha_{2k+1} = \alpha.$$

Proof. The first and the last equations are clear. We have

$$\begin{aligned} \mathrm{res}_{\Gamma_{2^n}}^{\bar{\Gamma}_{2^n}^\infty} \beta_{2k,\mathbf{v}} &= \mathrm{res}_{\Gamma_{2^n}}^{\bar{\Gamma}_{2^n}^\infty} (p'_V)^*\Big((\mathrm{ind}_{T^\infty/\{\pm I\}}^{\bar{\Gamma}_2^\infty/\{\pm I\}} \varrho_k)\bar{\otimes}\mathbf{v}\Big) \\ &= (p_V)^*\Big((\mathrm{res}_{\Gamma_2/\{\pm I\}}^{\bar{\Gamma}_2^\infty/\{\pm I\}} \mathrm{ind}_{T^\infty/\{\pm I\}}^{\bar{\Gamma}_2^\infty/\{\pm I\}} \varrho_k)\bar{\otimes}\mathbf{v}\Big) = (p_V)^*\Big((\mathrm{ind}_{T^{(2)}/\{\pm I\}}^{\Gamma_2/\{\pm I\}} \varrho_k)\bar{\otimes}\mathbf{v}\Big) \end{aligned}$$

Since $\Gamma_2/\{\pm I\} \cong \mathbb{F}_2\{a_0, b_0\}$, then we obtain

$$\mathrm{ind}_{T^{(2)}/\{\pm I\}}^{\Gamma_2/\{\pm I\}} \varrho_k = \begin{cases} \mathbf{0} \oplus (\mathbf{a_0} + \mathbf{b_0}) & \text{for } k \text{ even} \\ \mathbf{a_0} \oplus \mathbf{b_0} & \text{for } k \text{ odd.} \end{cases}$$

It proves the last equation. □

The next step is the classification of irreducible representations of wreath products $\Gamma \wr C_2^{t_1} \wr \cdots \wr C_2^{t_k}$, where Γ stands for either Γ_{2^n} or $\bar{\Gamma}_{2^n}^\infty$. We do not consider the general case (since it is not necessary for the proof of the main theorem), only the wreath products $\Gamma \wr C_2$ and $\{\pm 1\} \wr C_2^2$.

Representations of wreath products $\Gamma \wr C_2$

Definition 3.3.13. Let Γ be a locally finite group and let $\varphi, \psi \in \mathrm{IR}(\Gamma)$ be non-isomorphic irreducible representations. Define

$$\varphi \boxtimes \psi := \mathrm{ind}_{\Gamma\times\Gamma}^{\Gamma\wr C_2} \varphi \bar{\otimes} \psi \in \mathrm{IR}(\Gamma \wr C_2)$$

($\varphi \boxtimes \psi$ is irreducible by 2.3.32).

Let e and t denote the neutral and the non-neutral element of C_2.

Proposition 3.3.14. *Let Γ be a locally finite group and let $\varphi : \Gamma \to U(V)$ be an irreducible representation. Then there is a decomposition*

$$\mathrm{ind}_{\Gamma\times\Gamma}^{\Gamma\wr C_2} \varphi \bar{\otimes} \varphi \simeq \varphi^+ \oplus \varphi^-,$$

where $\varphi^+, \varphi^- \in \mathrm{IR}(\Gamma \wr C_2)$, and for $g_1, g_2 \in \Gamma$, $h \in C_2$

$$\chi_{\varphi^\mu}(g_1, g_2; h) = \begin{cases} \chi_\varphi(g_1)\chi_\varphi(g_2) & \text{for } h = e \\ \mu\chi_\varphi(g_1 g_2) & \text{for } h = t \end{cases}.$$

Proof. Let

$$V^+ := \{e \otimes v \otimes w + t \otimes w \otimes v \in \mathbb{C}(\Gamma \wr C_2) \otimes_{\mathbb{C}(\Gamma\times\Gamma)} (V \otimes V)\}$$
$$V^- := \{e \otimes v \otimes w - t \otimes w \otimes v \in \mathbb{C}(\Gamma \wr C_2) \otimes_{\mathbb{C}(\Gamma\times\Gamma)} (V \otimes V)\}.$$

The spaces V^+ and V^- are $\Gamma \wr C_2$-subrepresentations which are irreducible since

$$\mathrm{res}_{\Gamma\times\Gamma}^{\Gamma\wr C_2} V^+ \simeq \mathrm{res}_{\Gamma\times\Gamma}^{\Gamma\wr C_2} V^- \simeq \varphi \bar{\otimes} \varphi$$

is irreducible. Obviously $\chi_{\varphi^\mu}(g_1, g_2; 1) = \chi_\varphi(g_1)\chi_\varphi(g_2)$. Let $\{v_i\}_{i=1}^n$ be a basis of V and $M(g)$ the matrix of $\varrho(g)$ in this basis. Then the collection

$$\{v_{ij} := e \otimes v_i \otimes v_j + t \otimes v_j \otimes v_i\}_{i,j=1}^n$$

is a basis of V^+. We have

$$\begin{aligned}(g_1, g_2; t)v_{ij} &= (g_1, g_2; e)(t \otimes v_i \otimes v_j) + (g_1, g_2; e)(e \otimes v_j \otimes v_i) \\ &= t \otimes g_2 v_i \otimes g_1 v_j + e \otimes g_1 v_j \otimes g_2 v_i \\ &= e \otimes \sum_l M(g_1)_{lj} v_l \otimes \sum_k M(g_2)_{ki} v_k + t \otimes \sum_k M(g_2)_{ki} v_k \otimes \sum_l M(g_1)_{lj} v_l \\ &= \sum_{k,l} M(g_1)_{lj} M(g_2)_{ki} v_{lk}\end{aligned}$$

Finally we obtain

$$\chi_{\varphi^+}(g_1, g_2; t) = \sum_{i,j} M(g_1)_{ij} M(g_2)_{ji} = \sum_i M(g_1 g_2)_{ii} = \chi_\varphi(g_1 g_2).$$

Similarly $\chi_{\varphi^-}(g_1, g_2; t) = -\chi_\varphi(g_1 g_2)$. □

Representations of $K = \{\pm 1\} \wr C_2^2$

By Proposition 2.3.31 all irreducible representations of K are subrepresentations of $\operatorname{ind}_{\{\pm1\}^4}^{\{\pm1\}\wr C_2^2} \left(\bar{\bigotimes}_{a\in C_2^2} \varphi_a\right)$ for $\varphi_a \in \mathrm{IR}(\{\pm1\}) = \{\theta, \tau\}$. We will denote them by γ with some indices, and the lower index is the number of non-trivial representations among φ_a's.

Let $p : G \to C_2^2$ be the obvious projection. The representations

$$\gamma_1 := \operatorname{ind}_{\{\pm1\}^4}^{\{\pm1\}\wr C_2^2} \theta\bar{\otimes}\theta\bar{\otimes}\theta\bar{\otimes}\tau$$

and

$$\gamma_3 := \operatorname{ind}_{\{\pm1\}^4}^{\{\pm1\}\wr C_2^2} \theta\bar{\otimes}\tau\bar{\otimes}\tau\bar{\otimes}\tau$$

are irreducible. For $\varphi, \psi \in \mathrm{IR}(C_2) = \{\theta, \tau\}$ define

$$\gamma_0^{\varphi,\psi} := p^*(\varphi\bar{\otimes}\psi), \qquad \gamma_4^{\varphi,\psi} := p^*(\varphi\bar{\otimes}\psi) \otimes \det.$$

There are decompositions

$$\operatorname{ind}_{\{\pm1\}^4}^{\{\pm1\}\wr C_2^2} \theta\bar{\otimes}\theta\bar{\otimes}\theta\bar{\otimes}\theta \simeq \bigoplus_{\varphi,\psi\in\mathrm{IR}(C_2)} \gamma_0^{\varphi,\psi}$$

$$\operatorname{ind}_{\{\pm1\}^4}^{\{\pm1\}\wr C_2^2} \tau\bar{\otimes}\tau\bar{\otimes}\tau\bar{\otimes}\tau \simeq \bigoplus_{\varphi,\psi\in\mathrm{IR}(C_2)} \gamma_4^{\varphi,\psi}.$$

Now choose $a \in C_2^2 \setminus \{(0,0)\}$. Since $\{\pm1\} \wr C_2^2 \simeq (\{\pm1\}^{(C_2^2/\langle a\rangle)} \wr \langle a\rangle) \rtimes (C_2^2/\langle a\rangle)$ we can define the representations

$$\gamma_{2,a}^\mu := \operatorname{ind}_{(\{\pm1\}^{(C_2^2/\langle a\rangle)}\wr\langle a\rangle)}^{(\{\pm1\}^{(C_2^2/\langle a\rangle)}\wr\langle a\rangle)\rtimes(C_2^2/\langle a\rangle)} (\theta\bar{\otimes}\tau)^\mu$$

for $\mu \in \{+,-\}$. Again, we obtain the following decompositions

$$\operatorname{ind}_{\{\pm1\}^4}^{\{\pm1\}\wr C_2^2} \theta\bar{\otimes}\theta\bar{\otimes}\tau\bar{\otimes}\tau \simeq \gamma_{2,a}^+ \oplus \gamma_{2,a}^-,$$

where a is the difference between coordinates with the same isomorphism class of representation. As a consequence we obtain

Corollary 3.3.15.

$$\mathrm{IR}(\{\pm 1\} \wr C_2^2) = \{\gamma_1, \gamma_3\} \cup \{\gamma_0^{i,j}, \gamma_4^{i,j}\}_{i,j\in\{0,1\}} \cup \{\gamma_{2,a}^{\mu}\}_{a\in C_2^2\setminus\{(0,0)\}}^{\mu\in\{+,-\}}.$$

Since the action of $W_{O(4)}(G) \cong GL_2(\mathbb{F}_2) \simeq \Sigma_3$ on G is the obvious one, we have

Corollary 3.3.16. Here follow the orbits of an action of $W_{O(4)}(G)$ on $\mathrm{IR}(G)$:

$$\{\gamma_0^{0,0}\}, \quad \{\gamma_0^{0,1}, \gamma_0^{1,0}, \gamma_0^{1,1}\}, \quad \{\gamma_4^{0,0}\}, \quad \{\gamma_4^{0,1}, \gamma_4^{1,0}, \gamma_4^{1,1}\}, \quad \{\gamma_1\}, \quad \{\gamma_3\}$$

$$\{\gamma_{2,(0,1)}^{+}, \gamma_{2,(1,0)}^{+}, \gamma_{2,(1,1)}^{+}\}, \quad \{\gamma_{2,(0,1)}^{-}, \gamma_{2,(1,0)}^{-}, \gamma_{2,(1,1)}^{-}\}$$

Denote

$$\gamma_2^{+} := \gamma_{2,(0,1)}^{+} \oplus \gamma_{2,(1,0)}^{+} \oplus \gamma_{2,(1,1)}^{+} \tag{3.3.17}$$

$$\gamma_2^{-} := \gamma_{2,(0,1)}^{-} \oplus \gamma_{2,(1,0)}^{-} \oplus \gamma_{2,(1,1)}^{-} \tag{3.3.18}$$

Once we have listed irreducible representations of "pieces" which build up every 2-stubborn subgroup of $SO(7)$, we need to describe sets the $\mathrm{IR}_{ev}(P)$ for P being a 2-stubborn subgroup of $Spin(7)$ (or equivalently, $\mathrm{IR}(P/\langle u\rangle)$). Recall that every 2-stubborn subgroup of $SO(7)$ has the form $(\Pi P_i) \cap SO(7)$ where P_i's are isomorphic to one of the groups $\bar{\Gamma}_2 \wr C_2$, $\bar{\Gamma}_4$, Γ_4, $\Gamma_1 \wr C_2^2$, $\bar{\Gamma}_2$ or Γ_1.

Representations of groups L_i^{∞}

Let

$$U_k := \{(a_1, \ldots, a_k) \in \{\pm 1\}^k : a_1 \ldots a_k = 1\} \tag{3.3.19}$$

and let U_k^* be a dual group. For each sequence $(\mu_1, \ldots, \mu_k) \in \{\pm 1\}^k \cong \mathrm{IR}(C_2)^k$ define

$$\tau_{(\mu_1,\ldots,\mu_k)} := \mathrm{res}_{U_k}^{C_2^k}\ \mu_1 \bar{\otimes} \ldots \bar{\otimes} \mu_k. \tag{3.3.20}$$

Let $\eta_i^k := \bigoplus \tau_{(\mu_1,\ldots,\mu_k)}$, where the sum is taken over all sequences such that -1 appears exactly i times.

Proposition 3.3.21. *Assume that k is an odd integer, and that ϱ is an Σ_k-invariant representation of U_k. Then ϱ is isomorphic to a direct sum of representations η_i^k for $i < k/2$.*

Proof. It is clear that ϱ is a sum of η_i^k's. If k is odd, then $\tau_{(\mu_1,\ldots,\mu_k)}$ is isomorphic to $\tau_{(-\mu_1,\ldots,-\mu_k)}$. Therefore η_i^k is isomorphic to η_{k-i}^k and the lower index can be assumed to be less than $k/2$. □

We have

$$L_3^\infty = ((\bar{\Gamma}_2^\infty)^3 \times \Gamma_1) \cap SO(7) \simeq (\bar{\Gamma}_2^\infty)^3$$
$$L_1^\infty = (\bar{\Gamma}_2^\infty \times (\Gamma_1)^5) \cap SO(7) \simeq \bar{\Gamma}_2^\infty \times U_5$$
$$L_0 = (\Gamma_1)^7 \cap SO(7) \simeq U_7$$

These isomorphisms are given by the formulae

$$(\bar{\Gamma}_2^\infty)^3 \ni (g_1, g_2, g_3) \mapsto \Delta(g_1, g_2, g_3, \det\, g_1 g_2 g_3) \in SO(7) \tag{3.3.22}$$

$$\bar{\Gamma}_2^\infty \times U_5 \ni (g, v) \mapsto \Delta(g, u \cdot \det\, g) \in SO(7) \tag{3.3.23}$$

and the isomorphism $(\Gamma_1)^7 \cap SO(7) \simeq U_7$ is obvious ($\Delta(m_1, ..., m_n)$ denotes the matrix obtained by putting the matrices m_i on the diagonal).

Corollary 3.3.24. We have

$$\mathrm{IR}_{ev}(L_3^\infty) \cong \mathrm{IR}(\bar{\Gamma}_2^\infty)^3, \quad \mathrm{IR}_{ev}(L_1^\infty) \cong \mathrm{IR}(\bar{\Gamma}_2^\infty) \times U_5^*, \quad \mathrm{IR}_{ev}(L_0) \cong U_7^*$$

The group $W_G(L_3^\infty) \simeq \Sigma_3$ acts on $\mathrm{IR}(L_3^\infty)$ by permuting coordinates. The action of $W_G(L_1^\infty) \simeq \Sigma_5$ (resp. $W_G(L_0) \simeq \Sigma_7$) on $\mathrm{IR}_{ev}(L_1^\infty)$ (resp. $\mathrm{IR}_{ev}(L_0)$) is the natural action on U_5^* (resp. U_7^*). The orbits of $W_G(L_i)$ on $\mathrm{IR}_{ev}(L_i)$ are listed below:

$$\mathrm{IR}_{ev}(L_3) = \{\varrho\bar{\otimes}\varrho\bar{\otimes}\varrho\}_{\varrho\in\mathrm{IR}(\bar{\Gamma}_2)} \cup \{\varrho\bar{\otimes}\varrho\bar{\otimes}\sigma, \varrho\bar{\otimes}\sigma\bar{\otimes}\varrho, \sigma\bar{\otimes}\varrho\bar{\otimes}\varrho\}_{\varrho\neq\sigma\in\mathrm{IR}(\bar{\Gamma}_2)} \cup$$
$$\cup \{\varrho\bar{\otimes}\sigma\bar{\otimes}\tau, \varrho\bar{\otimes}\tau\bar{\otimes}\sigma, \sigma\bar{\otimes}\varrho\bar{\otimes}\tau, \sigma\bar{\otimes}\tau\bar{\otimes}\varrho, \tau\bar{\otimes}\varrho\bar{\otimes}\sigma, \tau\bar{\otimes}\sigma\bar{\otimes}\varrho\}_{\varrho\neq\sigma\neq\tau\in\mathrm{IR}(\bar{\Gamma}_2)}$$
$$\mathrm{IR}_{ev}(L_1) = \{\varrho\bar{\otimes}\eta_k^5\}_{\varrho\in\mathrm{IR}(\bar{\Gamma}_2)}^{k=0,1,2}, \qquad \mathrm{IR}_{ev}(L_0) = \{\eta_k^7\}^{k=0,1,2,3}.$$

Proposition 3.3.25. *Let $\varphi\bar{\otimes}\psi\bar{\otimes}\omega$ be an irreducible representation of $L_3^\infty \cong (\bar{\Gamma}_2^\infty)^3$, and let $i : \bar{\Gamma}_2^\infty \times U_5 \cong L_1^\infty \to L_3^\infty$ be an inclusion sending the factor $\bar{\Gamma}_2^\infty$ of L_1^∞ onto the first factor of L_3^∞. Define the sets $I_\theta = \{(1,1)\}$, $I_\tau = \{(-1,-1)\}$, $I_{\beta_k} = \{(1,1),(-1,-1)\}$, $I_{\alpha_k} = \{(1,-1),(-1,1)\}$. If exactly one of the representations ψ, ω is isomorphic to α_k for some $k \in \mathbb{Z}_2^\wedge$, then*

$$i^*(\varphi\bar{\otimes}\psi\bar{\otimes}\omega) \simeq (\varphi \otimes \det)\bar{\otimes}\Big(\bigoplus_{(\mu_1,\mu_2)\in I_\psi}\ \bigoplus_{(\mu_3,\mu_4)\in I_\omega} \tau_{(\mu_1,\mu_2,\mu_3,\mu_4,1)}\Big).$$

Otherwise

$$i^*(\varphi\bar{\otimes}\psi\bar{\otimes}\omega) \simeq \varphi\bar{\otimes}(\bigoplus_{(\mu_1,\mu_2)\in I_\psi}\bigoplus_{(\mu_3,\mu_4)\in I_\omega}\tau_{(\mu_1,\mu_2,\mu_3,\mu_4,1)}).$$

Proof. Note that the map $\bar{\Gamma}_2^\infty \times U_5 \to \bar{\Gamma}_2^\infty \times \bar{\Gamma}_2^\infty \times \bar{\Gamma}_2^\infty$ is given by the formula

$$(g,(e_1,e_2,e_3,e_4,e_5)) \mapsto (g, \det g \cdot \Delta(e_1,e_2), \det g \cdot \Delta(e_3,e_4)).$$

We have

$$\chi_{\varphi\bar{\otimes}\psi\bar{\otimes}\omega}(g,(1,\dots,1)) = \chi_\varphi(g)\chi_\psi(-I_2)\chi_\omega(-I_2) = \chi_\varphi(g)\chi_{\psi\bar{\otimes}\omega}(-I_4).$$

Thus at the first coordinate appears $\varphi\otimes\det$ if one of ψ, ω is isomorphic to α_k and φ otherwise. What appears at the second coordinate is clear, since

$$\operatorname{res}^{\bar{\Gamma}_2^\infty}_{\{\pm1\}^2}\sigma = \begin{cases} \tau_{(1,1)} & \text{for } \sigma\simeq\theta \\ \tau_{(-1,-1)} & \text{for } \sigma\simeq\tau \\ \tau_{(1,-1)}\oplus\tau_{(-1,1)} & \text{for } \sigma\simeq\alpha_{2k_1} \\ \tau_{(1,1)}\oplus\tau_{(-1,-1)} & \text{for } \sigma\simeq\beta_{2k}. \end{cases} \quad \square$$

Corollary 3.3.26. Let $\eta^5_{i(l,m)} = \bigoplus \tau_{\mu_1,\mu_2,\mu_3,\mu_4,\mu_5}$, where the sum is taken over all sequences such that exactly l of the numbers μ_1,μ_2 are equal -1, exactly m of the numbers μ_3,μ_4 are equal -1 and finally i of the numbers $\mu_1,\dots,\mu_5$ are equal -1. Then

$$\operatorname{res}^{L_3^\infty}_{L_1^\infty}\varphi\bar{\otimes}\alpha_{2k+1}\bar{\otimes}\alpha_{2k'+1} \simeq \varphi\bar{\otimes}\eta^5_{2(1,1)} \qquad \operatorname{res}^{L_3^\infty}_{L_1^\infty}\varphi\bar{\otimes}\alpha_{2k+1}\bar{\otimes}\theta \simeq \varphi\bar{\otimes}\eta^5_{1(1,0)}$$
$$\operatorname{res}^{L_3^\infty}_{L_1^\infty}\varphi\bar{\otimes}\alpha_{2k+1}\bar{\otimes}\tau \simeq \varphi\bar{\otimes}\eta^5_{2(1,0)} \qquad \operatorname{res}^{L_3^\infty}_{L_1^\infty}\varphi\bar{\otimes}\theta\bar{\otimes}\theta \simeq \varphi\bar{\otimes}\eta^5_0$$
$$\operatorname{res}^{L_3^\infty}_{L_1^\infty}\varphi\bar{\otimes}\theta\bar{\otimes}\tau \simeq \varphi\bar{\otimes}\eta^5_{2(0,2)} \qquad \operatorname{res}^{L_3^\infty}_{L_1^\infty}\varphi\bar{\otimes}\tau\bar{\otimes}\tau \simeq \varphi\bar{\otimes}\eta^5_{1(0,0)}.$$

and

$$\operatorname{res}^{L_3^\infty}_{L_1^\infty}\varphi\bar{\otimes}\beta_{2k}\bar{\otimes}\omega \simeq \operatorname{res}^{L_3^\infty}_{L_1^\infty}\varphi\bar{\otimes}(\theta\oplus\tau)\bar{\otimes}\omega,$$
$$\operatorname{res}^{L_3^\infty}_{L_1^\infty}\varphi\bar{\otimes}\psi\bar{\otimes}\beta_{2k} \simeq \operatorname{res}^{L_3^\infty}_{L_1^\infty}\varphi\bar{\otimes}\psi\bar{\otimes}(\theta\oplus\tau).$$

Proposition 3.3.27. *Let $\varphi\bar{\otimes}\tau_{(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5)}$ be an irreducible representation of L_1^∞. If $\mu_1\dots\mu_5 = 1$, then*

$$\operatorname{res}^{L_1^\infty}_{L_0}\varphi\bar{\otimes}\tau_{(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5)} \simeq \bigoplus_{(\mu_1',\mu_2')\in I_\varphi}\tau_{(\mu_1',\mu_2',\mu_1,\mu_2,\mu_3,\mu_4,\mu_5)}.$$

Otherwise,

$$\bigoplus_{(\mu'_1,\mu'_2)\in I_\varphi} \tau_{(-\mu'_1,-\mu'_2,\mu_1,\mu_2,\mu_3,\mu_4,\mu_5)}.$$

Proof. The conclusion follows from the fact that the map $U_7 \to \bar{\Gamma}_2^\infty \times U_5$ is given by the formula

$$(e_1, e_2, e_3, e_4, e_5, e_6, e_7) \mapsto (\Delta(e_1, e_2), (e_1e_2e_3, e_1e_2e_4, \ldots, e_1e_2e_7)). \qquad \square$$

Corollary 3.3.28. Let $\eta^7_{i(l)} = \bigoplus \tau_{\mu_1,\ldots,\mu_7}$, where the sum is taken over all sequences such that exactly l of the numbers μ_1, μ_2 are equal -1 and exactly i of the numbers $\mu_1, \ldots, \mu_7$ are equal -1. Σ_5-invariant representations of L_1^∞ restrict to L_0 as follows:

$$\begin{array}{lll} \alpha_k \bar{\otimes} \eta^5_0 \mapsto \eta^7_{1(1)} & \theta \bar{\otimes} \eta^5_0 \mapsto \eta^7_0 & \tau \bar{\otimes} \eta^5_0 \mapsto \eta^7_{2(0)} \\ \alpha_k \bar{\otimes} \eta^5_1 \mapsto \eta^7_{2(1)} & \theta \bar{\otimes} \eta^5_1 \mapsto \eta^7_{3(2)} & \tau \bar{\otimes} \eta^5_1 \mapsto \eta^7_{1(0)} \\ \alpha_k \bar{\otimes} \eta^5_2 \mapsto \eta^7_{3(1)} & \theta \bar{\otimes} \eta^5_2 \mapsto \eta^7_{2(0)} & \tau \bar{\otimes} \eta^5_2 \mapsto \eta^7_{3(0)} \end{array}$$

Moreover, $\mathrm{res}^{L_1^\infty}_{L_0}\ \beta_{2k} \bar{\otimes} \eta^5_i \simeq \mathrm{res}^{L_1^\infty}_{L_0}\ (\theta \oplus \tau) \bar{\otimes} \eta^5_i$.

Representations of groups N_i^∞, K_i^∞, M_i^∞ and $\bar{M}_i^\infty$

Let $J \in \mathcal{T}_{irr}(4) = \{N, K, \bar{M}, M\}$. Fix isomophisms

$$J^\infty \times U_3^* \ni (g, v) \mapsto \Delta(g, v \cdot \det g) \in J_0^\infty,$$

$$J^\infty \times \bar{\Gamma}_2^\infty \ni (g, h) \mapsto \Delta(g, h, \det g \cdot \det h) \in J_1^\infty.$$

Denote the representations corresponding to elements of U_3^* by

$$\tau_0 = \tau_{(1,1,1)}, \qquad \tau_i = \tau_{(1,-1,-1)}, \qquad \tau_j = \tau_{(-1,1,-1)}, \qquad \tau_k = \tau_{(-1,-1,1)}. \tag{3.3.29}$$

Proposition 3.3.30. *(a) There are bijections*

$$\mathrm{IR}_{ev}(J_1^\infty) \simeq \mathrm{IR}(J^\infty) \times \mathrm{IR}(\bar{\Gamma}_2), \quad \mathrm{IR}_{ev}(J_0^\infty) \simeq \mathrm{IR}(J^\infty) \times U_3^*$$

(b) The group $W_G(J_1^\infty)$ acts naturally on $\mathrm{IR}(J^\infty)$ and trivially on $\mathrm{IR}(\bar{\Gamma}_2^\infty)$.

(c) The action of $W_G(J_0^\infty) \simeq W_G(J_1^\infty) \times \Sigma_3$ on $\mathrm{IR}_{ev}(J_0^\infty)$ is product-wise.

(d) The restriction map

$$\mathrm{IR}_{ev}(J_1^\infty) \simeq \mathrm{IR}(J^\infty) \times \mathrm{IR}(\bar{\Gamma}_2) \longrightarrow \mathrm{IR}(J^\infty) \times U_3^* \simeq \mathrm{IR}_{ev}(J_0^\infty)$$

is given by the formula

$$\varrho \bar{\otimes} \omega \mapsto \begin{cases} \varrho \bar{\otimes} \tau_0 & \text{for } \omega \simeq \theta \\ \varrho \bar{\otimes} \tau_i & \text{for } \omega \simeq \tau \\ \varrho \bar{\otimes} (\tau_0 \oplus \tau_i) & \text{for } \omega \simeq \beta_{2k} \\ (\varrho \otimes \det) \bar{\otimes} (\tau_j \oplus \tau_k) & \text{for } \omega \simeq \alpha_{2k+1}. \end{cases}$$

Proof. The action of the Weyl group is clear. The inclusion $J^\infty \times U_3 \simeq J_0^\infty \subseteq J_1^\infty \simeq J^\infty \times \bar{\Gamma}_2^\infty$ is given by the formula

$$(g, (a_1, a_2, a_3)) \mapsto (g, \Delta((-1)^{a_1} \det g, (-1)^{a_2} \det g, (-1)^{a_3})).$$

It implies that the restriction formulae are valid. □

The previous proposition shows that we need to know the action on $\mathrm{IR}(J^\infty)$ given by tensoring by determinant representation. It is trivial in cases $J = M$ and $J = \bar{M}$ since these groups are contained in $SO(4)$. For $J = K$ it is given by

$$\gamma_1 \leftrightarrow \gamma_3, \qquad \gamma_0^{i,j} \leftrightarrow \gamma_4^{i,j}$$

and for $J = N$ by

$$\alpha_{2k+1} \boxtimes \theta \leftrightarrow \alpha_{2k+1} \boxtimes \tau, \qquad \beta_{2k} \boxtimes \theta \leftrightarrow \beta_{2k} \boxtimes \tau, \qquad \tau^+ \leftrightarrow \theta^+, \qquad \tau^- \leftrightarrow \theta^-$$

and it is trivial in other cases.

3.4 Generators-and-relations form of N_1^∞

Denote $G := Spin(7)$. In the next section we classify odd representations of discrete approximations of 2-stubborn subgroups of G. It requires different methods than the even case. The main reason is that stubborn subgroups of *Spin*-groups are not products of simpler groups. Hence we need to present the group N_1^∞ in the generators-and-relations form. Of course, it does not exists any finite presentation of N_1^∞. Instead, for each $n > 1$, we give a presentation of the finite subgroups $N_1^{(n)}$ (cf. 2.1.7). It is sufficient since every element of N_1^∞ can be written as a product of generators of $N_1^{(n)}$ (if n is large enough). We regard G as the double covering of $SO(7)$ and we represent elements of G by paths in $SO(7)$.

Let $u \in G$ be a non-trivial lift of the unity of $SO(7)$ and let e_{ij} be a matrix which has an (i,j)-th entry equal 1 and other entries equal 0.

Definition 3.4.1. Let $E(k,l;s)$ be an element of $Spin(n)$ given by the path

$$[0,1] \ni t \mapsto \sum_{i \neq k,l} e_{ii} + c(st)(e_{kk} + e_{ll}) + s(st)(e_{kl} - e_{lk}),$$

where s and c are defined in 3.3.4.

Here follows a list of relations between elements $E(k,l;s)$:

Proposition 3.4.2. *The following holds:*

(a) $E(k,l;s)E(k',l';s') = E(k',l';s')E(k,l;s)$ *for* k,k',l,l' *pairwise different*

(b) $E(k,l;s) = E(l,k;-s)$

(c) $E(k,l;1) = u$

(d) $E(k,l;\frac{1}{2})E(k,m;s) = E(k,m;-s)E(k,l;\frac{1}{2})$

(e) $E(k,l;\frac{1}{2})E(l,m;\frac{1}{2}) = E(k,m;\frac{1}{2})$

(f) $E(k,m;s)E(k,l;\frac{1}{4}) = E(k,l;\frac{1}{4})E(l,m;s)$

(g) $E(k,l;s)E(m,n;s)E(k,m;s')E(l,n;s') =$
$= E(k,m;s')E(l,n;s')E(k,l;s)E(m,n;s)$

Proof. For any paths ω, ω' in G a symbol $\omega \circ \omega'$ stands for a composition of ω and ω', and $\omega \sim \omega'$ means that these paths are homotopic.

(a) Since all elements of the paths $E(k,l;s)$ and $E(k',l';s')$ commute the paths commute too.

(b) Obvious.

(c) $E(k,l;1)$ is a generator of $\pi_1(SO(7))$.

(d) Let $j := \sum_{i \neq k,l,m} e_{ii}$.

$$\begin{aligned}
&E(k,l;\tfrac{1}{2})E(k,m;s)E(k,l;-\tfrac{1}{2})E(k,m;s) \\
&\qquad\sim (E(k,l;\tfrac{1}{2}) \circ e(k,l;\tfrac{1}{2}))(I \circ E(k,m;s)) \\
&\qquad\quad (E(k,l;-\tfrac{1}{2}) \circ e(k,l;\tfrac{1}{2}))(I \circ E(k,m;s)) \\
&\sim (E(k,l;\tfrac{1}{2}) \cdot I \cdot E(k,l;-\tfrac{1}{2}) \cdot I) \circ (e(k,l;\tfrac{1}{2})E(k,m;s)e(k,l;-\tfrac{1}{2})E(k,m;s)) \\
&\qquad\sim e(k,l;\tfrac{1}{2})E(k,m;s)e(k,l;-\tfrac{1}{2})E(k,m;s) \\
&\sim [(j + e_{mm} - e_{kk} - e_{ll})(j + e_{ll} + c(st)(e_{kk} - e_{mm}) + s(st)(e_{km} + e_{mk}))]^2 \\
&\qquad\sim [j - e_{ll} + c(st)(-e_{kk} + e_{mm}) + s(st)(-e_{kk} - e_{mm})]^2 \\
&\qquad\sim j + e_{ll} + (c(st)^2 + s(st)^2)(e_{kk} + e_{mm}) = I
\end{aligned}$$

e., f., g. Similar arguments.

□

Define

$$T := \{E(1,2;t_1)E(3,4;t_2)E(5,6;t_3) : t_1, t_2, t_3 \in \mathbb{R}\}. \tag{3.4.3}$$

Obviously, T is a maximal torus of G.

Remark. The homomorphism

$$\mathbb{R}^3 \ni (t_1, t_2, t_3) \mapsto E(1,2;t_1)E(3,4;t_2)E(5,6;t_3) \in T \tag{3.4.4}$$

induces isomorphisms

$$\mathbb{R}^3/\langle k_1, k_2, k_3 : k_i \in \mathbb{Z} \wedge k_1 + k_2 + k_3 \in 2\mathbb{Z}\rangle \simeq T \tag{3.4.5}$$

and

$$\mathbb{Z}[\tfrac{1}{2}]/\langle k_1, k_2, k_3 : k_i \in \mathbb{Z} \wedge k_1 + k_2 + k_3 \in 2\mathbb{Z}\rangle \simeq T^\infty \tag{3.4.6}$$

Proposition 3.4.7. *Let $N_1^{(n)}$ denotes the n-th discrete approximation of N_1 (cf. 2.1.8). Then $N_1^{(n)}$ is generated by the following elements*

$a = E(2,7;\tfrac{1}{2})$, $b = E(4,7;\tfrac{1}{2})$, $c = E(6,7;\tfrac{1}{2})$,
$x = E(1,2;2^{-n})$, $y = E(3,4;2^{-n})$, $z = E(5,6;2^{-n})$,
$s = E(3,4;\tfrac{1}{2})E(1,3;\tfrac{1}{4})E(2,4;\tfrac{1}{4})$

and the relations

$$a^2 = b^2 = c^2 = s^2 = x^{2^n} = y^{2^n} = z^{2^n} = u, \qquad u^2 = e,$$

$$ba = abu, \qquad cb = bcu, \qquad ca = acu,$$

$$sa = bs, \qquad sb = as, \qquad sc = cs,$$

$$xa = ax^{-1}, \qquad yb = by^{-1}, \qquad zc = cz^{-1},$$

$$xs = sy, \qquad ys = sx, \qquad zs = sz,$$

$$xb = bx, \qquad xc = cx, \qquad ya = ay, \qquad yc = cy, \qquad za = az, \qquad zb = bz,$$

$$yx = xy, \qquad zx = xz, \qquad zy = yz.$$

Proof. It follows from 3.4.2. □

Elementary calculations lead to the following:

Proposition 3.4.8. *Here follows the list of generators of n-th discrete approximations of 2-stubborn subgroups of $G = Spin(7)$:*

- $N_1^{(n)} = \langle a, b, c, s, x, y, z \rangle$,
- $N_0^{(n)} = \langle a, b, c, s, x, y, z^{2^{n-1}} \rangle$,
- $L_3^{(n)} = \langle a, b, c, x, y, z \rangle$,
- $L_1^{(n)} = \langle a, b, c, x, y^{2^{n-1}}, z^{2^{n-1}} \rangle$,
- $L_0^{(n)} = \langle a, b, c, x^{2^{n-1}}, y^{2^{n-1}}, z^{2^{n-1}} \rangle$,
- $K_1^{(n)} = \langle a, b, c, s, x^{2^{n-1}}, y^{2^{n-1}}, x^{2^{n-2}} y^{2^{n-2}}, z \rangle$,
- $K_0^{(n)} = \langle a, b, c, s, x^{2^{n-1}}, y^{2^{n-1}}, x^{2^{n-2}} y^{2^{n-2}}, z^{2^{n-1}} \rangle$,
- $\bar{M}_1^{(n)} = \langle ab, c, s, x^{2^{n-1}}, y^{2^{n-1}}, xy, z \rangle$,
- $\bar{M}_0^{(n)} = \langle ab, c, s, x^{2^{n-1}}, y^{2^{n-1}}, xy, z^{2^{n-1}} \rangle$,
- $M_1^{(n)} = \langle ab, c, s, x^{2^{n-1}}, y^{2^{n-1}}, x^{2^{n-2}} y^{2^{n-2}}, z \rangle$,
- $M_0^{(n)} = \langle ab, c, s, x^{2^{n-1}}, y^{2^{n-1}}, x^{2^{n-2}} y^{2^{n-2}}, z^{2^{n-1}} \rangle$. □

We conclude with presentations of groups $W_G(P)$ for some $P \in \mathcal{R}_2(G)$:

Proposition 3.4.9. *Fix $n \geq 4$. Let*

$$k := E(3,7;\tfrac{1}{2})E(2,3;\tfrac{1}{4}), \qquad l := E(6,7;\tfrac{1}{4}),$$
$$h_1 := x^{2^{n-3}}y^{2^{n-3}}, \qquad h_1 := x^{2^{n-3}}y^{-2^{n-3}}.$$

Then

$$W_G(N_1) = 1, \qquad W_G(K_1) = \langle k, x^{2^{n-2}}\rangle \simeq \Sigma_3,$$
$$W_G(\bar{M}_1) = \langle h_2, k^{-1}h_1k\rangle \simeq \Sigma_3, \qquad W_G(M_1) = \langle k, h_1, h_2\rangle \simeq \Sigma_3 \wr \Sigma_2,$$

and for every $J \in \mathcal{T}_{irr}(4) = \{N, K, M, \bar{M}\}$ we have

$$W_G(J_0) \cong W_G(J_1) \times \langle l, z^{2^{n-1}}\rangle \simeq W_G(J_1) \times \Sigma_3.$$

Proof. It follows from elementary calculations and from 3.2.8. □

3.5 Odd representations of 2-stubborn subgroups of $Spin(7)$

In this section we classify odd irreducible representations of discrete approximations of 2-stubborn subgroups of $G := Spin(7)$. In the construction of a faithful representation of $DI(4)$ we use only parts which are devoted to M_0 and L_0. The classification of odd representations of other groups is presented here for completeness only, however, it could be useful for finding other examples of representations of $DI(4)$.

Definition 3.5.1. Let P be any stubborn subgroup of G from the list in 3.2.8. A *root* (resp. *an even root*, *an odd root*) of P is any (resp. even, odd) irreducible representation of $T^\infty \cap P$. Denote by $Rt(P)$ ($Rt_{ev}(P)$, $Rt_{od}(P)$) the set of all roots of P (resp. the set of even roots, the set of odd roots).

By 2.3.28 any irreducible representation of $(\mathbb{Z}/2^\infty)^3 \simeq \mathbb{Z}[\frac{1}{2}]/\mathbb{Z}^3$ has the form ϱ_{k_1,k_2,k_3}, for $k_1, k_2, k_3 \in \mathbb{Z}_2^\wedge$. From the isomorphism 3.4.6 follows that any irreducible representation of T^∞ has the form ϱ_{k_1,k_1,k_3}, where

$$k_1, k_2, k_3 \equiv 0 \mod \tfrac{1}{2}, \quad \text{and} \quad k_1 \equiv k_2 \equiv k_3 \mod 1. \qquad (3.5.2)$$

Then $\mathrm{IR}(T^\infty) \simeq (k_1, k_2, k_3 \in \frac{1}{2}\mathbb{Z}_2^\wedge : k_1 \equiv k_2 \equiv k_3 \mod 1)$. It easily follows that $\mathrm{IR}_{od}(T^\infty) \simeq (\frac{1}{2} + \mathbb{Z}_2^\wedge)^3$. For each P the set of roots $Rt(P)$ is a quotient $\mathrm{IR}(T^\infty)/Lt(P)$, where

$$Lt(P) = \{\omega \in \mathrm{IR}(T^\infty) : \mathrm{res}^{T^\infty}_{T^\infty \cap P}\, \omega \text{ is trivial}\}. \tag{3.5.3}$$

By 2.3.31 every irreducible representation of P^∞ is a subrepresentation of $\mathrm{ind}^{P^\infty}_{T^\infty \cap P}\, \omega$ for $\omega \in Rt(P)$. Moreover, this representation does not depend on a choice of ω within its $P^\infty/(T^\infty \cap P)$-orbit. Our goal will be to decompose representations of this kind into irreducible summands. To do this we will use Propositions 2.3.32 and 2.3.33 to a pair of groups $T^\infty \cap P \subseteq P^\infty$ whenever possible. First, we need to calculate an action of $P^\infty/(T^\infty \cap P)$ on $Rt(P)$ by conjugation.

Proposition 3.5.4. *We have*

$$\begin{aligned}
Lt(N_1) &= Lt(L_3) = 0, & Lt(N_0) &= \langle (0,0,2) \rangle, \\
Lt(L_1) &= \langle (0,2,0), (0,0,2) \rangle, & Lt(L_0) &= \langle (2,0,0), (0,2,0), (0,0,2) \rangle, \\
Lt(K_1) &= \langle (4,0,0), (2,-2,0) \rangle, & Lt(K_0) &= \langle (4,0,0), (2,-2,0), (0,0,2) \rangle, \\
Lt(\bar{M}_1) &= \langle (2,-2,0) \rangle, & Lt(\bar{M}_0) &= \langle (2,-2,0), (0,0,2) \rangle, \\
Lt(M_1) &= \langle (4,0,0), (2,-2,0) \rangle, & Lt(M_0) &= \langle (4,0,0), (2,-2,0), (0,0,2) \rangle.
\end{aligned}$$

Moreover,

$$P^\infty/(T^\infty \cap P) = \begin{cases} \langle a, b, c, s \rangle & \text{for } P = N_1, N_0, K_1, K_0 \\ \langle ab, c, s \rangle & \text{for } P = \bar{M}_1, \bar{M}_0, M_1, M_0 \\ \langle a, b, c \rangle & \text{for } P = L_3, L_1, L_0, \end{cases}$$

and for $(t_1, t_2, t_3) \in T^\infty$ (here we use the isomorphism 3.4.6) we have

$$\begin{aligned}
a^*(t_1, t_2, t_3) &= (-t_1, t_2, t_3) \\
b^*(t_1, t_2, t_3) &= (t_1, -t_2, t_3) \\
c^*(t_1, t_2, t_3) &= (t_1, t_2, -t_3) \\
s^*(t_1, t_2, t_3) &= (t_2, t_1, t_3).
\end{aligned}$$

Proof. It follows from a strightforward calculation using 3.4.7 and 3.4.8. □

Odd representations of M_0

Since $Rt_{od}(M_0) = \mathrm{IR}_{od}(T^\infty)/\langle(4,0,0),(2,-2,0),(0,0,2)\rangle$ there are exactly 16 odd roots of M_0 and each has a unique representation as (α,β,γ) where $\alpha \in \{-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2}\}$, $\beta,\gamma \in \{-\frac{1}{2},\frac{1}{2}\}$. There are four orbits of the action of $M_0/T^\infty \cap M_0$ on $Rt_{od}(M_0)$, namely

$$O_1^{M_0} = \{(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}),(-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}),(-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})\}$$
$$O_2^{M_0} = \{(\tfrac{3}{2},-\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{3}{2},-\tfrac{1}{2},-\tfrac{1}{2}),(-\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}),(-\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2})\}$$
$$O_3^{M_0} = \{(\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2}),(-\tfrac{3}{2},-\tfrac{1}{2},\tfrac{1}{2}),(-\tfrac{3}{2},-\tfrac{1}{2},-\tfrac{1}{2})\}$$
$$O_4^{M_0} = \{(\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}),(-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}),(-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})\}$$

The stabilizer of the action of $M_0/(T^\infty \cap M_0)$ on the set of roots has 2 elements for each root ω. Therefore each representation $\mathrm{ind}_{T^\infty \cap M_0}^{M_0} \omega$ decomposes into two irreducible and non-isomorphic representations of dimension 4 (cf. 2.3.33). Now we construct these factors. Define the following subgroups of M_0:

$$H_1^+ = \langle x^{2^{n-2}} y^{3\cdot 2^{n-2}} u, s x^{2^{n-1}} \rangle, \qquad H_1^- = \langle x^{2^{n-2}} y^{3\cdot 2^{n-2}} u, s x^{2^{n-1}} u \rangle,$$
$$H_2^+ = \langle x^{2^{n-2}} y^{3\cdot 2^{n-2}}, s x^{2^{n-1}} \rangle, \qquad H_2^- = \langle x^{2^{n-2}} y^{3\cdot 2^{n-2}}, s x^{2^{n-1}} u \rangle,$$
$$H_3^+ = \langle x^{2^{n-2}} y^{2^{n-2}} u, absx^{2^{n-1}} \rangle, \qquad H_3^- = \langle x^{2^{n-2}} y^{2^{n-2}} u, absx^{2^{n-1}} u \rangle,$$
$$H_4^+ = \langle x^{2^{n-2}} y^{2^{n-2}}, absx^{2^{n-1}} \rangle, \qquad H_4^- = \langle x^{2^{n-2}} y^{2^{n-2}}, absx^{2^{n-1}} u \rangle$$

Note that these groups are normal in M_0. Let $\mathcal{H} := \{H_i^\varepsilon\}_{i\in\{1,2,3,4\}}^{\varepsilon\in\{+,-\}}$.

For $i \in \{1,2,3,4\}$ and $\varepsilon \in \{+,-\}$ define the representations

$$\vartheta_i^\varepsilon := \mathrm{ind}_{\langle H_i^\varepsilon, u\rangle}^{M_0} \mathrm{res}_{\langle H_i^\varepsilon, u\rangle}^{\langle H_i^\varepsilon, u\rangle/H_i^\varepsilon} \tau,$$

where τ is the non-trivial irreducible representation of $\langle H_i^\varepsilon, u\rangle/H_i^\varepsilon \simeq C_2$.

Proposition 3.5.5. *For each $\omega \in O_i^{M_0}$ there is the following decomposition:*

$$(\mathrm{ind}_{T^\infty \cap M_0}^{M_0} \omega)^{\oplus 4} \simeq \vartheta_i^+ \oplus \vartheta_i^-.$$

Proof. Since both H_i^ε, $\langle H_i^\varepsilon, u\rangle$ are normal subgroups of M_0, we have

$$\chi_{\vartheta_i^\varepsilon}(g) = \begin{cases} 16 & \text{for } g \in H_i^\varepsilon \\ -16 & \text{for } g \in uH_i^\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

Moreover, $H_i^+ \cap T^\infty \cap M_0 = H_i^- \cap T^\infty \cap M_0$ and

$$H_i^+ \cap (M_0 \setminus (T^\infty \cap M_0)) = uH_i^- \cap (M_0 \setminus (T^\infty \cap M_0)).$$

Then

$$\chi_{\vartheta_i^+}(g) + \chi_{\vartheta_i^-}(g) = \begin{cases} 32 & \text{for } g \in H_i^\varepsilon \cap T^\infty \cap M_0 \\ -32 & \text{for } g \in uH_i^\varepsilon \cap T^\infty \cap M_0 \\ 0 & \text{otherwise} \end{cases}$$

and thus it is equal to four times character of $\mathrm{ind}_{T^\infty \cap M_0}^{M_0}\, \omega$. The equity of characters implies the isomorphism of the corresponding representations. □

Proposition 3.5.6. *For each $i \in \{1,2,3,4\}$ and $\varepsilon \in \{+,-\}$*

$$\vartheta_i^\varepsilon \simeq (\lambda_i^\varepsilon)^{\oplus 4}$$

where λ_i^ε is an irreducible four-dimensional representation. Moreover, for $\omega \in O_i^{M_0}$

$$\mathrm{ind}_{T^\infty \cap M_0}^{M_0}\, \omega \simeq \lambda_i^+ \oplus \lambda_i^-.$$

Proof. Existence of λ_i^ε's follows from 3.5.5 and their irreducibility follows by checking its characters. □

Corollary 3.5.7.

$$\mathrm{IR}_{od}(M_0) = \{\lambda_1^+, \lambda_1^-, \lambda_2^+, \lambda_2^-, \lambda_3^+, \lambda_3^-, \lambda_4^+, \lambda_4^-\}$$

Proof. By 2.3.31 all irreducible representations of M_0 are subrepresentations of $\mathrm{ind}_{T^\infty \cap M_0}^{M_0}\, \omega$ where $\omega \in Rt(M_0)$. □

Now we calculate the action of $W_G(M_0)$ on $\mathrm{IR}_{od}(M_0)$. Recall (3.4.9) that

$$W_G(M_0) = \langle h_1, h_2, k, l, z^{2^{n-2}}\rangle \simeq (\Sigma_3 \wr \Sigma_2) \times \Sigma_3$$

Proposition 3.5.8. *The conjugations by the elements k, h_1, h_2 and a induce the following action on $\mathcal{H}$ (and hence on* $\mathrm{IR}_{od}(M_0)$*):*

$$k : \begin{array}{c} H_1^+ \leftrightarrow H_3^- \\ H_1^- \leftrightarrow H_4^- \\ H_2^+ \leftrightarrow H_3^+ \\ H_2^- \leftrightarrow H_4^+ \end{array} \qquad h_1 : H_3^+ \leftrightarrow H_3^- \qquad h_2 : H_2^+ \leftrightarrow H_2^- \qquad a : \begin{array}{c} H_1^+ \leftrightarrow H_4^+ \\ H_1^- \leftrightarrow H_4^- \\ H_2^+ \leftrightarrow H_3^+ \\ H_2^- \leftrightarrow H_3^- \end{array}$$

The conjugations by elements l and $z^{2^{n-2}}$ act trivially on $\mathcal{H} \simeq \mathrm{IR}_{od}(M_0)$.

Proof. It is straightforward from the computations below:

$$k^{-1}(sx^{2^{n-1}})k = abx^{2^{n-2}}y^{2^{n-2}}uabx^{2^{n-1}}y^{2^{n-1}}u = x^{2^{n-2}}y^{2^{n-2}}u$$

$$k^{-1}(absx^{2^{n-1}})k = k^{-1}abkx^{2^{n-2}}y^{2^{n-2}}u = y^{2^{n-1}}ux^{2^{n-2}}y^{2^{n-2}}u = x^{2^{n-2}}y^{3\cdot 2^{n-2}}.$$

$$a^{-1}(sx^{2^{n-1}})a = a^{-1}bsx^{2^{n-1}}u = absx^{2^{n-1}}$$

$$a^{-1}(x^{2^{n-2}}y^{2^{n-2}})a = a^{-1}ax^{3\cdot 2^{n-2}}y^{2^{n-2}}u = (x^{2^{n-2}}y^{3\cdot 2^{n-2}}u)^3.$$ □

Corollary 3.5.9. There exist two orbits of the action of $W_G(M_0)$ on the set $\mathrm{IR}_{od}(M_0)$, namely $\{\lambda_1^-, \lambda_4^-\}$ and $\{\lambda_1^+, \lambda_2^+, \lambda_2^-, \lambda_3^+, \lambda_3^-, \lambda_4^+\}$. In paticular, every $W_G(M_0)$-invariant odd representation of M_0 is isomorphic to

$$(\lambda_1^- \oplus \lambda_4^-)^{\oplus r_1} \oplus (\lambda_1^+ \oplus \lambda_2^+ \oplus \lambda_2^- \oplus \lambda_3^+ \oplus \lambda_3^- \oplus \lambda_4^+)^{\oplus r_2}.$$

for non-negative integers r_1 and r_2.

Odd representations of K_0

The restriction $Rt_{od}(K_0) \to Rt_{od}(M_0)$ is an isomorphism (since $T^\infty \cap M_0 = T^\infty \cap K_0$) but the group $K_0/(T^\infty \cap K_0)$ acting on $Rt_{od}(K_0)$ is larger than $M_0/(T^\infty \cap M_0)$. The roots lie in two $K_0/(T^\infty \cap K_0)$-orbits:

$$O_1^{K_0} := O_1^{M_0} \cup O_4^{M_0} \qquad \text{and} \qquad O_2^{K_0} := O_2^{M_0} \cup O_3^{M_0}.$$

Proposition 3.5.10. *We have* $\mathrm{IR}_{od}(K_0) = \{\mu_1^+, \mu_1^-, \mu_2^+, \mu_2^-\}$, *where*

$$\mu_1^\varepsilon \cong \mathrm{ind}_{M_0}^{K_0}\, \lambda_1^\varepsilon \cong \mathrm{ind}_{M_0}^{K_0}\, \lambda_4^\varepsilon \qquad \textit{and} \qquad \mu_2^\varepsilon \cong \mathrm{ind}_{M_0}^{K_0}\, \lambda_2^\varepsilon \cong \mathrm{ind}_{M_0}^{K_0}\, \lambda_3^\varepsilon$$

Proof. Since the action of $K_0/M_0 \simeq C_2$ on $\mathrm{IR}_{od}(M_0)$ is free (cf. 3.5.8, the action of a), then $\mathrm{ind}_{M_0}^{K_0}\, \lambda_i^\varepsilon$ is irreducible for each i and each ε. □

Corollary 3.5.11. The restrictions of irreducible representations of K_0 to M_0 are as follows:

$$\operatorname{res}^{K_0}_{M_0} \mu_1^+ \simeq \lambda_1^+ \oplus \lambda_4^+ \qquad \operatorname{res}^{K_0}_{M_0} \mu_1^- \simeq \lambda_1^- \oplus \lambda_4^-$$

$$\operatorname{res}^{K_0}_{M_0} \mu_2^+ \simeq \lambda_2^+ \oplus \lambda_3^+ \qquad \operatorname{res}^{K_0}_{M_0} \mu_2^- \simeq \lambda_2^- \oplus \lambda_3^-$$

Recall that $W_G(K_0) = \langle k, x^{2^{n-2}} \rangle \simeq \Sigma_3$. Since $x^{2^{n-2}} = h_1 h_2$ then $x^{2^{n-2}}$ acts on $\mathcal{H}$ by swapping H_3^+ with H_3^- and H_2^+ with H_2^-. This implies (cf. 3.5.8)

Proposition 3.5.12. *The generators of $W_G(K_0)$ act on $\mathrm{IR}_{od}(K_0)$ as follows:*

$$k : \mu_1^+ \leftrightarrow \mu_2^- \qquad x^{2^{n-2}} : \mu_2^+ \leftrightarrow \mu_2^- . \quad \square$$

Corollary 3.5.13. There are two orbits of the action of $W_G(K_0)$ on the set $\mathrm{IR}_{od}(K_0)$, namely $\{\mu_1^-\}$ and $\{\mu_1^+, \mu_2^+, \mu_2^-\}$. Each odd $W_G(K_0)$-equivariant representation ϱ of K_0 is isomorphic to

$$(\mu_1^-)^{\oplus r_1} \oplus (\mu_1^+ \oplus \mu_2^+ \oplus \mu_2^-)^{\oplus r_2}$$

(for some integers r_1, r_2) and $\varrho \in \mathrm{Rep}_{od}(K_0)$ is $W_G(K_0)$-equivariant if and only if $\operatorname{res}^{K_0}_{M_0} \varrho$ is $W_G(M_0)$-equivariant.

Odd representations of $\bar{M}_0^\infty$

Recall that $Rt_{od}(\bar{M}_0) = \mathrm{IR}_{od}(T^\infty)/\langle (2,-2,0), (0,0,2) \rangle$ and $\bar{M}_0^\infty/(T^\infty \cap \bar{M}_0) = \langle ab, c, s \rangle$. Note that each odd root of $\bar{M}_0$ has a unique presentation as (α, β, γ) where $\beta, \gamma \in \{-\frac{1}{2}, \frac{1}{2}\}$. Moreover, each orbit of an $\bar{M}_0^\infty/(T^\infty \cap \bar{M}_0)$-action on $Rt_{od}(\bar{M}_0)$ has a representative (possibly non-unique) having a presentation of the form $(\alpha, \frac{1}{2}, \frac{1}{2})$.

Let $\omega_\alpha := (\alpha, \frac{1}{2}, \frac{1}{2})$ and let $O_\alpha^{\bar{M}_0}$ be an orbit containing ω_α.

Proposition 3.5.14. *The orbit $O_\alpha^{\bar{M}_0}$ is free if and only if $\alpha \equiv -\frac{1}{2} \mod 2$ and $\alpha \neq -\frac{1}{2}$. Otherwise its stabilizer has two elements.*

Proof. The subgroup $\langle ab, c \rangle$ acts freely on odd roots of $\bar{M}_0$. To check whether the action of $\bar{M}_0^\infty/(T^\infty \cap \bar{M}_0)$ is free we need to find if $s\omega_\alpha \in \langle ab, c \rangle \omega_\alpha$. If $\alpha \equiv \frac{1}{2} \mod 2$, then

$$s\omega_\alpha = s(\alpha, \tfrac{1}{2}, \tfrac{1}{2}) = (\tfrac{1}{2}, \alpha, \tfrac{1}{2}) = (\tfrac{1}{2}, \alpha, \tfrac{1}{2}) + \tfrac{1}{2}(\alpha - \tfrac{1}{2})(2, -2, 0) = (\alpha, \tfrac{1}{2}, \tfrac{1}{2}) = \omega_\alpha$$

and if $\alpha \equiv -\frac{1}{2} \mod 2$ then

$$s\omega_\alpha = s(\alpha, \tfrac{1}{2}, \tfrac{1}{2}) = (\tfrac{1}{2}, \alpha, \tfrac{1}{2}) \overset{ab}{\sim} (-\tfrac{1}{2}, -\alpha, \tfrac{1}{2}) = \\ (-\tfrac{1}{2}, -\alpha, \tfrac{1}{2}) - \tfrac{1}{2}(\alpha + \tfrac{1}{2})(2, -2, 0) = (-\alpha - 1, \tfrac{1}{2}, \tfrac{1}{2}) = \omega_{-\alpha-1}.$$

Hence the orbit $O_\alpha^{\bar{M}_0}$ is free only for $\alpha \equiv -\frac{1}{2}$ and $\alpha \neq -\alpha - 1$ (i.e. $\alpha \neq -\frac{1}{2}$). □

Let $\nu_\alpha := \operatorname{ind}_{T_{\bar{M}_0}}^{\bar{M}_0} \omega_\alpha$. ν_α is an 8-dimensional representation and each $\varrho \in \mathrm{IR}_{od}(\bar{M}_0)$ is a subrepresentation of ν_α for some α. The following proposition lists some properties of the representations ν_α:

Proposition 3.5.15. *The following holds:*

(a) If $\alpha \equiv \frac{1}{2} \mod 4$, then there exist $\nu_\alpha^+, \nu_\alpha^- \in \mathrm{IR}_{od}(\bar{M}_0^\infty)$ such that

$$\nu_\alpha \simeq \nu_\alpha^+ \oplus \nu_\alpha^-, \qquad \operatorname{res}_{M_0}^{\bar{M}_0} \nu_\alpha^+ \simeq \lambda_1^+, \qquad \operatorname{res}_{M_0}^{\bar{M}_0} \nu_\alpha^- \simeq \lambda_1^-,$$

and both ν_α^+ and ν_α^- are irreducible.

(b) If $\alpha \equiv -\frac{3}{2} \mod 4$, then there exist $\nu_\alpha^+, \nu_\alpha^- \in \mathrm{IR}_{od}(\bar{M}_0^\infty)$ such that

$$\nu_\alpha \simeq \nu_\alpha^+ \oplus \nu_\alpha^-, \qquad \operatorname{res}_{M_0}^{\bar{M}_0} \nu_\alpha^+ \simeq \lambda_2^+, \qquad \operatorname{res}_{M_0}^{\bar{M}_0} \nu_\alpha^- \simeq \lambda_2^-,$$

and both ν_α^+ and ν_α^- are irreducible.

(c) If $\alpha \equiv \frac{3}{2} \mod 2$ then

$$\operatorname{res}_{M_0}^{\bar{M}_0} \nu_\alpha \simeq \lambda_3^+ \oplus \lambda_3^-,$$

and ν_α is irreducible.

(d) If $\alpha \equiv -\frac{1}{2} \mod 2$ and $\alpha \neq -\frac{1}{2}$ then

$$\operatorname{res}_{M_0}^{\bar{M}_0} \nu_\alpha \simeq \lambda_4^+ \oplus \lambda_4^-,$$

and ν_α is irreducible.

(e) If $\alpha = -\frac{1}{2}$, then there exist $\nu_\alpha^+, \nu_\alpha^- \in \mathrm{IR}_{od}(\bar{M}_0^\infty)$ such that

$$\nu_\alpha \simeq \nu_\alpha^+ \oplus \nu_\alpha^-, \qquad \operatorname{res}_{M_0}^{\bar{M}_0} \nu_\alpha^+ \simeq \lambda_4^+, \qquad \operatorname{res}_{M_0}^{\bar{M}_0} \nu_\alpha^- \simeq \lambda_4^-,$$

and both ν_α^+ and ν_α^- are irreducible.

Proof. In cases c) and d) ν_α is irreducible and in cases a), b) and e) it decomposes for a sum of two irreducible and non-isomorphic 4-dimensional representations. Now note that

$$\operatorname{res}^{\bar{M}_0^\infty}_{M_0^\infty} \nu_\alpha = \operatorname{res}^{\bar{M}_0^\infty}_{M_0^\infty} \operatorname{ind}^{\bar{M}_0^\infty}_{T^\infty\cap\bar{M}_0} \omega_\alpha = \operatorname{ind}^{M_0^\infty}_{T^\infty\cap M_0} \operatorname{res}^{T^\infty\cap\bar{M}_0}_{T^\infty\cap M_0} \omega_\alpha = \lambda_i^+ \oplus \lambda_i^-$$

where $i = 1$ (2, 3, 4) if $\alpha \equiv \frac{1}{2}$ (resp. $-\frac{3}{2}$, $\frac{3}{2}$, $-\frac{1}{2}$). The statements in cases c) and d) are clear; in a), b) and e) ν_α^+ the factor which restricts to λ_i^+ and ν_α^- is the factor which restricts to λ_i^-. □ □

Proposition 3.5.16. *Each odd irreducible representation of $\bar{M}_0^\infty$ is determined by its restrictions to M_0 and to $\langle xy\rangle$.*

Proof. Let ϱ_k, $k \in \mathbb{Z}_2^\wedge$ be the irreducible representation of C_{2^∞} which corresponds to k. Then

$$\operatorname{res}^{\bar{M}_0^\infty}_{\langle xy\rangle} \nu_\alpha \simeq \varrho^{\oplus 4}_{\alpha+\frac{1}{2}} \oplus \varrho^{\oplus 4}_{-\alpha-\frac{1}{2}}$$

$$\operatorname{res}^{\bar{M}_0^\infty}_{\langle xy\rangle} \nu_\alpha^+ \simeq \operatorname{res}^{\bar{M}_0^\infty}_{\langle xy\rangle} \nu_\alpha^- \simeq \varrho^{\oplus 2}_{\alpha+\frac{1}{2}} \oplus \varrho^{\oplus 2}_{-\alpha-\frac{1}{2}}$$

Hence the restrictions of irreducible odd representations of $\bar{M}_0^\infty$ to $\langle xy\rangle$ are isomorphic only in the following cases:

$$\nu_\alpha \simeq \nu_{-\alpha-1} \qquad \text{and} \qquad \nu_\alpha^+ \simeq \nu_\alpha^- \simeq \nu_{-\alpha-1}^+ \simeq \nu_{-\alpha-1}^-$$

If $\alpha \equiv \frac{1}{2} \mod 2$ then restriction of these representations to M_0 are not isomorphic since $\alpha \not\equiv -\alpha-1 \mod 4$. If $\alpha \equiv -\frac{1}{2} \mod 2$ then ν_α (ν_α^+, ν_α^-) is isomorphic to $\nu_{-\alpha-1}$ (resp. $\nu_{-\alpha-1}^+$, $\nu_{-\alpha-1}^-$). □

Proposition 3.5.17. *The generators $z^{2^{n-2}}$ and l act on $\mathrm{IR}_{od}(\bar{M}_0)$ trivially and h_2, $k^{-1}h_1k$ act in the following way:*

- *$h_2(\nu_\alpha^+) = \nu_{-\alpha-1}^+$ for $\alpha \equiv \frac{1}{2} \mod 2$ and $h_2(\varrho) = \varrho$ for other irreducible representations $\varrho \in \mathrm{IR}_{od}(\bar{M}_0)$,*
- *if $\alpha \equiv -\frac{3}{2} \mod 4$, then $k^{-1}h_1k(\nu_\alpha^+) = \nu_\alpha^-$ and $k^{-1}h_1k(\nu_\alpha^-) = \nu_\alpha^+$. For other representations $\varrho \in \mathrm{IR}_{od}(\bar{M}_0)$ we have $k^{-1}h_1k(\varrho) = \varrho$.*

Proof. Both h_2 and $k^{-1}h_1k$ act trivially on $\langle xy\rangle$ and hence on $\mathrm{IR}(\langle xy\rangle)$. The action on $\mathrm{IR}_{od}(M_0)$ is given by

$$h_2 : \lambda_2^+ \leftrightarrow \lambda_2^- \qquad k^{-1}h_1k : \lambda_1^+ \leftrightarrow \lambda_2^+$$

and $z^{2^{n-2}}$, l act trivially on both $\mathrm{IR}_{od}(\langle xy\rangle)$ and $\mathrm{IR}_{od}(M_0)$. Now the application of the previous lemma gives the conclusion. □

Now let us give some corollaries:

Corollary 3.5.18.

$$\mathrm{IR}_{od}(\bar{M}_0) = \{\nu^+_{2a+\frac{1}{2}}, \nu^-_{2a+\frac{1}{2}}\}_{a\in\mathbb{Z}_2^\wedge} \cup \{\nu_{2a-\frac{1}{2}}\}_{a\in\mathbb{Z}_2^\wedge\setminus\{0\}} \cup \{\nu^+_{-\frac{1}{2}}, \nu^-_{-\frac{1}{2}}\}$$

Corollary 3.5.19. Let ϱ be an odd $W_G(\bar{M}_0)$-invariant representation of $\bar{M}_0^\infty$. Then ϱ is a direct sum of the following summands:

- $\nu_{2k-\frac{1}{2}}$, $k \in \mathbb{Z}_2^\wedge\setminus\{0\}$,
- $\nu^+_{-\frac{1}{2}}$,
- $\nu^-_{-\frac{1}{2}}$,
- $\nu^-_{4k+\frac{1}{2}}$, $k \in \mathbb{Z}_2^\wedge$,
- $\nu^+_{4k+\frac{1}{2}} \oplus \nu^+_{-4k-\frac{3}{2}} \oplus \nu^-_{-4k-\frac{3}{2}}$, $\mathbb{Z}_2^\wedge \in A$.

Corollary 3.5.20. Each odd $W_G(\bar{M}_0)$-invariant representation of $\bar{M}_0^\infty$ such that its restriction to M_0 is $W_G(M_0)$-invariant is isomorphic to a direct sum of the following summands:

- $\nu^-_{-\frac{1}{2}} \oplus \nu^-_{4k+\frac{1}{2}}$
- $\nu^+_{-\frac{1}{2}} \oplus \nu^+_{4k+\frac{1}{2}} \oplus \nu^+_{-4k-\frac{3}{2}} \oplus \nu^-_{-4k-\frac{3}{2}} \oplus \nu_{4l+\frac{3}{2}}$
- $\nu^+_{4k+\frac{1}{2}} \oplus \nu^+_{-4k-\frac{3}{2}} \oplus \nu^-_{-4k-\frac{3}{2}} \oplus \nu_{4l+\frac{3}{2}} \oplus \nu_{4m-\frac{1}{2}} \oplus \nu^-_{4n+\frac{1}{2}}$

Odd representations of N_0^∞

Since $Rt_{od}(N_0) \simeq \mathrm{IR}_{od}(T^\infty)/\langle(0,0,2)\rangle$ and $N_0^\infty/(T^\infty \cap N_0) \simeq \langle a,b,c,s\rangle$, then each root $\omega \in Rt_{od}(N_0)$ can be presented uniquely in the form $(\alpha, \beta, \pm\frac{1}{2})$ and each $N_0^\infty/(T^\infty \cap N_0)$-orbit has a representant having the form $(\alpha, \beta, \frac{1}{2})$ where $\alpha, \beta \equiv \frac{1}{2} \mod 2$ (this presentation is not unique unless $\alpha = \beta$).

Proposition 3.5.21.

$$\mathrm{IR}_{od}(N_0^\infty) = \{\xi_{\{2a+\frac{1}{2},2b+\frac{1}{2}\}}\}_{a\neq b\in\mathbb{Z}_2^\wedge} \cup \{\xi^+_{2a+\frac{1}{2}}, \xi^-_{2a+\frac{1}{2}}\}_{a\in\mathbb{Z}_2^\wedge}$$

Proof. Let $\xi_\omega := \mathrm{ind}_{T^\infty\cap N_0}^{N_0^\infty}\,\omega$. Every $\varrho \in \mathrm{IR}_{od}(N_0^\infty)$ is a subrepresentation of ξ_ω for some ω of the form $(\alpha, \beta, \frac{1}{2})$, $\alpha, \beta \equiv \frac{1}{2} \mod 2$. If $\alpha \neq \beta$ then ω lies in a free W_{N_0}-orbit hence ξ_ω is irreducible (will denote it by $\xi_{\{\alpha,\beta\}}$; note that $\xi_{\{\alpha,\beta\}} \simeq \xi_{\{\beta,\alpha\}}$). If $\alpha = \beta$ then $s \in N_0^\infty/(T^\infty \cap N_0)$ stabilizes ω and hence ξ_ω decomposes into the sum of two 8-dimensional representations ξ^+_α and ξ^-_α. □

Proposition 3.5.22. *The following table lists the restrictions of irreducible odd representations of N_0 to K_0 and $\bar{M}_0$ (for $\alpha \equiv \beta \equiv \frac{1}{2} \mod 2$):*

ϱ	$\mathrm{res}_{K_0}^{N_0^\infty}\,\varrho$	$\mathrm{res}_{\bar{M}_0^\infty}^{N_0^\infty}\,\varrho$
$\xi_{\{\alpha,\beta\}}$, $\alpha \equiv \beta \mod 4$	$\mu_1^+ \oplus \mu_1^-$	$\nu^+_{\alpha+\beta-\frac{1}{2}} \oplus \nu^-_{\alpha+\beta-\frac{1}{2}} \oplus \nu_{-\alpha+\beta-\frac{1}{2}}$
$\xi_{\{\alpha,\beta\}}$, $\alpha \not\equiv \beta \mod 4$	$\mu_2^+ \oplus \mu_2^-$	$\nu^+_{\alpha+\beta-\frac{1}{2}} \oplus \nu^-_{\alpha+\beta-\frac{1}{2}} \oplus \nu_{-\alpha+\beta-\frac{1}{2}}$
ξ^+_α	μ_1^+	$\nu^+_{2\alpha-\frac{1}{2}} \oplus \nu^+_{-\frac{1}{2}}$
ξ^-_α	μ_1^-	$\nu^-_{2\alpha-\frac{1}{2}} \oplus \nu^-_{-\frac{1}{2}}$

Proof. For $\alpha \neq \beta$ we have $\xi_{\{\alpha,\beta\}} = \mathrm{ind}_{T^\infty\cap N_0}^{N_0^\infty}\,\omega$, where $\omega = (\alpha, \beta, \frac{1}{2})$. Hence

$$\begin{aligned}\mathrm{res}_{K_0}^{N_0^\infty}\,\xi_{\{\alpha,\beta\}} &= \mathrm{res}_{K_0}^{N_0^\infty}\,\mathrm{ind}_{T^\infty\cap N_0}^{N_0}\,\omega = \mathrm{ind}_{T^\infty\cap K_0}^{K_0}\,\mathrm{res}_{T^\infty\cap K_0}^{T^\infty\cap N_0}\,\omega \\ &= \mathrm{ind}_{T^\infty\cap K_0}^{K_0}\,\eta = \mu_i^+ \oplus \mu_i^-,\end{aligned}$$

where $\eta = \mathrm{res}_{T^\infty\cap K_0}^{T^\infty\cap N_0}\,\omega \in Rt_{od}(K_0)$ lies in the orbit $O_1^{K_0}$ if $\alpha \equiv \beta \mod 4$ and $O_2^{K_0}$ otherwise. Similarly,

$$\begin{aligned}\mathrm{res}_{\bar{M}_0^\infty}^{N_0^\infty}\,\xi_{\{\alpha,\beta\}} &= \mathrm{res}_{\bar{M}_0^\infty}^{N_0^\infty}\,\mathrm{ind}_{T^\infty\cap N_0}^{N_0^\infty}\,\omega \\ &= \bigoplus_{g\in W_GN_0/W_G\bar{M}_0} \mathrm{ind}_{T^\infty\cap\bar{M}_0}^{\bar{M}_0^\infty}\,\mathrm{res}_{T^\infty\cap\bar{M}_0}^{T^\infty\cap N_0}\,g^*\omega \\ &= \mathrm{ind}_{T^\infty\cap\bar{M}_0}^{\bar{M}_0^\infty}\,\mathrm{res}_{T^\infty\cap\bar{M}_0}^{T^\infty\cap N_0}\,\omega \oplus \mathrm{ind}_{T^\infty\cap\bar{M}_0}^{\bar{M}_0^\infty}\,\mathrm{res}_{T^\infty\cap\bar{M}_0}^{T^\infty\cap N_0}\,a^*\omega \\ &= \nu^+_{\alpha+\beta-\frac{1}{2}} \oplus \nu^-_{\alpha+\beta-\frac{1}{2}} \oplus \nu_{-\alpha+\beta-\frac{1}{2}}.\end{aligned}$$

For roots of the form $\omega = (\alpha, \alpha, \frac{1}{2})$ the representation ξ_ω splits into two irreducible ones. We choose ξ_α^+ to be the one which restricts to μ_1^+ and ξ_α^- to be the one which restricts to μ_1^- (since $\mathrm{res}_{K_0}^{N_0} \xi_\alpha \simeq \mu_1^+ \oplus \mu_1^-$). Similarly to the case $\alpha \neq \beta$ we see that

$$\mathrm{res}_{\bar{M}_0^\infty}^{N_0^\infty} \xi_\alpha^+ \simeq \nu^\varepsilon_{2\alpha - \frac{1}{2}} \oplus \nu^\zeta_{-\frac{1}{2}} \qquad \mathrm{res}_{\bar{M}_0^\infty}^{N_0^\infty} \xi_\alpha^- \simeq \nu^\eta_{2\alpha - \frac{1}{2}} \oplus \nu^\theta_{-\frac{1}{2}}$$

where the signs ε, ζ, η, θ must be equal to $+$, $+$, $-$ and $-$ respectively, since

$$\mathrm{res}_{M_0}^{N_0^\infty} \xi_\alpha^\varepsilon \simeq \mathrm{res}_{M_0}^{K_0} \mu_1^\varepsilon \simeq \lambda_1^\varepsilon \oplus \lambda_4^\varepsilon. \qquad \square$$

Odd representations of groups J_1^∞

Let J be one of the symbols N, K, $\bar{M}$ or M. Recall that $W_G(J_0) \simeq W_G(J_1) \times \Sigma_3$ where the factor Σ_3 is generated by $z^{2^{n-2}}$ and l. Moreover, the elements $z^{2^{n-2}}$ and l act trivially on $\mathrm{IR}_{od}(J_0^\infty)$. Now we classify odd irreducible representations of J_1^∞.

Proposition 3.5.23. *An odd representation of J_1^∞ is irreducible if and only if its restriction to J_0^∞ is irreducible.*

Proof. Let $\omega \in Rt_{od}(J_1)$. The $J_1^\infty/(T^\infty \cap J_1)$-orbit of ω is isomorphic to the $J_1^\infty/(T^\infty \cap J_1)$-orbit (and hence $J_0^\infty/(T^\infty \cap J_0)$-orbit) of $\omega|_{J_0}$. Then both of them are either free (in that case both $\mathrm{ind}_{T^\infty \cap J_1}^{J_1^\infty} \omega$ and $\mathrm{ind}_{T^\infty \cap J_0}^{J_0^\infty} \omega$ are irreducible) or has an order 2 stablizer. In this case $\mathrm{ind}_{T^\infty \cap J_1}^{J_1^\infty} \omega$ decomposes into two irreducible representations and this splitting agrees with tha splitting of $\mathrm{ind}_{T^\infty \cap J_0}^{J_0^\infty} \omega$. $\square$

Proposition 3.5.24.

$$\mathrm{IR}_{od}(J_1^\infty) \simeq \mathrm{IR}_{od}(J_0^\infty) \times (\tfrac{1}{2} + \mathbb{Z}_2^\wedge)/\{\pm 1\}.$$

Proof. The map $\mathrm{res}_{T^\infty \cap J_0}^{T^\infty \cap J_1} : Rt_{od}(J_1) \to Rt_{od}(J_0)$ is a surjection and the counterimage of any root $(\alpha, \beta, \pm\frac{1}{2})$ is the set $(\alpha, \beta, \pm(2k + \frac{1}{2}))$. $\square$

Definition 3.5.25. Fix $\varrho \in \mathrm{IR}_{od}(J_0^\infty)$. Let $(\alpha, \beta, \frac{1}{2})$ be a root of ϱ. Denote by $\varrho_{;\gamma}$ the irreducible representation of J_1^∞ with root (α, β, γ) such that $\mathrm{res}_{J_0^\infty}^{J_1^\infty} \varrho_{;\gamma} \simeq \varrho$ (the previous lemma asserts the this is unique up to isomorphism).

Proposition 3.5.26. *Fix* $J \in \mathcal{T}_{irr}(4) = \{N, K, \bar{M}, M\}$, $\varrho \in \mathrm{IR}_{od}(J_0^\infty)$ *and* $\gamma \in \frac{1}{2} + \mathbb{Z}_2^\wedge$. *For any g representing an element of* $W_G(J_1)$ *we have* $g^*(\varrho_{;\gamma}) \simeq (g^*\varrho)_{;\gamma}$ *(obviously* $W_G(J_1) \subseteq W_G(J_0)$*).*

Proof. By 3.4.9 we can assume that $g \in \langle k, h_1, h_2 \rangle$. Then $g^{-1}zg = z$ and the conclusion follows. □

Odd representations of groups L_i^∞

We conclude with a classification of odd representations of groups L_i, $i = 0, 1, 3$. The sets of odd roots of these groups are respectively $\mathrm{IR}_{od}(T^\infty)$, $\mathrm{IR}_{od}(T^\infty)/\langle(0,2,0),(0,0,2)\rangle$ and $\mathrm{IR}_{od}(T^\infty)/\langle(2,0,0),(0,2,0),(0,0,2)\rangle$. Let

$$\kappa_{\{\alpha,\beta,\gamma\}} := \mathrm{ind}_{T^\infty \cap L_3}^{L_3^\infty} \omega, \quad \omega = (\alpha, \beta, \gamma) \in Rt_{od}(L_3)$$

$$\kappa_{\{\alpha\}} := \mathrm{ind}_{T^\infty \cap L_1}^{L_1^\infty} \omega, \quad \omega = (\alpha, \tfrac{1}{2}, \tfrac{1}{2}) \in Rt_{od}(L_1)$$

$$\kappa := \mathrm{ind}_{T^\infty \cap L_0}^{L_0} \omega, \quad \omega = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) \in Rt_{od}(L_0).$$

Proposition 3.5.27.

$$\mathrm{IR}_{od}(L_3) = \{\kappa_{\{\alpha,\beta,\gamma\}} : \alpha, \beta, \gamma \equiv \tfrac{1}{2} \mod 2\}$$

$$\mathrm{IR}_{od}(L_1) = \{\kappa_{\{\alpha\}} : \alpha \equiv \tfrac{1}{2} \mod 2\}$$

$$\mathrm{IR}_{od}(L_0) = \{\kappa\}$$

The group $W_G(L_3) \simeq \Sigma_3$ *permutes indexes at* $\kappa_{\{\alpha,\beta,\gamma\}}$ *and the groups* $W_G(L_1)$ *and* $W_G(L_0)$ *act trivially on the corresponding sets* $\mathrm{IR}_{od}(L_i)$

Proof. Obvious since the action of $L_i^\infty/(T^\infty \cap L_i)$ on the set of roots is free. □

Chapter 4

Homotopy representations of $Spin(7)$ and $SU(2)^n$

Recall that an m-dimensional homotopy complex representation of a compact Lie group G at prime p is a map $BG \to BU(m)^\wedge_p$. This chapter contains a partial classification of homotopy (complex) representations at 2 of some classical Lie groups. Of course, we are mainly interested in groups $Spin(7)$ and $SU(2)^3$, which appear in the centralizer homotopy decomposition of the 2-compact group $DI(4)$.

4.1 A spectral sequence calculating cohomology of EI-categories

Let $\mathcal{C}$ be a small category and $M : \mathcal{C} \to \mathbf{Ab}$ a contravariant functor. *The n-th cohomology group of $\mathcal{C}$ with coefficients M* is the n-th derived functor of the inverse limit functor evaluated on M. In this section we state some properties of cohomology groups of categories. Throughout this section R is a commutative ring.

Definition 4.1.1. Let $\mathcal{C}$ be a small category. *An $R[\mathcal{C}]$-module* is a contravariant functor $\mathcal{C} \to \mathbf{Mod}_R$.

Remark. The category of $R[\mathcal{C}]$-modules, denoted by $\mathbf{Mod}_{R[\mathcal{C}]}$, is abelian and has enough injectives and projectives.

Remark. Let G be a group and $\beta(G)$ the category with the single object $*$ and morphisms $\mathrm{Mor}_{\beta(G)}(*,*) = G$. Then $R[\beta(G)]$-modules are exactly $R[G]$-modules and $H^*(\beta(G); M) = H^*(G; M)$.

Cohomology groups are functorial with respect to both a category and a coefficient module. If

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence of $R[\mathcal{C}]$-modules, then there exists the long exact sequence of cohomology groups

$$\ldots \longrightarrow H^i(\mathcal{C}; M') \longrightarrow H^i(\mathcal{C}; M) \longrightarrow H^i(\mathcal{C}; M'') \longrightarrow H^{i+1}(\mathcal{C}; M') \longrightarrow \ldots$$

Definition 4.1.2. A category $\mathcal{C}$ is *an EI-category* if each endomorphism in $\mathcal{C}$ is actually an automorphism. The set of isomorphism classes of objects of an EI-category $\mathcal{C}$ admits the partial order given by

$$C \geq C' \Leftrightarrow \mathrm{Mor}_{\mathcal{C}}(C, C') \neq \emptyset. \tag{4.1.3}$$

Definition 4.1.4. Let $\mathcal{C}$ be an EI-category. A subcategory $\mathcal{D} \subseteq \mathcal{C}$ is *closed* if each morphism $c : C \to C'$ of $\mathcal{C}$ such that $C \in \mathcal{D}$ lies in $\mathcal{D}$. Note that each closed subcategory is full.

Definition 4.1.5. Let $\mathcal{C}$ be a small EI-category and M an $R[\mathcal{C}]$-module. *The support* of M is the set $\mathrm{supp}(M) := \{C \in \mathcal{C} : M(C) \neq 0\}$.

Here follows the motivation for introducing supports and closed subcategories:

Proposition 4.1.6. *Let M, N be $R[\mathcal{C}]$-modules and $\mathcal{D}$ be a closed subcategory of $\mathcal{C}$. Assume that the support of N is contained in* $\mathrm{Ob}(\mathcal{D})$*. Then the restriction*

$$\mathrm{Hom}_{\mathcal{C}}(M, N) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(\mathrm{res}^{\mathcal{C}}_{\mathcal{D}} M, \mathrm{res}^{\mathcal{C}}_{\mathcal{D}} N)$$

is an isomorphism.

Proof. The inverse of this homomorphism maps $f : \mathrm{res}^{\mathcal{C}}_{\mathcal{D}} M \to \mathrm{res}^{\mathcal{C}}_{\mathcal{D}} N$ into f', where $f'(C) = f(C)$ for $C \in \mathcal{D}$ and $f'(C) = 0$ otherwise. □

Proposition 4.1.7. *Let $\mathcal{C}$ be a small EI-category, $\mathcal{D}$ a closed subcategory of $\mathcal{C}$ and M an $R[\mathcal{C}]$-module. Assume that* $\mathrm{supp}(M) \subseteq \mathrm{Ob}(\mathcal{D})$*. Then*

$$H^*(\mathcal{C}; M) \cong H^*(\mathcal{D}; \mathrm{res}^{\mathcal{C}}_{\mathcal{D}} M).$$

Proof. Let Q^* be an injective resolution of an $R[\mathcal{C}]$-module N. By 4.1.6 $\operatorname{res}^{\mathcal{C}}_{\mathcal{D}} Q^*$ is an injective resolution of $\operatorname{res}^{\mathcal{C}}_{\mathcal{D}} N$, and

$$H^n(\mathcal{D}; \operatorname{res}^{\mathcal{C}}_{\mathcal{D}} M) = H^n(\operatorname{Hom}_{\mathcal{D}}(R, \operatorname{res}^{\mathcal{C}}_{\mathcal{D}} Q^*)) \stackrel{(4.1.6)}{=} H^n(\operatorname{Hom}_{\mathcal{C}}(R, Q^*)) = H^n(\mathcal{C}; M),$$

where R stands for the constant functor. □

Proposition 4.1.7 allows, in some cases, for replacing category whose cohomology we want to calculate by a smaller one.

The rest of the present section is devoted to the construction of a spectral sequence which converges to the cohomology groups of a given EI-category. Fix an EI-category $\mathcal{C}$. We begin with the construction of an exact sequence which calculates the cohomology of $\mathcal{C}$.

For each $R[\mathcal{C}]$-module M let $(V(M)^*, d^*_M)$ be the cochain complex defined by the formulas

$$V(M)^r = \prod_{C_0 \to \ldots \to C_r \in N(\mathcal{C}^{op})_r} M(C_0) \tag{4.1.8}$$

$$d^r_M(v)(C_0 \stackrel{m}{\to} C_1 \to \ldots \to C_{r+1}) = M(m)(v(C_1 \to \ldots \to C_{r+1})) + \sum_{i=1}^{r+1} (-1)^i v(C_0 \to \ldots \to C_{i-1} \to C_{i+1} \to \ldots \to C_{r+1})$$

Proposition 4.1.9. ([O2, Lemma 2])

$$H^*(\mathcal{C}; M) = H^*((V(M)^*, d^*_M)).$$

Definition 4.1.10. Let $\mathcal{C}$ be an EI-category. *A gradation* on $\mathcal{C}$ is a function $|\cdot| : \operatorname{Ob}(\mathcal{C}) \to \mathbb{Z}$, such that for each morphism $C \to C'$ holds $|C| \geq |C'|$. A gradation $|\cdot|$ is *strict* if each morphism $C \to C'$ such that $|C| = |C'|$ is an isomorphism. A gradation is *non-negative* if $|C| \geq 0$ for each $C \in \operatorname{Ob}(\mathcal{C})$. A category with a gradation is said to be *a graded category.*

Definition 4.1.11. Let $\mathcal{C}$ be a category. A set $X \subseteq \operatorname{Ob}(\mathcal{C})$ is *convex* iff for each sequence of morphisms $C \to C' \to C''$ in $\mathcal{C}$ such that $C, C'' \in X$ holds $C' \in X$.

Let $\mathcal{C}$ be a graded category and X a convex subset of $\mathrm{Ob}(\mathcal{C})$. For any $R[\mathcal{C}]$-module M define the following $R[\mathcal{C}]$-modules:

$$M_X(C) := \begin{cases} M(C) & \text{for } C \in X \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.12}$$

$$M_i^j := M_{\{C \in \mathrm{Ob}(\mathcal{C}) : i \leq |C| \leq j\}}.$$

Remark. If the gradation on $\mathcal{C}$ is strict, then

$$M_i^i = \bigoplus_{|K|=i} M_{\{K\}}.$$

Theorem 4.1.13. *Let $\mathcal{C}$ be a graded EI-category and M an $R[\mathcal{C}]$-module. There is a spectral sequence whose first term is*

$$E_1^{s,t} := H^{s+t}(\mathcal{C}; M_s^s)$$

which converges to $H^{s+t}(\mathcal{C}; M)$. The differentials of the first term are the differentials of the exact sequences

$$0 \longrightarrow M_{s+1}^{s+1} \longrightarrow M_s^{s+1} \longrightarrow M_s^s \longrightarrow 0.$$

Proof. Consider the following filtration on $V(M)^*$ (cf. 4.1.8):

$$F^s V(M)^r := \prod_{\substack{C_0 \to \ldots \to C_r \in N(\mathcal{C}^{op})_r \\ |C_0| \geq s}} M(C_0)$$

Note that

$$F^s V(M)^* / F_{s+1} V(M)^* = \prod_{\substack{C_0 \to \ldots \to C_r \in N(\mathcal{C}^{op})_r \\ |C_0| = s}} M(C_0) = V(M_s^s)^*$$

Thus we obtain the spectral sequence

$$\begin{aligned} E_1^{s,t} = H^{s+t}(F^s V(M)^* / F^{s+1} V(M)^*) &= H^{s+t}(V(M_s^s)^*) \\ &\cong H^{s+t}(\mathcal{C}; M_s^s) \Rightarrow H^{s+t}(V(M)^*) \cong H^{s+t}(\mathcal{C}; M). \end{aligned}$$

□

Let us state two obvious corollaries of theorem 4.1.13:

Proposition 4.1.14. *If the gradation on $\mathcal{C}$ is strict, then the first term of the spectral sequence is*

$$E_1^{s,t} = \bigoplus_{|K|=s} H^{s+t}(\mathcal{C}; M_{\{K\}}).$$

The differential $\delta_1^{s,t} : E_1^{s,t} \to E_1^{s+1,t}$ is the sum $\bigoplus_{|K|=s} \bigoplus_{|K'|=s+1} \delta^s_{K,K'}$, where $\delta^{s,t}_{K,K'} : H^{s+t}(\mathcal{C}; M_{\{K\}}) \to H^{s+t+1}(\mathcal{C}; M_{\{K'\}})$ is the differential of the short exact sequence

$$0 \longrightarrow M_{\{K'\}} \longrightarrow M_{\{K,K'\}} \longrightarrow M_{\{K\}} \longrightarrow 0.$$

Proposition 4.1.15. *Assume that the gradation on $\mathcal{C}$ is strict and for each $K \in \mathrm{Ob}(\mathcal{C})$ the cohomology $H^*(\mathcal{C}; M_{\{K\}})$ is concentrated in dimension $|K|$. Then the spectral sequence degenerates to the cochain complex (C^*_M, δ^*_M), where*

$$C^s_M = \bigoplus_{|K|=s} H^s(\mathcal{C}; M_{\{K\}}) \qquad \delta^s_M = \bigoplus_{|K|=s} \bigoplus_{|K'|=s+1} \delta^s_{K,K'}$$

and $\delta^s_{K,K'} : H^s(\mathcal{C}; M_{\{K\}}) \to H^{s+1}(\mathcal{C}; M_{\{K'\}})$ is the differential of the short exact sequence

$$0 \longrightarrow M_{\{K'\}} \longrightarrow M_{\{K,K'\}} \longrightarrow M_{\{K\}} \longrightarrow 0.$$

4.2 Cohomology of the categories $\mathcal{R}_p(G)$

In this section we present some methods of calculating cohomology of categories $\mathcal{R}_p(G)$. Most of them were provided by [JMO1]. Throughout this section p is a prime integer, G is either a Lie group or a finite group, and A is a ring of p-adic integers. Recall (1.3.8) that $\mathcal{R}_p(G)$ denotes the category of G-orbits with p-stubborn isotropy groups and $\mathcal{O}_p(G)$ the category of G-orbits with p-toral isotropy groups.

Definition 4.2.1. Let Γ be a finite group and M an $A[\Gamma]$-module. Define the following $A[\mathcal{O}_p(\Gamma)]$-modules :

$$F_M(\Gamma/P) = \begin{cases} M^P & \text{for } P = \{1\} \\ 0 & \text{for } P \neq \{1\} \end{cases},$$

$$F'_M(\Gamma/P) = M^\Gamma$$

$$F''_M(\Gamma/P) = \begin{cases} 0 & \text{for } P = \{1\} \\ M^P & \text{for } P \neq \{1\} \end{cases}.$$

Obviously, there exists an exact sequence

$$0 \longrightarrow F_M \longrightarrow F'_M \longrightarrow F''_M \longrightarrow 0. \qquad (4.2.2)$$

Definition 4.2.3. ([JMO1, 5.3]) Let Γ be a finite group and M an $A[\Gamma]$-module. Let

$$\Lambda^*(\Gamma; M) := H^*(\mathcal{O}_p(\Gamma); F_M).$$

Remark. If $\Gamma/\{1\} \in \mathcal{R}_p(G)$, then $\Lambda^*(\Gamma; M) \simeq H^*(\mathcal{R}_p(\Gamma); F_M)$ (F_M here denotes formally the restriction of F to $\mathcal{R}_p(\Gamma)$).

Here follows the motivation for introducing functors Λ:

Definition 4.2.4. Let $\mathcal{C}$ be a category. An $A[\mathcal{C}]$-module M is *atomic* if there exists $C \in \mathrm{Ob}(\mathcal{C})$ such that $M(D) = 0$ if D is not isomorphic to C. In other words, a functor is atomic if its support contains only one isomorphism class of objects.

Theorem 4.2.5. ([JMO1, 5.4]) *Let G be a compact Lie group and M an atomic $A[\mathcal{R}_p(G)]$-module concentrated on an object Q. Then*

$$H^*(\mathcal{R}_p(G); M) \cong \Lambda^*(\mathrm{Aut}_{\mathcal{R}_p(G)}(G/Q); M(G/Q)).$$

Recall some properties of groups $\Lambda^*(\Gamma; M)$.

Proposition 4.2.6. ([JMO1, 6.1]) *Let Γ be a finite group and let M be an $A[\Gamma]$-module. Then*

(a) If p divides $|\Gamma|$, then $\Lambda^0(\Gamma; M) = 0$. Otherwise $\Lambda^0(\Gamma; M) = M^\Gamma$ and $\Lambda^i(\Gamma; M) = 0$ for $i > 0$,

(b) If Γ is not p-reduced, or if p divides the kernel of the Γ-action on M, then $\Lambda^i(\Gamma; M) = 0$ for $i > 0$.

(c) If Γ' is the kernel of the Γ-action on M and $p \not| |H|$, then $\Lambda^*(\Gamma; M) = \Lambda^*(\Gamma/\Gamma'; M)$,

(d) *Assume that M is A-projective. Then for any finite group Γ' and any $A[\Gamma']$-module M', there is an exact sequence (Künneth formula)*

$$0 \longrightarrow \bigoplus_{i+j=n+1} \mathrm{Tor}_A(\Lambda^i(\Gamma; M), \Lambda^j(\Gamma'; M')) \longrightarrow \Lambda^n(\Gamma \times \Gamma'; M \otimes_A M')$$
$$\longrightarrow \bigoplus_{i+j=n} \Lambda^i(\Gamma; M) \otimes_A \Lambda^j(\Gamma'; M') \longrightarrow 0$$

Proposition 4.2.7. ([JMO1, 6.2.(ii)]) *Let Γ be a finite group, M an $A[\Gamma]$-module and Γ_p a Sylow p-subgroup of Γ. Let $\sim$ be the equivalence relation among p-Sylow subgroups generated by nontrivial intersection, and set*

$$\Delta := \{g \in \Gamma : g^{-1}\Gamma_p g \sim \Gamma_p\}.$$

Then $\Lambda^1(\Gamma; M) \cong M^\Delta / M^\Gamma$.

The following proposition is very useful for calculating Λ^*-groups:

Proposition 4.2.8. ([JMO1, 5.2.(ii)]) *For each finite group Γ and $A[\Gamma]$-module M*

$$H^n(\mathcal{R}_p(\Gamma); F'_M) = \begin{cases} M^\Gamma & \text{for } n = 0 \\ 0 & \text{for } n > 0. \end{cases}$$

Acyclicity of the module F'_M can be used to reduce calculations of groups $\Lambda^*(\Gamma; M) = H^*(\mathcal{R}_p(\Gamma); M)$ to calculations of $H^*(\mathcal{R}_p(\Gamma); F''_M)$. It is easier, since by 4.1.6 we have

$$H^*(\mathcal{R}_p(\Gamma); F''_M) \cong H^*(\mathcal{R}_p(\Gamma) \setminus \{\Gamma/\{1\}\}; F''_M)$$

and the category $\mathcal{R}_p(\Gamma) \setminus \{\Gamma/\{1\}\}$ (a full subcategory of $\mathcal{R}_p(\Gamma)$ on all objects but $\Gamma/\{1\}$) is smaller than $\mathcal{R}_p(\Gamma)$. Here follows the simplest application of this procedure:

Proposition 4.2.9. *Let $p = 2$ and let M be an $A[\Sigma_3]$-module. Then*

$$\Lambda^n(\Sigma_3; M) = \begin{cases} M^{\Sigma_2}/M^{\Sigma_3} & \text{for } n = 1 \\ 0 & \text{for } n \neq 1. \end{cases}$$

Proof. The long exact sequence induced by the short exact sequence 4.2.2 reduces to

$$0 \longrightarrow M^{\Sigma_3} \longrightarrow M^{\Sigma_2} \longrightarrow \Lambda^1(\Sigma_3; M) \longrightarrow 0,$$

since $H^n(\mathcal{R}_2(\Sigma_3)); F''_M) = H^n(\{1\}; M^{\Sigma_2}) = 0$ for $n > 0$. □

The functors Λ^* are relatively easy to calculate and allow to compute the cohomology of $\mathcal{R}_p(G)$ with coefficients in any atomic module. In general (i.e. if coefficients are not atomic), cohomology groups can be calculated by the spectral sequence 4.1.14. Here follows an application of this technique (the following proposition was provided by [JMO1, p.229]):

Proposition 4.2.10. *Fix a finite group* Γ *and an* $A[\mathcal{R}_p(\Gamma)]$*-module* M*. Assume that the* p*-Sylow subgroup of* Γ *has an order* p^l*. Then* $\Lambda^n(\Gamma; M) = 0$ *for* $n > l$*.*

Proof. The proof is by induction on the order of p-Sylow subgroup of Γ. For $l = 0$ the conclusion follows from 4.2.6.(a), so assume that $l > 0$. Define a gradation on $\mathcal{R}_p(\Gamma)$ by putting $|\Gamma/P| = -k$ if $|P| = p^k$. The first term of the spectral sequence 4.1.14 for $A[\mathcal{R}_p(\Gamma)]$ module F''_M (cf. 4.2.1) is

$$E_1^{s,t} = \bigoplus_{|P|=p^s} H^{s+t}(\mathcal{R}_p(\Gamma); (F''_M)_{\{\Gamma/P\}}) = \bigoplus_{|P|=p^s} \Lambda^{s+t}(N_\Gamma(P)/P; (F''_M)(\Gamma/P)).$$

If $P \neq \{1\}$, then the rank of p-Sylow subgroup of $N_\Gamma(P)/P$ is less than l and therefore by induction $\Lambda^{s+t}(N_\Gamma(P)/P; (F''_M)(\Gamma/P)) = 0$ for $s + t > l$. Moreover, $(F''_M)(\Gamma/\{1\}) = 0$. Hence $E_1^{s,t} = 0$ for $s + t \geq l$ and then $H^n(\mathcal{R}_p(\Gamma); F''_M) = 0$ for $n > l$. The conclusion follows from acyclicity of F'_M and the exact sequence 4.2.2. □

Remark. The similar argument allows to prove that if for each p-stubborn subgroup $P \subseteq \Gamma$ the groups $\Lambda^n(P; -)$ vanish for $n \geq l$, then the groups $\Lambda^n(\Gamma; -)$ vanish for $n > l$.

Definition 4.2.11. *A* Λ*-dimension* of a finite group Γ at p, denoted by $\Lambda\mathrm{d}_p(\Gamma)$, is the greatest integer n such that there exists an $A[\Gamma]$-module M such that $\Lambda^n(\Gamma; M) \neq 0$. By 4.2.10 such integer always exists, and is not greater than the rank of p-Sylow subgroup of Γ.

Corollary 4.2.12. For $\mathcal{R} = \mathcal{R}_p(\Gamma)$ we have

$$\Lambda\mathrm{d}_p(\Gamma) \leq 1 + \sup_{\Gamma/1 \neq \Gamma/P \in \mathcal{R}} \Lambda\mathrm{d}_p(\mathrm{Aut}_{\mathcal{R}}(\Gamma/P)).$$

Theorem 4.2.13. *Let G be a compact Lie group. Denote $\mathcal{R} := \mathcal{R}_p(G)$. Then* $\mathrm{cd}_p(\mathcal{R}) = \sup_{G/P\in\mathcal{R}}(\Lambda\mathrm{d}_p(\mathrm{Aut}_{\mathcal{R}}(G/P)))$.

Proof. Put $n = \sup_{G/P\in\mathcal{R}}(\Lambda\mathrm{d}_p(\mathrm{Aut}_{\mathcal{R}}(G/P)))$. Choose any $A[\mathcal{R}]$-module M and a strict gradation on $\mathcal{R}$. Let $E_*^{*,*}$ be a spectral sequence converging to $H^*(\mathcal{R}; M)$ associated to this gradation (4.1.13). Since

$$E_1^{s,t} = \bigoplus_{|G/P|=s} \Lambda^{s+t}(\mathrm{Aut}_{\mathcal{R}}; M(G/P)),$$

then for $s+t > n$ holds $E_1^{s+t} = 0$ and therefore $H^k(\mathcal{R}; M) = 0$ for $k > n$. It proves that $\mathrm{cd}_p(\mathcal{R}) \leq n$. To prove $\mathrm{cd}_p(\mathcal{R}) \geq n$ consider an atomic functor concentrated on the object with the maximal Λ-dimension. □

4.3 Calculations of Λ^*-functors.

The present section contains calculations of groups $\Lambda^*(\Sigma_n; M)$ for some integers n and $A[\Sigma_n]$-modules M. We begin with calculating cohomology of the category $\mathcal{J} := \mathcal{R}_2(\Sigma_3)$ with any coefficients.

Recall (4.2.9) that

$$\Lambda^n(\Sigma_3; M) = \begin{cases} M^{\Sigma_2}/M^{\Sigma_3} & \text{for } n = 1 \\ 0 & \text{for } n \neq 1. \end{cases}$$

The category $\mathcal{J}$ has two objects: $\Sigma_3/1$ and Σ_3/Σ_2. Its morphisms are given by

$$\begin{aligned} \mathrm{Aut}_{\mathcal{J}}(\Sigma_3/1) &\simeq \Sigma_3 & \qquad \mathrm{Mor}_{\mathcal{J}}(\Sigma_3/1, \Sigma_3/\Sigma_2) &\simeq \Sigma_3/\Sigma_2 \\ \mathrm{Aut}_{\mathcal{J}}(\Sigma_3/\Sigma_2) &\simeq 1 & \qquad \mathrm{Mor}_{\mathcal{J}}(\Sigma_3/\Sigma_2, \Sigma_3/1) &= \emptyset. \end{aligned}$$

The category $\mathcal{J}$ has a strict gradation given by $|\Sigma_3/1| = 1$ and $|\Sigma_3/\Sigma_2| = 0$.

Proposition 4.3.1. *If M is an $A[\mathcal{J}]$-module, then the spectral sequence 4.1.14 degenerates to an exact sequence*

$$\begin{aligned} 0 \longrightarrow H^0(\mathcal{J}; M) \longrightarrow M(\Sigma_3/\Sigma_2)& \\ \xrightarrow{d} M(\Sigma_3/1)^{\Sigma_2}/M(\Sigma_3/1)^{\Sigma_3} &\longrightarrow H^1(\mathcal{J}; M) \longrightarrow 0, \end{aligned}$$

where d is the composition

$$M(\Sigma_3/\Sigma_2) \xrightarrow{M(1\Sigma_2)} M(\Sigma_3/1)^{\Sigma_2} \twoheadrightarrow M(\Sigma_3/1)^{\Sigma_2}/M(\Sigma_3/1)^{\Sigma_3}.$$

Proof. By 4.2.9 all entries of the first term of the spectral sequence vanish except $E_1^{0,0} = \Lambda^0(1; M(\Sigma_3/\Sigma_2)) \cong M(\Sigma_3/\Sigma_2)$ and $E_1^{1,0} = \Lambda^1(\Sigma_3; M(\Sigma_3/1))$. To prove the second part consider the canonical homomorphism of $A[\mathcal{J}]$-modules $M \to F'_{M(\Sigma_3/1)}$. We obtain a commutative diagram

$$\begin{array}{ccc} M(\Sigma_3/\Sigma_2) & \xrightarrow{d} & M(\Sigma_3/1)^{\Sigma_2}/M(\Sigma_3/1)^{\Sigma_3} \\ \downarrow & & \| \\ M(\Sigma_3/1)^{\Sigma_2} & \xrightarrow{d} & M(\Sigma_3/1)^{\Sigma_2}/M(\Sigma_3/1)^{\Sigma_3} \end{array}$$

Hence the upper d is the suitable composition. □

Definition 4.3.2. For each n let $M(n,k)$ be a free A-module generated by elements x_B, where B runs over all subsets of $\{1,\dots,n\}$ having k elements. The natural action of Σ_n produces a structure of $A[\Sigma_n]$-module on $M(n,k)$. Note that $M(n,1) \cong A^n$ with a natural Σ_n-action and $M(n,k) \cong M(n,n-k)$.

Proposition 4.3.3. *If $n \geq 4$, then $\Lambda^1(\Sigma_n; M) = 0$ for each $A[\Sigma_n]$-module M.*

Proof. The relation $\sim$ introduced in 4.2.7 is transitive. □

The groups $\Lambda^*(\Sigma_5; -)$

Let $a = (12), b = (34), s = (13)(24) \in \Sigma_5$. Recall (3.1.18) that there exist, up to conjugacy, four stubborn subgroups of Σ_5, namely 1, $\langle a \rangle \simeq C_2$, $\langle ab, s \rangle \simeq C_2^2$ and $\langle a, b, s \rangle \simeq Q_8$. Its Weyl groups (in Σ_5) are respectively $\Sigma_5, \Sigma_3, \Sigma_3$ and 1.

Proposition 4.3.4. *Let M be an $A[\Sigma_5]$-module. Then*

$$\Lambda^n(\Sigma_5; M) = \begin{cases} M^{\Sigma_2\times\Sigma_2\times 1}/M^{\Sigma_2\times\Sigma_3} + M^{\Sigma_4\times 1} & \text{for } n = 2 \\ 0 & \text{for } n \neq 2. \end{cases}$$

Proof. By 4.2.6.(a) and 4.3.3 $\Lambda^n(\Sigma_5; M) = 0$ for $n = 0, 1$. Let $\mathcal{R} := \mathcal{R}_2(\Sigma_5)$. Since $\Lambda^n(\Sigma_5; M) \cong H^{n-1}(\mathcal{R}; F''_M)$ for $n \geq 2$ it is sufficient to calculate Λ^2 only. Introduce a gradation on $\mathcal{R}$ by $|\Sigma_5/\langle a, b, s\rangle| = 0$, $|\Sigma_5/\langle ab, s\rangle| = |\Sigma_5/\langle a\rangle| = 1$

and $|\Sigma_5/1| = 2$. Again, the spectral sequence which computes $H^*(\mathcal{R}; F''_M)$ degenerates to an exact sequence

$$0 \longrightarrow H^0(\mathcal{R}; F''_M) \longrightarrow M^{\langle a,b,s\rangle} \xrightarrow{d} M^{\langle a,b\rangle}/M^{\langle a\rangle\times\Sigma_3} \oplus M^{\langle a,b,s\rangle}/M^{\Sigma_4\times 1} \longrightarrow H^1(\mathcal{R}; F''_M) \longrightarrow 0.$$

Since the full subcategories with object sets respectively $\{\langle a,b,s\rangle, \langle ab,s\rangle\}$ and $\{\langle a,b,s\rangle, \langle a\rangle\}$ are both isomorphic to $\mathcal{J}$, then the differential d is the composition of the sum of inclusions $M^{\langle a,b,s\rangle} \subseteq M^{\langle a,b\rangle}$ and $M^{\langle a,b,s\rangle} \subseteq M^{\langle a,b,s\rangle}$ with the suitable projection. Then $\Lambda^2(\Sigma_5; M) \cong H^1(\mathcal{R}; F''_M) \simeq M^{\Sigma_2\times\Sigma_2\times 1}/M^{\Sigma_2\times\Sigma_3} + M^{\Sigma_4\times 1}$. □

Corollary 4.3.5. For each n we have $\Lambda^n(\Sigma_5; M(5,1)) = 0$.

Proof. Since

$$\begin{aligned} M^{\langle a,b\rangle} &= A\{x_1 + x_2, x_3 + x_4, x_5\} \\ M^{\langle a\rangle\times\Sigma_3} &= A\{x_1 + x_2, x_3 + x_4 + x_5\} \\ M^{\Sigma_4\times 1} &= A\{x_1 + x_2 + x_3 + x_4, x_5\}, \end{aligned}$$

then the conclusion follows from 4.3.4. □

Note that $\Lambda^2(\Sigma_5; M)$ not always vanishes. For example, $\Lambda^2(\Sigma_5; M(5,2)) \neq 0$.

The groups $\Lambda^*(\Sigma_7; -)$

Λ^*-groups of Σ_7 are much more difficult to calculate.

Denote $\mathcal{R} := \mathcal{R}_2(\Sigma_7)$, and let $a = (12)$, $b = (34)$, $c = (56)$, $s = (13)(24)$. As shown in Section 3.1, there is up to conjugacy 7 classes of stubborn subgroups of Σ_7, namely $L_0 := 1$, $L_1 := \langle a\rangle$, $L_3 := \langle a,b,c\rangle$, $K_0 = \langle ab,s\rangle$, $K_1 = \langle ab,s,c\rangle$, $N_0 = \langle a,b,s\rangle$, $N_1 = \langle a,b,s,c\rangle$ ($\mathcal{R}_2(\Sigma_7)$ is a subcategory of $\mathcal{R}_2(Spin(7))$ and this notation coincides with the notation introduced in 3.2.8). Its automorphism groups are respectively 1 (for N_1), Σ_3 (for N_0, L_3 and K_1), Σ_5 (for L_1) and $\Sigma_3\times\Sigma_3$ (for K_0).

Proposition 4.3.6. *If M is an $A[\Sigma_7]$-module, then $\Lambda^n(\Sigma_7; M) = 0$ for $n \neq 2, 3$.*

Proof. For $n = 0, 1$ the conclusion is clear. Since $\Lambda d_2(\mathrm{Aut}_{\mathcal{R}}(\Sigma_7/P)) \le 2$ for all orbits $\Sigma_7/P \in \mathcal{R}$ except $\Sigma_7/1$, then for each $n > 3$

$$\Lambda^n(\Sigma_7; M) \cong H^{n-1}(\mathcal{R}; F''_M) \cong H^{n-1}(\mathcal{R} \setminus \{\Sigma_7/1\}; F''_M) = 0.$$

□

Proposition 4.3.7. $\Lambda^3(\Sigma_7; M(7,1)) = 0$.

Proof. It is enough to prove that for each $\Sigma_7/P \in \mathcal{R}$ and each $n > 1$ holds

$$\Lambda^n_P := \Lambda^n(\mathrm{Aut}_{\mathcal{R}}(\Sigma_7/P); F''_{M(7,1)}(\Sigma_7/P)) = 0.$$

For $P = L_0 = 1$ we have $F''_{M(7,1)}(\Sigma_7/1) = 0$ and for $P = N_1, N_0, K_1, L_3$ we have $\Lambda d_2(P) < 2$ and therefore $\Lambda^n_P = 0$ for $n \ge 2$. Moreover

$$F''_{M(7,1)}(\Sigma_7/L_1) = M(7,1)^{L_1} \cong A\{x_1 + x_2\} \oplus M(5,1)\{x_i\}_{i=3}^7$$
$$F''_{M(7,1)}(\Sigma_7/K_0) \cong A\bar{\otimes}A\{x_1 + x_2 + x_3 + x_4\} \oplus A\bar{\otimes}M(3,1)\{x_i\}_{i=5}^7$$

Then, for $n > 1$, we have $\Lambda^n_{L_1} = 0$ by 4.3.4 and $\Lambda^n_{K_0} = 0$ by 4.2.6.(d). □

The groups $\Lambda^*(\Sigma_3 \wr C_2; -)$

The last part of this section is devoted to the group $\Sigma_3 \wr C_2$ which appears as the automorphism group of one of the objects of $\mathcal{R}_2(Spin(7))$. There exist (up to conjugacy) three 2-stubborn subgroups of $\Sigma_3 \wr C_2$, namely 1, Σ_2 and $\Sigma_2 \wr C_2$. Its Weyl groups are respectively $\Sigma_3 \wr C_2$, Σ_3 and 1. Note that the full subcategory of $\mathcal{R} := \mathcal{R}_2(\Sigma_3 \wr C_2)$ containing Σ_2 and $\Sigma_2 \wr C_2$ is isomorphic to $\mathcal{J}$.

Proposition 4.3.8. *Let M be an $A[\Sigma_3 \wr C_2]$-module. Then*

$$\Lambda^n(\Sigma_3 \wr C_2; M) = \begin{cases} M^{\Sigma_2 \times \Sigma_2}/M^{\Sigma_2 \times \Sigma_3} + M^{\Sigma_2 \wr C_2} & \text{for } n = 2 \\ 0 & \text{for } n \neq 2. \end{cases}$$

Proof. Since the rank of $\Sigma_3 \wr C_2$ is even, and the relation $\sim$ from 4.2.7 is transitive, then $\Lambda^n(\Sigma_3 \wr C_2; M) = 0$ for $n = 0, 1$. For $n > 1$ we have

$$\Lambda^n(\Sigma_3 \wr C_2; M) \cong H^{n-1}(\mathcal{R}_2(\Sigma_3 \wr C_2); F''_M) \cong H^{n-1}(\mathcal{J}; \mathrm{res}^{\mathcal{R}}_{\mathcal{J}} F''_M).$$

By 4.3.1 we have

$$H^{n-1}(\mathcal{J}; \mathrm{res}^{\mathcal{R}}_{\mathcal{J}} F''_M) \cong \begin{cases} M^{\Sigma_2 \times \Sigma_2}/M^{\Sigma_2 \times \Sigma_3} + M^{\Sigma_2 \wr C_2} & \text{for } n = 2 \\ 0 & \text{for } n \neq 2. \end{cases}$$ □

4.4 Homotopy representations of $Spin(7)$

Let ϱ be an $\mathcal{R}_2(Spin(7))$-invariant complex representation of N_1^∞ (which is a 2-discrete approximation of 2-normalizer of the maximal torus of $Spin(7)$, cf. 3.2.8). In the present section we show that, under some mild conditions, $B\varrho_2^\wedge$ extends to a map $BSpin(7)_2^\wedge \to BU(m)_2^\wedge$, where m is the dimension of ϱ.

Denote $G := Spin(7)$, $\mathcal{R} := \mathcal{R}_2(G)$, $A := \mathbb{Z}_2^\wedge$ and let $\Xi = \Xi^\varrho$ be an $A[\mathcal{R}]$-module introduced in 2.4.4. By Theorem 2.4.9.(a) we need to prove that $H^3(\mathcal{R}; \Xi^\varrho)$ vanishes, and that the cohomology of $\mathcal{R}$ in coefficients in any $A[\mathcal{R}]$-module vanishes in dimensions above 3.

Proposition 4.4.1. *If M is an $A[\mathcal{R}]$-module, then $H^n(\mathcal{R}; M) = 0$ for each $n > 3$.*

Proof. The groups of automorphisms of the objects of $\mathcal{R}$ are 1, Σ_3, $\Sigma_3 \times \Sigma_3$, Σ_5, $\Sigma_3 \wr C_2$, $\Sigma_3 \wr C_2 \times \Sigma_3$ and Σ_7 (cf. 3.2.8). We have $\Lambda\mathrm{d}_2(1) = 0$ (obvious), $\Lambda\mathrm{d}_2(\Sigma_3) = 1$ (by 4.2.9), $\Lambda\mathrm{d}_2(\Sigma_5) = 2$ (by 4.3.4), $\Lambda\mathrm{d}_2(\Sigma_3 \wr C_2) \leq 2$ (by 4.3.8) and $\Lambda\mathrm{d}_2(\Sigma_7) \leq 3$ (by 4.3.6). Moreover, $\Lambda\mathrm{d}_2(\Sigma_3 \times \Sigma_3) = 2$ and $\Lambda\mathrm{d}_2(\Sigma_3 \wr C_2 \times \Sigma_3) = \Lambda\mathrm{d}_2(\Sigma_3 \wr C_2) + 1 \leq 3$. $\square$

Since ϱ is $\mathcal{R}$-invariant, it splits into the even part ϱ_{ev} and the odd part ϱ_{od}. Moreover, $\Xi^\varrho = \Xi^{\varrho_{ev}} \oplus \Xi^{\varrho_{od}}$. Then we can consider separately the odd case and the even case.

Theorem 4.4.2. *If ϱ is an odd $\mathcal{R}$-invariant complex representation of N_1^∞, then the map $B\varrho_2^\wedge : (BN_1^\infty)_2^\wedge \to BU(m)_2^\wedge$ extends to a map $BSpin(7)_2^\wedge \to BU(m)_2^\wedge$.*

Proof. By 4.4.1 and 2.4.9 we need to prove that for each 2-stubborn subgroup $P \subseteq Spin(7)$ we have $\Lambda^3(W_G(P); \Xi^\varrho(P)) = 0$. If $P \notin \{L_0, M_0\}$, then $\Lambda\mathrm{d}_p(\mathrm{Aut}_{\mathcal{R}}(G/P)) < 3$ and the conclusion is obvious. Since there exists only one (up to isomorphism) odd representation of L_0 3.5.27, then $\Xi^\varrho(L_0) \cong A$ and, by 4.2.6.(c), we obtain $\Lambda^*(W_G(L_0); \Xi^\varrho(L_0)) = 0$. There exists more than one odd representation of M_0 but there is an element of order 2 in $W_G(M_0)$ which acts trivially on $\mathrm{IR}_{od}(M_0)$ (cf. 3.5.8). Again, by 4.2.6.(c) we obtain that the suitable groups Λ^* vanish. $\square$

The even case is significantly more difficult. Assume that ϱ is even.

Proposition 4.4.3. *If ϱ is an even complex $\mathcal{R}$-invariant representation of N_1^∞, then $\Lambda^3(\Sigma_3 \wr C_2; \Xi^\varrho(M_0)) = 0$.*

Proof. By 3.3.3 and 3.3.30 $\mathrm{res}_{M_0}^{N_1^\infty} \varrho$ is a direct sum of representations having the form $\varphi \bar{\otimes} \psi$, where

$$\varphi \in \{\tau_{\mathbf{0}}, \tau_{\mathbf{a_0}+\mathbf{b_0}} \oplus \tau_{\mathbf{a_1}+\mathbf{b_1}} \oplus \tau_{\mathbf{a_0}+\mathbf{b_0}+\mathbf{a_1}} \oplus \tau_{\mathbf{a_0}+\mathbf{b_0}+\mathbf{b_1}} \oplus \tau_{\mathbf{a_0}+\mathbf{a_1}+\mathbf{b_1}} \oplus \tau_{\mathbf{b_0}+\mathbf{a_1}+\mathbf{b_1}},$$
$$\tau_{\mathbf{a_0}} \oplus \tau_{\mathbf{b_0}} \oplus \tau_{\mathbf{a_1}} \oplus \tau_{\mathbf{b_1}} \oplus \tau_{\mathbf{a_0}+\mathbf{a_1}} \oplus \tau_{\mathbf{a_0}+\mathbf{b_1}} \oplus \tau_{\mathbf{b_0}+\mathbf{a_1}} \oplus \tau_{\mathbf{b_0}+\mathbf{b_1}} \oplus \tau_{\mathbf{a_0}+\mathbf{b_0}+\mathbf{a_1}+\mathbf{b_1}}\}$$

and $\psi \in \{\tau_0, \tau_i \oplus \tau_j \oplus \tau_k\}$. Therefore $\Xi(M_0)$ is a direct sum of modules $N \bar{\otimes} N'$, where N is an $A[\Sigma_3 \wr C_2]$-module corresponding to one of the sums listed above, and N' is an $A[\Sigma_3]$-module isomorphic either to A or to $M(3,1)$. If $\varphi \simeq \tau_{\mathbf{0}}$, then $N \simeq A$, if φ is the first sum then $N \simeq \mathrm{res}_{\Sigma_3 \wr C_2}^{\Sigma_6} M(6,1)$, and if φ is the last sum then N is a submodule of $\mathrm{res}_{\Sigma_3 \wr C_2}^{\Sigma_6} M(6,2)$ generated by elements x_{ij} where $1 \le i \le 3$ and $4 \le j \le 6$. By 4.3.8 in the first case $\Lambda^2(\Sigma_3 \wr C_2; A) = 0$, in the second one we have

$$\begin{aligned}\Lambda^2(\Sigma_3 \wr C_2; \mathrm{res}_{\Sigma_3 \wr C_2}^{\Sigma_6} M(6,1)) &\cong M(6,1)^{\Sigma_2 \times \Sigma_2} / M(6,1)^{\Sigma_2 \times \Sigma_3} + M(6,1)^{\Sigma_2 \wr C_2} \\ &= A\{x_1 + x_2, x_3, x_4 + x_5, x_6\}/ \\ /A\{x_1 + x_2, x_3, x_4 + x_5 + x_6\} &+ A\{x_1 + x_2 + x_4 + x_5 + x_6, x_3, x_6\} = 0\end{aligned}$$

and finally in the third case

$$\begin{aligned}\Lambda^2(\Sigma_3 \wr C_2; N) &\cong N^{\Sigma_2 \times \Sigma_2} / N^{\Sigma_2 \times \Sigma_3} + N^{\Sigma_2 \wr C_2} \\ &= A\{x_{14} + x_{15} + x_{24} + x_{25}, x_{16} + x_{26}, x_{34} + x_{35}, x_{36}\}/ \\ &/A\{x_{14} + x_{15} + x_{16} + x_{24} + x_{25} + x_{26}, x_{34} + x_{35} + x_{36}\} \\ &\quad + A\{x_{14} + x_{15} + x_{24} + x_{25}, x_{16} + x_{26} + x_{34} + x_{35}, x_{36}\} = 0\end{aligned}$$

Then for each N we obtain $\Lambda^2(\Sigma_3 \wr C_2; N) = 0$ and, since $\Lambda d_2(\Sigma_3) = 1$, from 4.2.6.(d) follows $\Lambda^3(\Sigma_3 \wr C_2 \times \Sigma_3; N \bar{\otimes} N') = 0$. □

Let $\mathcal{C}$ be the full subcategory of $\mathcal{R}$ containing objects L_i, K_i and N_i. By 3.1.27 it is isomorphic to $\mathcal{R}_2(\Sigma_7)$, and the correspondence between 2-stubborn subgroups of Σ_7 and $Spin(7)$ is as follows:

$$\begin{array}{ccc} 1 \mapsto L_0 & \langle a \rangle \mapsto L_1 & \langle a,b,c \rangle \mapsto L_3 \\ \langle ab, s \rangle \mapsto K_0 & \langle ab, s, c \rangle \mapsto K_1 & \\ \langle a, b, s \rangle \mapsto N_0 & \langle a, b, s, c \rangle \mapsto N_1 & \end{array}$$

where, as before, $a = (12)$, $b = (34)$, $c = (56)$, $s = (13)(24)$. Note that $\mathcal{C}$ is a closed subcategory of $\mathcal{R}$ in the sense of 4.1.4.

Proposition 4.4.4. *There is an isomorphism* $H^3(\mathcal{R};\Xi) \cong H^3(\mathcal{C};\operatorname{res}^{\mathcal{R}}_{\mathcal{C}}\Xi)$.

Proof. Since $\mathcal{C}$ is closed in $\mathcal{R}$, there is an exact sequence of $A[\mathcal{R}]$-modules

$$0 \longrightarrow \Xi|_{\mathcal{R}\setminus\mathcal{C}} \longrightarrow \Xi \longrightarrow \Xi|_{\mathcal{C}} \longrightarrow 0$$

which induces the long exact sequence

$$\ldots \longrightarrow H^3(\mathcal{R};\Xi|_{\mathcal{R}\setminus\mathcal{C}}) \longrightarrow H^3(\mathcal{R};\Xi) \longrightarrow H^3(\mathcal{R};\Xi|_{\mathcal{C}}) \longrightarrow H^4(\mathcal{R};\Xi|_{\mathcal{R}\setminus\mathcal{C}}) \longrightarrow$$

The only object of $\operatorname{Ob}(\mathcal{R}) \setminus \operatorname{Ob}(\mathcal{C})$ having Λ-dimension larger than 2 is M_0, but by 4.4.3 we have $\Lambda^3(\Sigma_3 \wr C_2 \times \Sigma_3;\Xi(M_0)) = 0$. Then $H^3(\mathcal{R};\Xi|_{\mathcal{R}\setminus\mathcal{C}}) = H^4(\mathcal{R};\Xi|_{\mathcal{R}\setminus\mathcal{C}}) = 0$. Since $\mathcal{C}$ is closed we have $H^*(\mathcal{R};\Xi|_{\mathcal{C}}) \cong H^*(\mathcal{C};\operatorname{res}^{\mathcal{R}}_{\mathcal{C}}\Xi)$ (cf. 4.1.7). □

From now on we denote the restriction $\operatorname{res}^{\mathcal{R}}_{\mathcal{C}}\Xi$ just by Ξ. This should not lead to confusion since all further calculations will be made in the category $\mathcal{C}$.

Define a gradation of $\mathcal{C}$ by $|N_1| = 0$, $|N_0| = |L_3| = |K_1| = 1$, $|L_1| = |K_0| = 2$ and $|L_0| = 3$. It is clear that the gradation is strict. The first term of the spectral sequence (4.1.13) converging to $H^*(\mathcal{C};\Xi)$ is

$$\begin{array}{ccccccc} \Xi(N_1) & \xrightarrow{d_1^{0,0}} & E_1^{1,0} & \xrightarrow{d_1^{1,0}} & E_1^{2,0} & \xrightarrow{d_1^{2,0}} & E_1^{3,0} \\ \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & E_1^{3,-1} \end{array} \tag{4.4.5}$$

where

$$\begin{aligned} E_1^{1,0} &= \Lambda^1(\Sigma_3;\Xi(L_3)) \oplus \Lambda^1(\Sigma_3;\Xi(N_0)) \oplus \Lambda^1(\Sigma_3;\Xi(K_1)) \\ E_1^{2,0} &= \Lambda^2(\Sigma_5;\Xi(L_1)) \oplus \Lambda^2(\Sigma_3^2;\Xi(K_0)) \\ E_1^{3,i} &= \Lambda^{3+i}(\Sigma_7;\Xi(L_0)) \end{aligned}$$

for $i = 0, -1$. Although, possibly, the second term is not equal to the infinite term, we see that $E_2^{3,0} = E_\infty^{3,0} = \operatorname{coker} d_0^{2,0}$. Then $H^3(\mathcal{C};\Xi) = 0$ if and only if $d_1^{2,0}$ is an epimorphism.

Let $F := F'_{\Xi(L_0)}$ (see 4.2.1). Since

$$\operatorname{res}^{N_1^\infty}_{L_0} \varrho \simeq (\eta_0^7)^{\oplus l_0} \oplus (\eta_1^7)^{\oplus l_1} \oplus (\eta_2^7)^{\oplus l_2} \oplus (\eta_3^7)^{\oplus l_3},$$

(cf. 3.3.21) then $\Xi(L_0)$ is a direct sum of some of the modules $M(7,i)$, $i = 0,\ldots,3$, where $M(7,i)$ appears as a summand of $\Xi(L_0)$ if and only if $\operatorname{res}^{N_1^\infty}_{L_0} \varrho$ contains a subrepresentation isomorphic to η_i^7.

Proposition 4.4.6. *We have*

$$
\begin{aligned}
F'_{M(7,1)}(L_1) &\simeq A\{x_1+x_2\} \oplus M(5,1)\{x_k\}_{k\geq 3}\\
F'_{M(7,1)}(K_0) &\simeq (A\bar{\otimes}A)\{x_1+x_2+x_3+x_4\} \oplus (A\bar{\otimes}M(3,1))\{x_k\}_{k\geq 5}\\
F'_{M(7,2)}(L_1) &\simeq A\{x_{12}\} \oplus M(5,1)\{x_{1k}+x_{2k}\}_{k\geq 3} \oplus M(5,2)\{x_{kl}\}_{k,l\geq 3}\\
F'_{M(7,2)}(K_0) &\simeq (M(3,1)\bar{\otimes}A)\{x_{12}+x_{34}, x_{13}+x_{24}, x_{14}+x_{23}\}\\
&\quad \oplus (A\bar{\otimes}M(3,1))\{x_{1k}+x_{2k}+x_{3k}+x_{4k}\}_{k\geq 5}\\
&\quad \oplus (A\bar{\otimes}M(3,1))\{x_{kl}\}_{k,l\geq 5}\\
F'_{M(7,3)}(L_1) &\simeq M(5,1)\{x_{12k}\}_{k\geq 3} \oplus M(5,2)\{x_{1kl}+x_{2kl}\}_{k,l\geq 3}\\
&\quad \oplus M(5,2)\{x_{klm}\}_{k,l,m\geq 3}\\
F'_{M(7,3)}(K_0) &\simeq (A\bar{\otimes}A)\{x_{123}+x_{124}+x_{134}+x_{234}\}\\
&\quad \oplus (M(3,1)\bar{\otimes}M(3,1))\{x_{12k}+x_{34k}, x_{13}+x_{24k}, x_{14}+x_{23k}\}_{k\geq 5}\\
&\quad \oplus (A\bar{\otimes}M(3,1))\{x_{1kl}+x_{2kl}+x_{3kl}+x_{4kl}\}_{k,l\geq 5}(A\bar{\otimes}A)\{x_{567}\}
\end{aligned}
$$

Let $\bar{E}_*^{*,*}$ be the spectral sequence which converges to $H^*(\mathcal{C};F)$ (the gradation on $\mathcal{C}$ is the same as before). The obvious homomorphisms of $A[\mathcal{C}]$-modules $f:\Xi\to F$ induces the morphism of spectral sequences $E_*^{*,*}\to\bar{E}_*^{*,*}$. Of course, the differential $\bar{d}_1^{2,0}:\bar{E}_1^{2,0}\to\bar{E}_1^{3,0}$ is surjective because F is acyclic.

Proposition 4.4.7. *Assume that at least one of the following conditions is satisfied:*

(a) $\operatorname{res}^{N_1^\infty}_{L_0}\varrho$ *does not contain a subrepresenation isomorphic to* η_2^7*,*

(b) $\operatorname{res}^{N_1^\infty}_{L_0}\varrho$ *does not contain a subrepresenation isomorphic to* η_3^7*,*

(c) $\operatorname{res}^{N_1^\infty}_{L_1}\varrho$ *contains a subrepresenation isomorphic to either* $\theta\bar{\otimes}\eta_2^5$*, or to* $\tau\bar{\otimes}\eta_2^5$*.*

Then the differential $d_1^{2,0}$ *is an epimorphism.*

Proof. In all the cases it suffices to prove that both groups

$$\Lambda^2(\Sigma_5;\operatorname{coker}(\Xi(L_1)\to F(L_1)=\Xi(L_0)^{\langle a\rangle}))$$

and

$$\Lambda^2(\Sigma_3\times\Sigma_3;\operatorname{coker}(\Xi(K_0)\to F(K_0)=\Xi(K_0)^{\langle ab,s\rangle}))$$

vanish. Since $\Lambda \mathrm{d}_2(\Sigma_5) = \Lambda \mathrm{d}_2(\Sigma_3 \times \Sigma_3) = 2$, then the maps

$$\Lambda^2(\Sigma_5; \Xi(L_1)) \longrightarrow \Lambda^2(\Sigma_5; F(L_1))$$
$$\Lambda^2(\Sigma_3 \times \Sigma_3; \Xi(K_0)) \longrightarrow \Lambda^2(\Sigma_3 \times \Sigma_3; F(K_0))$$

are epimorphisms. Hence $E_1^{2,0} \to \bar{E}_1^{2,0}$ is an epimorphism. The conclusion follows from the commutativity of the diagram

$$\begin{array}{ccc} E_1^{2,0} & \longrightarrow & \bar{E}_1^{2,0} \\ \downarrow d_1^{2,0} & & \downarrow \bar{d}_1^{2,0} \\ E_1^{3,0} & = & \bar{E}_1^{3,0} \end{array}$$

Case (a).

If $\mathrm{res}_{L_0}^{N_1^\infty} \varrho$ does not contain η_3^7, then by 4.3.6 we have $E_1^{3,0} = 0$, so assume otherwise. Let x_{ijk}, $1 \le i < j < k \le 7$ be the generators of the submodule of $\Xi(L_0)$ corresponding to η_3^7. By 3.3.28 there exists a subrepresentation of $\mathrm{res}_{L_1^\infty}^{N_1^\infty} \varrho$ which is isomorphic either to $\tau \bar{\otimes} \eta_2^5$ or to $\beta_{2k} \bar{\otimes} \eta_2^5$ for some $k \in A$, and another subrepresentation isomorphic to $\alpha_{2k'+1} \bar{\otimes} \eta_2^5$, where $k' \in A$. Then $\Xi(L_1)$ contains the direct sum of two $A[\Sigma_5]$-submodules isomorphic to $M(5,2)$: the one generated by irreducible subrepresentations of $\tau \bar{\otimes} \eta_2^5$ (or $\beta_{2k} \bar{\otimes} \eta_2^5$) maps onto the summand

$$M(5,2)\{x_{klm}\}_{3 \le k < l < m \le 7} \subseteq F(L_1)$$

and the one generated by irreducible subrepresentations of $\alpha_{2k'+1} \bar{\otimes} \eta_2^5$ maps onto

$$M(5,2)\{x_{1kl} + x_{2kl}\}_{3 \le k < l \le 7} \subseteq F(L_1).$$

By 4.4.6 $\mathrm{coker}(\Xi(L_1^\infty) \to F(L_1^\infty))$ is a quotient of a direct sum of $A[\Sigma_5]$-modules isomorphic either to $M(5,1)$ or to A. Hence by 4.3.4 we obtain that $\Lambda^2(\Sigma_5; \mathrm{coker}(\Xi(L_1^\infty) \to F(L_1^\infty)) = 0$. Similarly, there is a subrepresentation of $\mathrm{res}_{K_0}^{N_1^\infty} \varrho$ which is isomorphic either to $\gamma_2^+ \bar{\otimes} \tau$ or to $\gamma_2^- \bar{\otimes} \tau$, generates a submodule of $\Xi(K_0)$ isomorphic to $M(3,1) \bar{\otimes} M(3,1)$ and maps onto

$$M(3,1) \bar{\otimes} M(3,1)\{x_{12k} + x_{34k}, x_{13} + x_{24k}, x_{14} + x_{23k}\}_{5 \le k \le 7} \subseteq F(K_0).$$

Again by 4.4.6 we obtain that $\Lambda^2(\Sigma_3 \times \Sigma_3; \mathrm{coker}(\Xi(K_0) \to F(K_0)) = 0$.

Case (b).

The argument is quite similar. Assume that $\mathrm{res}^{N_1^\infty}_{L_0}\ \varrho$ contains a subrepresentation isomorphic to η_2^7 (otherwise the conclusion is obvious). Then there exists a subrepresentation of $\mathrm{res}^{N_1^\infty}_{L_1^\infty}\ \varrho$ which is isomorphic either to $\theta\bar{\otimes}\eta_2^5$ or to $\beta_{2k}\bar{\otimes}\eta_2^5$ for some $k \in A$. The submodule of $\Xi(L_1)$ generated by its irreducible subrepresentations maps onto the summand

$$M(5,2)\{x_{kl}\}_{3\le k<l\le 7} \subseteq F(L_1^\infty).$$

Like in Case (a) it implies that all summands of $F(L_1)$ having $\Lambda^2 \neq 0$ lie in the image of the homomorphism $\Xi(L_1) \to F(L_1)$. Since from 4.4.6 we have $\Lambda^2(\Sigma_3 \times \Sigma_3; F(K_0)) = 0$, then the conclusion follows.

Case (c).

Assume that $\mathrm{res}^{N_1^\infty}_{L_0}\ \varrho$ contains a subrepresentation isomorphic to η_2^7 and a subrepresentation isomorphic to η_3^7 (otherwise the assumptions of either (a) or (b) are satisfied). The argument given in the proof of Case (a) shows that $\Lambda^2(\Sigma_3 \times \Sigma_3; \mathrm{coker}(\Xi(K_0) \to F(K_0)) = 0$. By 3.3.28 and the assumptions we know that $\mathrm{res}^{N_1^\infty}_{L_1^\infty}\ \varrho$ contains a subrepresentation isomomorphic to $\alpha_{2k'+1}\bar{\otimes}\eta_2^5$, and at least two of three following representations:

$$\theta\bar{\otimes}\eta_2^5, \qquad \tau\bar{\otimes}\eta_2^5, \qquad \beta_{2k}\bar{\otimes}\eta_2^5.$$

There are three summands of $F(L_1^\infty)$ having $\Lambda^2 \neq 0$. These are

$$M(5,2)\{x_{1kl} + x_{2kl}\}_{3\le k<l\le 7} \subseteq F(L_1),$$
$$M(5,2)\{x_{klm}\}_{3\le k<l<m\le 7} \subseteq F(L_1),$$
$$M(5,2)\{x_{kl}\}_{3\le k<l\le 7} \subseteq F(L_1).$$

The image of the submodule of $\Xi(L_1^\infty)$ generated by $\alpha_{2k'+1}\bar{\otimes}\eta_2^5$ is the first of them. Similarly, the submodule corresponding to $\tau\bar{\otimes}\eta_2^5$ maps onto the second module, and finally the submodule corresponding to $\theta\bar{\otimes}\eta_2^5$ maps onto the last one. The image of the submodule generated by irreducible subrepresentations of $\beta_{2k}\bar{\otimes}\eta_2^5$ is generated by elements

$$x_{klm} + x_{no} \in M(5,2)\{x_{klm}\}_{3\le k<l<m\le 7} \oplus M(5,2)\{x_{kl}\}_{3\le k<l\le 7} \subseteq F(L_1^\infty),$$

such that $\{k,l,m,n,o\} = \{1,2,3,4,5\}$. Hence we see that if two of the representations $\theta\bar{\otimes}\eta_2^5$, $\tau\bar{\otimes}\eta_2^5$, $\beta_{2k}\bar{\otimes}\eta_2^5$ are contained if $\mathrm{res}^{N_1^\infty}_{L_1^\infty}\ \varrho$, then

$$\Lambda^2(\Sigma_5; \mathrm{coker}(\Xi(L_1^\infty) \to F(L_1^\infty)) = 0. \qquad \square$$

As a corollary we obtain the following theorem:

Theorem 4.4.8. *Assume that ϱ is an m-dimensional complex $\mathcal{R}$-invariant representation of N_1^∞. If at least one of the following conditions is satisfied:*

(a) $\mathrm{res}_{L_0}^{N_1^\infty} \varrho$ *does not contain a subrepresenation isomorphic to* η_2^7*,*

(b) $\mathrm{res}_{L_0}^{N_1^\infty} \varrho$ *does not contain a subrepresenation isomorphic to* η_3^7*,*

(c) $\mathrm{res}_{L_1}^{N_1^\infty} \varrho$ *contains a subrepresenation isomorphic to either* $\theta\bar{\otimes}\eta_2^5$*, or to* $\tau\bar{\otimes}\eta_2^5$*,*

then the map $B\varrho_2^\wedge : (BN_1)_2^\wedge \to BU(m)_2^\wedge$ *extends to a map* $BSpin(7)_2^\wedge \to BU(m)_2^\wedge$*.*

4.5 Homotopy representations of $SU(2)^n$ and $SU(2)^n/\{\pm 1\}$

Fix a positive integer n. In the present section we give a partial classification of 2-complex homotopy representations of the Lie groups $SU(2)^n$ and $SU(2)^n/\{\pm 1\}$. Let A be a ring of 2-adic integers. From Theorem 3.1.11 follows that there are two (up to conjugacy) 2-stubborn subgroups of $SU(2) = Spin(3)$, namely

$$N := \pi^{-1}((\bar{\Gamma}_2 \times \{\pm 1\}) \cap SO(3)) \tag{4.5.1}$$

which is in fact the 2-normalizer of the maximal torus, and

$$Q := \pi^{-1}(\{\pm 1\}^3 \cap SO(3)) \cong \pi^{-1}(U_3^*), \tag{4.5.2}$$

(see 3.3.19) which is a group of order 8 isomorphic to the quaternion group. Let $T \subseteq N$ be a maximal torus of $SU(2) \cong Spin(3)$. The category $\mathcal{R}_2(SU(2))$ is isomorphic to the category $\mathcal{J} = \mathcal{R}_2(\Sigma_3)$ considered in Section 4.3 (the isomorphism is given in 3.1.27, however in this case it is obvious). Therefore we will denote $\mathcal{R}_2(SU(2))$ for $\mathcal{J}$ for short. We have $\mathcal{R}_2(SU(2)^n) \cong \mathcal{J}^n$, and by 3.2.2 also $\mathcal{R}_2(SU(2)^n/\{\pm 1\}) \cong \mathcal{J}^n$.

We begin with a classification of irreducible representations of N^∞ and Q. As before, let u be a non-trivial lift of unity in $Spin(3)$. The homomorphism

$$\bar{\Gamma}_2 \ni g \mapsto \begin{pmatrix} g & 0 \\ 0 & \det g \end{pmatrix} \in (N^\infty)/\{u\} \subseteq SO(3)$$

(cf. 3.1.2) is an isomorphism. It allows to identify even representations of N^∞ with representations of $\bar{\Gamma}_2$. Moreover, for any 2-adic integer l the representation

$$\lambda_{l+\frac{1}{2}} := \mathrm{ind}_{T^\infty}^{N^\infty}\ \varrho_{l+\frac{1}{2}}$$

is irreducible (by 2.3.32). By combining it with 3.3.7 we obtain

Proposition 4.5.3.

$$\mathrm{IR}(N^\infty) = \{\theta, \tau\} \cup \{\beta_{2l}\}_{l\in A} \cup \{\alpha_{2l+1}\}_{l\in A} \cup \{\lambda_{2l+\frac{1}{2}}\}_{k\in A}$$

$$\mathrm{IR}(Q) = \{\theta, \tau_i, \tau_j, \tau_k, \lambda\}$$

and

$$\mathrm{res}_Q^{N^\infty}\ \varrho = \begin{cases} \theta & \text{for } \varrho = \theta \\ \tau_i & \text{for } \varrho = \tau \\ \theta \oplus \tau_i & \text{for } \varrho = \beta_{2l} \\ \tau_j \oplus \tau_k & \text{for } \varrho = \alpha_{2l+1} \\ \lambda & \text{for } \varrho = \lambda_{l+\frac{1}{2}}, \end{cases} \tag{4.5.4}$$

(τ_i, τ_j, τ_k are defined on p.76 and λ is the only irreducible 2-dimensional representation of Q).

Denote $\tau_* := \tau_i \oplus \tau_j \oplus \tau_k$.

Proposition 4.5.5. *If ϱ is an $\mathcal{J}^n$-invariant representation of $(N^\infty)^n/\{\pm 1\}$, then there is a natural equivalence of $\mathcal{J}^n$-modules $\Xi^\varrho \cong \Xi^{\mathrm{res}_{P^\infty}^{P^\infty/\{\pm1\}}\varrho}$ (cf. 2.4.4):*

$$\Xi^\varrho(P/\{\pm 1\}) \ni [\sigma] \mapsto [\mathrm{res}_{P^\infty}^{P^\infty/\{\pm1\}}\ \sigma] \in \Xi^{\mathrm{res}_{P^\infty}^{P^\infty/\{\pm1\}}\varrho}(P).$$

Proof. It follows from the definition of modules Ξ and the observation that the restriction of an irreducible representation of $(P^\infty)/\{\pm 1\}$ to P^∞ is irreducible. □

We begin, of course, with the case $n = 1$. It shows that there is a significant difference between the representations of Lie groups and homotopy representations — there is much more homotopy representations than algebraic ones.

Proposition 4.5.6. *For each $\mathcal{J}$-invariant m-dimensional complex representation ϱ of N^∞ the map $B\varrho_2^\wedge$ extends uniquely, up to homotopy, to a map $BSU(2)_2^\wedge \to BU(m)_2^\wedge$.*

Proof. By 4.3.1 the assumptions of 2.4.9.(a) and 2.4.9.(b) are satisfied. □

Proposition 4.5.7. *A representation $\varrho \in \mathrm{Rep}(N^\infty)$ is $\mathcal{J}$-invariant if and only if the total multiplicity of subrepresentations isomorphic to α_{2l+1} is equal to the total multiplicity of subrepresentations isomorphic either to β_{2l} or to τ.*

Proof. The representation ϱ is $\mathcal{J}$-invariant if and only if $\mathrm{res}_Q^{N^\infty} \varrho$ is Σ_3-invariant. It happens only if the multiplicities of τ_i, τ_j and τ_k in $\mathrm{res}_Q^{N^\infty} \varrho$ are all equal. Then the conclusion follows from 4.5.4. □

We say that a $\mathcal{J}^n$-invariant represenation of $(N^\infty)^n$ is $\mathcal{J}^n$-irreducible if it does not contain any non-trivial $\mathcal{J}^n$-invariant subrepresentation. Of course, $\mathcal{J}^n$-irreducible representations are not necessarily irreducible.

Corollary 4.5.8. Each $\mathcal{J}$-invariant representation ϱ of N^∞ splits into a direct sum of $\mathcal{J}$-irreducible representations (since if $\varrho = \varrho_1 \oplus \varrho_2$ and ϱ, ϱ_1 are $\mathcal{J}$-invariant, then ϱ_2 also is), but the splitting is not necessarily unique. If ϱ is $\mathcal{J}$-irreducible, then it is isomorphic to one of the following:

$$\theta, \qquad \lambda_{2l+\frac{1}{2}}, \qquad \alpha_{2l+1} \oplus \beta_{2l'}, \qquad \alpha_{2l+1} \oplus \tau.$$

Consider the general case. It comes out that $\mathcal{J}^n$-irreducible representations of $(N^\infty)^n$ are not necessarily tensor products of $\mathcal{J}$-irreducible representations of N^∞.

Let $\mathbf{n} := \{1, 2, \ldots, n\}$ and let ϱ be a representation of $(N^\infty)^n$. For each $k \in \mathbf{n}$ there is a unique presentation

$$\varrho \simeq \bigoplus_{\varphi \in \mathrm{IR}(N^\infty)^{\mathbf{n}\setminus\{k\}}} \varphi \bar{\otimes} \psi_\varphi.$$

Proposition 4.5.9. *The representation ϱ is $\mathcal{J}^n$-invariant if and only if for each k and each $\varphi \in \mathrm{IR}(N^\infty)^{\mathbf{n}\setminus\{k\}}$ the representation ψ_φ is $\mathcal{J}$-invariant.*

Proof. If ϱ is $\mathcal{J}^n$-invariant, then it obviously satisfies the conditions above. Since each morphism of $\mathcal{J}^n$ is a composition of mophisms with all coordinates but one being an identity, the inverse follows. □

Fix an $\mathcal{J}^n$-invariant representation ϱ of $(N^\infty)^n$. We intend to use the spectral sequence 4.1.14 to check when the module Ξ^ϱ satisfies the assumptions of 2.4.9. We will use the gradation on $\mathcal{J}$, given by $|N| = 0$, $|Q| = 1$ extended additively to a gradation on $\mathcal{J}^n$.

Proposition 4.5.10. *Fix $B \subseteq \mathbf{n}$ be a finite set. Then*

$$\Lambda^l(\Sigma_3^B; \bar{\bigotimes_{a\in B}} M(3,1)) = 0$$

for each $l \neq |B|$ and the homomorphism

$$A \simeq A^{\otimes B} \cong \bigotimes_{a\in B} \Lambda^1(\Sigma_3; M(3,1)) \longrightarrow \Lambda^{|B|}(\Sigma_3^B; \bar{\bigotimes_{a\in B}} M(3,1))$$

is an isomorphism.

Proof. It follows from 4.2.6.(d) and 4.2.9. □

Let $E_*^{*,*}$ be the spectral sequence 4.1.14 calculating $H^*(\mathcal{J}^n; \Xi^\varrho)$.

Corollary 4.5.11. Since

$$E_1^{s,t} = \bigoplus_{\substack{B\subseteq \mathbf{n}\\ |B|=s}} \Lambda^{s+t}(\Sigma_3^B; \Xi^\varrho(Q^B \times N^{\mathbf{n}\setminus B}))$$

we have $E_1^{s,t} = 0$ if $t \neq 0$, or if $s > n$. In particular, the spectral sequence degenerates to the following exact sequence (cf. 4.1.15):

$$0 \longrightarrow E_1^{0,0} \longrightarrow E_1^{1,0} \longrightarrow \ldots \longrightarrow E_1^{n-1,0} \longrightarrow E_1^{n,0} \longrightarrow 0.$$

Now we define some basis of modules $E_1^{l,0}$.

Definition 4.5.12. Fix an $\mathcal{J}^{\mathbf{n}}$-invariant representation ϱ of $(N^\infty)^{\mathbf{n}}$. Let $Z_\varrho(n)$ be a subset of $(\mathrm{IR}(N^\infty) \cup \{*\})^{\mathbf{n}}$ containing all elements $(\omega_1, \ldots, \omega_n)$ such that there exists $(\eta_1, \ldots, \eta_n) \in \mathrm{IR}(N^\infty)^{\mathbf{n}}$ satisfying conditions:

(a) $\eta_1 \bar{\otimes} \ldots \bar{\otimes} \eta_n \subseteq \varrho$,

(b) If $\omega_l \in \mathrm{IR}(N^\infty)$, then $\eta_l = \omega_l$,

(c) If $\omega_l = *$, then $\eta_l \in \{\beta_{2m}\}_{m\in\mathbb{Z}_2^\wedge} \cup \{\tau\} \subseteq \mathrm{IR}(N^\infty)$.

Note that (c) is equivalent to the condition that $\mathrm{res}_Q^{N^\infty}\ \eta_l$ contains a subrepresentation isomorphic to τ_i (or τ_j or τ_k). For each $B \subseteq \mathbf{n}$ let

$$Z_\varrho^B(n) := \{(\omega_1, \dots, \omega_n) \in Z_\varrho(n) : (\omega_l = *) \Leftrightarrow (l \in B)\}$$

$$Z_\varrho^r(n) := \bigcup_{|B|=r} Z_\varrho^B(n)$$

Let $\Xi_{\{B\}}^\varrho$ be a suitable atomic submodule of Ξ^ϱ (cf. 4.1.12)

Proposition 4.5.13.

$$H^r(\mathcal{J}^{\mathbf{n}}; \Xi_{\{B\}}^\varrho) \cong \begin{cases} A[Z_\varrho^B(n)] & \textit{for } r = |B| \\ 0 & \textit{for } r \neq |B| \end{cases}$$

Proof. Let M be the restriction of Ξ^ϱ to the subcategory $\mathcal{J}^B \times \{N\}^{\mathbf{n}\setminus B}$. Note that (by 4.1.6)

$$H^r(\mathcal{J}^{\mathbf{n}}; \Xi_{\{B\}}^\varrho) \cong H^r(\mathcal{J}^B; M).$$

$M(Q^B)$ is a free A-module with basis $\mathrm{IR}(Q^B \times (N^\infty)^{\mathbf{n}\setminus B}, \mathrm{res}_{Q^B \times (N^\infty)^{\mathbf{n}\setminus B}}^{(N^\infty)^{\mathbf{n}}}\ \varrho)$ (cf. 2.4.1). Since $\mathrm{res}_{Q^B \times (N^\infty)^{\mathbf{n}\setminus B}}^{(N^\infty)^{\mathbf{n}}}\ \varrho$ is Σ_3^B-invariant, then we have a presentation

$$\mathrm{res}_{Q^B \times (N^\infty)^{\mathbf{n}\setminus B}}^{(N^\infty)^{\mathbf{n}}}\ \varrho \cong \bigoplus_{\substack{(\mu_a) \in (\mathrm{IR}(Q)^{\Sigma_3})^B \\ (\nu_b) \in \mathrm{IR}(N^\infty)^{\mathbf{n}\setminus B}}} \left(\Big(\bar{\bigotimes_{a \in B}} \mu_a \Big) \bar{\otimes} \Big(\bar{\bigotimes_{b \in \mathbf{n}\setminus B}} \nu_b \Big) \right)^{\oplus l((\mu_a),(\nu_b))}$$

where $\mathrm{IR}(Q)^{\Sigma_3} = \{\theta, \lambda, \tau_i \oplus \tau_j \oplus \tau_k\}$. Then we obtain

$$M = \bigoplus_{\substack{(\mu_a) \in (\mathrm{IR}(Q)^{\Sigma_3})^B \\ (\nu_b) \in \mathrm{IR}(N^\infty)^{\mathbf{n}\setminus B} \\ l((\mu_a),(\nu_b))>0}} \Big(\bar{\bigotimes}_{\substack{a \in B \\ \mu_a = (\tau_i \oplus \tau_j \oplus \tau_k)}} M(3,1) \Big)$$

Finally, by 4.5.10

$$H^{|B|}(\mathcal{R}^B; M) = \bigoplus_{\substack{(\mu_a) = \tau_i \oplus \tau_j \oplus \tau_k \\ (\nu_b) \in \mathrm{IR}(N^\infty)^{\mathbf{n}\setminus B} \\ l((\mu_a),(\nu_b))>0}} A = A[Z_\varrho^B(n)]$$

and $H^r(\mathcal{J}^B; M) = 0$ if $r \neq |B|$. ∎

For each $\omega \in \mathrm{IR}(N^\infty) \cup \{*\}$ let

$$|\omega| = \begin{cases} 0 & \text{for } \omega \in \mathrm{IR}(N^\infty) \\ 1 & \text{for } \omega = * \end{cases}.$$

Then

$$E_1^{r,0} \cong \bigoplus_{|B|=r} A[Z_\varrho^B(n)] = A[Z_\varrho^r(n)]$$

and $d_1^{i,0} = \bigoplus_{j=1}^n d_l^i$ where

$$d_l^i(\omega_1, ..., \omega_n) = \begin{cases} (-1)^{|\omega_1|+...+|\omega_{l-1}|}(\omega_1, ..., \omega_{l-1}, *, \omega_{l+1}, ..., \omega_n) & \omega_j = \beta_k, \tau \\ (-1)^{|\omega_1|+...+|\omega_{l-1}|+1}(\omega_1, ..., \omega_{l-1}, *, \omega_{l+1}, ..., \omega_n) & \omega_j = \alpha_k \\ 0 & \omega_j = *, \theta, \lambda_k \end{cases}$$

As a corollary we obtain

Proposition 4.5.14. $H^*(E_1^{*,0}, d_1^{*,0}) \cong H^*(\mathcal{R}^n; \Xi^\varrho)$.

Proof. Straightforward from 4.1.15, 4.5.4 and 4.5.13. □

Proposition 4.5.15. *Let ϱ be an $\mathcal{J}^n$-invariant representation of $(N^\infty)^n$. Then $H^k(\mathcal{R}^n; \Xi^\varrho) = 0$ for $k \geq n$.*

Proof. By 4.5.14 it suffices to show that $H^k(A[Z_\varrho^*(n)]) = 0$. For $k > n$ it is clear since $Z_\varrho^k(n) = \emptyset$. Moreover $Z_\varrho^n(n)$ is either empty or it contains only $(*, \dots, *)$. But if $Z_\varrho^n(n) \neq \emptyset$, then there is $\alpha_l \in \mathrm{IR}(N^\infty)$ such that $(\alpha_l, *, \dots, *) \in Z_\varrho^{n-1}(n)$. Hence $(*, \dots, *) = \pm d_1^{n-1,0}(\alpha_l, *, \dots, *)$. □

Corollary 4.5.16. Let ϱ be an $\mathcal{J}^n$-invariant representation of $(N^\infty)^n$. If $n \leq 3$, then the representation ϱ extends to a homotopy representation of $SU(2)^n$. If $n \leq 2$, then the extension is unique.

Proof. It is straightforward from 4.5.15 and 2.4.9. □

Corollary 4.5.17. Let ϱ be an $\mathcal{J}^n$-invariant representation of $(N^\infty)^n/\{\pm 1\}$. If $n \leq 3$, then the representation ϱ extends to a homotopy representation of $SU(2)^n/\{\pm 1\}$. If $n \leq 2$, then the extension is unique.

Proof. By 4.5.5 it is an immediate consequence of 4.5.16. □

Now we will concentrate on the case $n = 3$. Each $\mathcal{R}^3$-invariant representation of $(N^\infty)^3$ extends to a map $(BSU(2)^3)^\wedge_2 \to BU(m)^\wedge_2$, although it possibly exists more than one extension.

Definition 4.5.18. Let

$$Z^2_\varrho(3)^\pm := \{(*,\ldots,*,\omega,*,\ldots,*) \in Z^2_\varrho(3) : \omega \in \mathrm{IR}(N^\infty) \setminus \{\lambda_{k+\frac{1}{2}}\}\}$$

$$Z^2_\varrho(3)^0 := Z^2_\varrho(3) \setminus Z^2_\varrho(3)^\pm$$

Let $\approx$ be a relation on $Z^2_\varrho(3)^\pm$ given by

$$(\sigma_1,*,*) \approx (*,\sigma_2,*) \Leftrightarrow (\sigma_1,\sigma_2,*) \in Z^1_\varrho(3)$$

$$(\sigma_1,*,*) \approx (*,*,\sigma_3) \Leftrightarrow (\sigma_1,*,\sigma_3) \in Z^1_\varrho(3)$$

$$(*,\sigma_2,*) \approx (*,*,\sigma_3) \Leftrightarrow (*,\sigma_2,\sigma_3) \in Z^1_\varrho(3)$$

Let $\sim$ be an equivalence relation spanned by $\approx$.

Let $\bar{A}[Z^2_\varrho(3)^\pm/\sim]$ be the kernel of the augmentation $A[Z^2_\varrho(3)^\pm/\sim] \to A$.

Proposition 4.5.19. $H^2(\mathcal{J}^3; A[Z^*_\varrho(3)]) \simeq \bar{A}[Z^2_\varrho(3)^\pm/\sim]$.

Proof. Let $\varphi : A[Z^2_\varrho(3)^\pm] \to A[Z^2_\varrho(3)^\pm]$ be an automorphism defined as follows:

$$\varphi(*,\ldots,*,\omega_i,*,\ldots,*) = (-1)^{i-1} sgn(\omega_i)(*,\ldots,*,\omega_i,*,\ldots,*)$$

where $sgn(\alpha_k) = -1$ and $sgn(\beta_k) = sgn(\tau) = 1$. If $b \approx b'$, then we have

$$\begin{aligned}\varphi(b-b') &= (-1)^{i-1} sgn(\omega_i)(*,\ldots,*,\omega_i,*,\ldots,*)+\\ &\quad + (-1)^{i'} sgn(\omega_{i'})(*,\ldots,*,\omega_{i'},*,\ldots,*) =\\ &\quad\quad = d_1^{1,0}(*,\ldots,*,\omega_i,*,\ldots,*,\omega_{i'},*,\ldots,*) \in \mathrm{im}\, d_1^{1,0}\end{aligned}$$

and hence $\mathrm{im}\, d_0^{1,0} = \varphi(\langle b-b'\rangle_{b\approx b'}) = \varphi(\langle b-b'\rangle_{b\sim b'})$. The argument similar to the one used in proof of 4.5.15 shows that $\bar{A}[Z^2_\varrho(3)^0] \subseteq d_1^{1,0}$. Moreover,

$$d_1^{2,0}(\varphi((*,\ldots,*,\omega_i,*,\ldots,*)) = (*,\ldots,*) \in Z^3_\varrho(3).$$

It implies that $\varphi(\sum b_i) \in \ker d_1^{2,0}$ if and only if $\sum b_i \in \bar{A}[Z^2_\varrho(3)^\pm]$. □

The following theorem constitutes a crucial step in the construction of a faithful representation of the 2-compact group $DI(4)$:

Theorem 4.5.20. *(a) Let ϱ be an $\mathcal{J}^3$-invariant representation of $(N^\infty)^3$. Then there is an bijection between the set of extensions of $B\varrho_2^\wedge$ to a map $BSU(2)^3 \to BU(m)_2^\wedge$ and the set $\bar{A}[Z_\varrho^2(3)^\pm/\sim]$.*

(b) Let ϱ' be an $\mathcal{J}^3$-invariant representation of $(N^\infty)^3/\{\pm 1\}$ and let $\varrho := \mathrm{res}_{(N^\infty)^3}^{(N^\infty)^3/\{\pm 1\}} \varrho'$. Then there is a bijection between the set of extensions of $(B\varrho')_2^\wedge$ to a map $B(SU(2)^3/\{\pm 1\}) \to BU(m)_2^\wedge$ and the set $\bar{A}[Z_\varrho^2(3)^\pm/\sim]$.

Proof. Case (a) follows immediately from 4.5.19 and 1.4.13. Case (b) reduces to Case (a) by 4.5.5. □

In Sections 1.4 and 5.4 we prove that the set of extensions has a structure of free and transitive $\bar{A}[Z_\varrho^2(3)^\pm/\sim]$-set and that this structure is functorial in some sense.

Chapter 5

A faithful representation of $DI(4)$

The present chapter contains a construction of a faithful complex representation of the 2-compact group $DI(4)$. Recall (cf. 1.5) The classifying space $BDI(4)$ is the 2-completion of the homotopy colimit of the following diagram :

$$\overset{GL_4(\mathbb{F}_2)}{\curvearrowright} B\{\pm 1\}^4 \rightrightarrows \overset{GL_3(\mathbb{F}_2)}{\curvearrowright} B((S^1)^3 \ltimes \{\pm 1\})^\wedge_2 \rightrightarrows \overset{GL_2(\mathbb{F}_2)}{\curvearrowright} B(SU(2)^3/\{\pm 1\})^\wedge_2 \rightrightarrows BSpin(7)^\wedge_2.$$

To simplify the notation, we denote $D := DI(4)$, $G := Spin(7)$, $L := SU(2)$ $H := L^3/\{\pm 1\}$. Let T be the torus defined in 3.4.3. The group T is a maximal torus of both G and H and its 2-completion is a maximal torus of D. Let W_D, W_G, W_H be the Weyl groups of D, G and H respectively. The group $N_G := N_1$ is a 2-normalizer of T in G (cf. 3.2.8). As usual, for any 2-toral group P its discrete approximation is denoted P^∞. Let $\mathcal{A} := \mathcal{A}_2(DI(4))$ be a centralizer decomposition category of $DI(4)$.

Outline of the construction

The construction is made in the following steps:

- *Choose a suitable faithful unitary representation ϱ of N_G^∞.* The term "suitable" means that we will be able to perform the next steps. This representation has to satisfy some obvious conditions, namely its restriction to T^∞ is W_D-invariant, its restriction to $N_H(T)^\infty$ is Σ_3-invariant and finally its restriction to $(\mathbb{Z}/2)^4$ is $\mathrm{GL}_4(\mathbb{F}_2)$-invariant. Author thinks that any such representation extends to D, however he knows only few examples. The one used in the construction is the easiest to describe. One would expect a representation of dimension $\dim(D) = 45$ which would be an analogue of the adjoint representation. Unfortunately, there is no such a representation (5.3.6).

- *Check that ϱ is $\mathcal{R}_2(G)$-invariant.* We use (5.3) an argument which does not use the classification of representations of stubborn subgroups of G. We obtain a homotopy compatible family of maps from the stubborn decomposition diagram of G into $BU(m)^\wedge_2$.

- *Extend this family to a map $f_G : BG^\wedge_2 \to BU(m)^\wedge_2$.* It follows easily from Theorem 4.4.8.

- *Prove that f_G is $\mathcal{A}$-invariant.* The difficult part is to prove that $f_H := f_G|_{BH^\wedge_2}$ is Σ_3-invariant. For any choice of ϱ the problem we face here cannot be solved using the traditional obstruction theory because the restriction of f_H to $N_H(T)^\infty_2$ has infinitely many non-homotopic extensions to $BH^\wedge_2$. We use here more subtle methods, developed in Sections 1.4, 5.4 and 5.5.

- *Extend f_G to a map $f_D : BDI(4) \to BU(m)^\wedge_2$* — see Section 5.6.

5.1 Bases of T^∞

Let x, y and z be the generators of T^∞ given in 3.5 (recall that in fact they depend on n and they are generators of the finite subgroup $T^{(n)} \subset T^\infty$). The formulae

$$(2^{-n}\mathbb{Z})^3 \ni (q_1, q_2, q_3) \mapsto x^{q_1 2^n} y^{q_2 2^n} z^{q_3 2^n} \in T^{(n)}$$

define a homomorphism $\mathbb{Z}[\frac{1}{2}] \to T^\infty$, which induces an isomorphism

$$\mathbb{Z}[\tfrac{1}{2}]^3/\{(k_1, k_2, k_3) : k_i \in \mathbb{Z} \wedge k_1 + k_2 + k_3 \equiv 0 \mod 2\} \cong T^\infty.$$

In order to present an automorphism of T^∞ as a matrix, or to present an irreducible representation of T^∞ as a sequence of numbers, we need to choose a basis of $\mathbb{Z}[\frac{1}{2}]^3$. We will use three different bases of $\mathbb{Z}[\frac{1}{2}]^3$, namely:

$$e_1 = (1,0,0), \qquad e_2 = (0,1,0), \qquad e_3 = (0,0,1), \tag{5.1.1}$$
$$e'_1 = (0,1,1), \qquad e'_2 = (1,0,1), \qquad e'_3 = (1,1,0), \tag{5.1.2}$$
$$e''_1 = (\tfrac{1}{2},\tfrac{1}{2},0), \qquad e''_2 = (\tfrac{1}{2},-\tfrac{1}{2},0), \qquad e''_3 = (0,0,-\tfrac{k-1}{2}), \tag{5.1.3}$$

where k is an odd 2-adic integer such that $k^2 - k + 2 = 0$ (note that $-\frac{k-1}{2} = k^{-1}$ and $k \equiv 3 \mod 8$). The bases $\{e_i\}$ and $\{e'_i\}$ may be regarded as "natural bases" of the maximal torus of $SO(7)$ and $Spin(7)$ respectively. The basis $\{e''_i\}$ is convenient for considering homotopy representations of $H = L^3/\{\pm 1\}$.

By *a root* we mean an irreducible representation of a 2-discrete torus. As shown in 2.3.30, each irreducible representation of the standard 2-discrete torus $(\mathbb{Z}/2^\infty)^r \simeq \mathbb{Z}[\frac{1}{2}]^r/\mathbb{Z}^r$ has the form $\varrho_{k_1,\ldots,k_r}$, where $k_i \in \mathbb{Z}_2^\wedge$ (and the sequence $(k_1,\ldots,k_r)$ will be also called a root). In general, for a given irreducible representation the corresponding sequence of 2-adic numbers depends on the choice of a basis. Further we use these bases as dual bases of a space of roots.

Example 5.1.4. The set of roots of T^∞ in the basis $\{e_i\}$ is (cf. 3.5.2)

$$\{(k_1,k_2,k_3 \in \tfrac{1}{2}\mathbb{Z}_2^\wedge) : k_1 \equiv k_2 \equiv k_3 \mod 1\}.$$

The set of roots of T^∞ in the basis $\{e'_i\}$ is $\mathbb{Z}_2^\wedge$. The set of roots of T^∞ in the basis $\{e''_i\}$ is

$$\{(k_1,k_2,k_3 \in \tfrac{1}{2}\mathbb{Z}_2^\wedge) : k_1 + k_2 + k_3 \equiv 0 \mod 1\}.$$

5.2 The Weyl group of $DI(4)$

Recall (1.5) that the Weyl group $W := W_D$ of $DI(4)$ is isomorphic to the group $\{\pm 1\} \times \mathrm{GL}_3(\mathbb{F}_2)$. The group W_D acts on the group $(\mathbb{Z}/2^\infty)^3$ (and this action induces the action on the classifying space of the maximal torus $BT_2^\wedge \simeq K(\mathbb{Z}_2^\wedge, 2)^3$). The representation $W_D \to \mathrm{GL}_3(\mathbb{Z}_2^\wedge)$ satisfies the following conditions:

- The generator of the first factor $\{\pm 1\}$ maps to the matrix $-I$.
- The composition $\mathrm{GL}_3(\mathbb{F}_2) \subset W_D \to \mathrm{GL}_3(\mathbb{Z}_2^\wedge) \xrightarrow{q} \mathrm{GL}_3(\mathbb{F}_2)$, where q is induced by the reduction $\mathbb{Z}_2^\wedge \to \mathbb{F}_2$, is an identity.

- The composition $W_G \subset W_D \to \mathrm{GL}_3(\mathbb{Z}_2^\wedge)$ induces the completed action of the Weyl group of G on T.

The present section contains a detailed description of this action.

In basis $\{e_i\}$ the group W_G acts on T^∞ by ± 1-permutation matrices (since the action is inherited from $SO(7)$). By changing a basis to $\{e_i'\}$ we obtain the following

Corollary 5.2.1. In basis $\{e_i'\}$ the Weyl group W_G is generated by the following matrices:

$$-I, \qquad a = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$r = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad s = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The homomorphism $W_G \to \{\pm 1\} \times \mathrm{GL}_3(\mathbb{F}_2) \cong W_D$, which maps $-I$ onto $(-I, I)$ and the generators a, b, r and s onto their mod 2 reductions is an embedding.

Proposition 5.2.2. *The group* $\mathrm{GL}_3(\mathbb{F}_2)$ *is generated by mod* 2 *reductions of elements* a, b, r, s, *and by the element*

$$v = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

They satisfy the following relations:

$$a^2 = b^2 = r^3 = s^2 = v^7 = I$$

$$ba = ab, \quad ra = abr, \quad rb = ar, \quad sa = bs, \quad sb = as, \quad sr = r^2 s,$$
$$va = r^2 sv, \quad vb = brsv^5, \quad vr = r^2 v^6, \quad vs = br^2 v^6.$$

Proof. It is clear that $\langle a, b, r, s\rangle$ is a subgroup of order 24, and that v has order 7. Therefore the group $\langle a, b, r, s, v\rangle$ has an order at least 168 and hence it is equal $\mathrm{GL}_3(\mathbb{F}_2)$. Elementary computations show that the relations above are satisfied. Moreover, the list of relations is complete, since they allow to write every element of $\mathrm{GL}_3(\mathbb{F}_2)$ in the form $a^\alpha b^\beta r^\varrho s^\sigma v^\omega$ in the unique way, where $0 \le \alpha < 2$, $0 \le \beta < 2$, $0 \le \varrho < 3$, $0 \le \sigma < 2$, $0 \le \omega < 7$. □

Proposition 5.2.3. *The generator $v \in \mathrm{GL}_3(\mathbb{F}_2) \subset W_D$ lifts to the matrix*

$$\begin{pmatrix} 0 & -3h+1 & -h \\ -1 & h-1 & -h \\ 0 & h & 3h-1 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{Z}_2^\wedge),$$

where h is the only root of the equation $4x^2 - 3x + 1 = 0$ in $\mathbb{Z}_2^\wedge$ (we still use the basis $\{e_i'\}$).

Proof. It is sufficient to check that this matrix and the matrices enlisted in 5.2.1 satisfy the relations in 5.2.2. □

By changing basis again from $\{e_i'\}$ to $\{e_i\}$ we obtain

Corollary 5.2.4. In the basis $\{e_i\}$ the generator $v \in W_D$ is given by the matrix

$$\frac{1}{2}\begin{pmatrix} 4h-1 & -1 & -1 \\ 0 & 4h-2 & -4h+2 \\ -4h+1 & -1 & -1 \end{pmatrix}$$

5.3 A representation of N_G^∞.

In the present section we construct a complex representation φ of $N_G^\infty \subset G$, which will be extended later to a map $BG_2^\wedge \to BU(m)_2^\wedge$. We use a representation having a huge dimension. The main reason for this particular choice is that it is easy to prove $\mathcal{R}_2(G)$-invariance of φ. Moreover, as we will see (5.3.5, 5.3.6) there is no analogue of an adjoint representation.

Throughout the present section we use the basis $\{e_i\}$.

Define a representation of N_G^∞ by

$$\varphi := \mathrm{ind}_{T^\infty}^{N_G^\infty} \left(\bigotimes_{w \in W_D/(W_D)_{\varrho_{1,0,0}}} (\theta \oplus w^* \varrho_{1,0,0}) \right). \tag{5.3.1}$$

(θ stands for a trivial one-dimensional representation). Obviously φ is not a restriction of a continuous representation of N_G since coordinates of its roots are not integers (in any basis).

Proposition 5.3.2. *The character of φ is W_D-invariant on T^∞ and vanishes on $N_G^\infty \setminus T^\infty$.*

Proof. It is an immediate consequence of the definition. □

Proposition 5.3.3. *The isotropy group* $(W_D)_{\varrho_{1,0,0}} \subseteq W_D$ *has order* 8. *An* W_D*-orbit of* $\varrho_{1,0,0}$ *contains exactly the representations* ϱ_{k_1,k_2,k_3}*, where*

$$\begin{aligned}(k_1,k_2,k_3) \in \{&(\pm1,0,0),(0,\pm1,0),(0,0,\pm1),\\ &(\pm\tfrac{4h-1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2}),(\pm\tfrac{1}{2},\pm\tfrac{4h-1}{2},\pm\tfrac{1}{2}),(\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{4h-1}{2}),\\ &(\pm(2h-1),\pm(2h-1),0),(\pm(2h-1),0,\pm(2h-1)),(0,\pm(2h-1),\pm(2h-1))\}\end{aligned}$$

and h *is defined in 5.2.3.*

Proof. By acting with $v \in W_D$ (cf. 5.2.4) and W_D we can easily produce the representations listed above. On the other hand, there are eight obvious elements in W_G which fix $\varrho_{1,0,0}$. Therefore the orbit contains exactly 42 elements. □

Corollary 5.3.4. $\dim\varphi = 2^{46}$.

It is a convenient place to prove the non-existence of an "adjoint" representation of D.

Proposition 5.3.5. *Let* ψ *be a representation of* N_G^∞ *containing root* $(1,0,0)$*. Assume that* $\dim\psi = \dim D = 45$*, and that* $\operatorname{res}^{N_G^\infty}_{T^\infty}\psi$ *is* W_D*-invariant. Then* ψ *is* not $\mathcal{R}_2(G)$*-invariant.*

Proof. By 5.3.3 an odd part of ψ contains the following roots

$$(\pm\tfrac{4h-1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2}),\quad (\pm\tfrac{1}{2},\pm\tfrac{4h-1}{2},\pm\tfrac{1}{2}),\quad (\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{4h-1}{2}).$$

Therefore $\psi \simeq \xi_{\{\frac{1}{2},-\frac{4h-1}{2}\};\frac{1}{2}} \oplus \xi^{\varepsilon}_{\frac{1}{2};-\frac{4h-1}{2}}$ for $\varepsilon \in \{+,-\}$ (cf. 3.5.21, 3.5.24). Then $\operatorname{res}^{N_G^\infty}_{K_1^\infty}\psi$ is isomorphic to $\mu^+_{2;\frac{1}{2}} \oplus \mu^-_{2;\frac{1}{2}} \oplus \mu^{\varepsilon}_{1;-\frac{4h-1}{2}}$ (cf. 3.5.22, 3.5.24) and it is not $W_G(K_1)$-invariant (cf. 3.5.13, 3.5.26). □

Corollary 5.3.6. A classifying map of a complexification of an adjoint representation of G does not extend to any 2-compact unitary representation of D.

The remaining part of this section is devoted to a proof of $\mathcal{R}_2(G)$-invariance of φ. Fix a positive integer n. Let x, y, z be elements of $N_G^\infty = N_1^infty$ defined in 3.4.8. The representation $\operatorname{res}^{N_G^\infty}_{T^\infty}\varphi$ is W_G-invariant, i.e. invariant under any permutation of the generators $x,y,z \in T^\infty$ and any of the involutions $x \leftrightarrow x^{-1}$, $y \leftrightarrow y^{-1}$, $z \leftrightarrow z^{-1}$.

Define $Z_\varphi := \{t \in T^\infty : \chi_\varphi(t) = 0\}$.

Proposition 5.3.7. *Fix integers k, l, m. If any of numbers k, l, m, $k \pm l$, $k \pm m$, $l \pm m$ is equal 2^{n-1} (mod 2^n), then $x^k y^l z^m \in Z_\varphi$.*

Proof. If $k \equiv 2^{n-1} \mod 2^n$, then $\chi_{\varrho_{1,0,0}}(x^k y^l z^m) = e(k) = -1$. Then

$$\chi_{\theta \oplus \varrho_{1,0,0}}(x^k y^l z^m) = 0 \Rightarrow \chi_\varphi(x^k y^l z^m) = 0.$$

By replacing succesively $\varrho_{1,0,0}$ by $\varrho_{0,1,0}$, $\varrho_{0,0,1}$, $\varrho_{2h-1,\pm(2h-1),0}$, $\varrho_{2h-1,0,\pm(2h-1)}$, $\varrho_{0,2h-1,\pm(2h-1)}$ and using an analogous argument we obtain the conclusion in the other cases (note that $h \equiv -1 \mod 8$). □

Proposition 5.3.8. *Let P be one of stubborn subgroups listed in 3.2.8 and $g \in G$. Assume that $g^{-1}Pg \subseteq N_G$. Then there exists $w \in W_G$ such that for each $t \in T(P^\infty)$ (cf. 2.1.2) holds $g^{-1}tg = w(t)$.*

Proof. By 3.1.12 the conjugation by g is a composition of some permutation of irreducible factors of P (formally, irreducible factors of the corresponding stubborn subgroup of $SO(7)$, cf. 3.2.1), some automorphism of P (in the category $\mathcal{R}$) and an inclusion $P \subseteq N_1 = N_G$. If P is finite (i.e. $P = L_0, M_0, K_0$), then the conclusion is obvious. For $P = L_1, L_3$ we see that g permutes generators x, y, z. If $P = N_i, \bar{M}_i, M_1, K_1$, then each permutation of irreducible factors and each $\mathcal{R}_2(G)$-automorphism of P acts trivially on $T(P)$. □

Proposition 5.3.9. *Let P be one of stubborn subgroups listed in 3.2.8 and $g \in G$. Assume that $g^{-1}Pg \subseteq N_1$. Then for any $t \in P^\infty \setminus T(P^\infty)$ either $g^{-1}tg \in Z_\varphi$, or $g^{-1}tg \notin T^\infty$.*

Proof. Define the set

$$Y(P) := \bigcup_{g: g^{-1}Pg \subseteq N_1} g^{-1}(P^\infty \setminus T(P^\infty))g \cap T^\infty.$$

It is sufficient to prove that $Y(P) \subseteq Z_\varphi$. Using the presentation (3.4.8) and the classification of morphisms in $\mathcal{R}_2(G)$ (3.1.12, 3.2.8) we obtain that:

$$Y(N_1) = Y(L_3) = \emptyset,$$

and for $P \neq N_1, L_3$ we have an inclusion

$$\begin{aligned} Y(P) \subseteq \{x^{\pm 2^{n-1}} y^l z^m\}_{l,m \in \mathbb{Z}} \cup \{x^k y^{\pm 2^{n-1}} z^m\}_{k,m \in \mathbb{Z}} \cup \{x^k y^l z^{\pm 2^{n-1}}\}_{k,l \in \mathbb{Z}} \\ \cup \{x^{\pm 2^{n-2}} y^{\pm 2^{n-2}} z^m\}_{m \in \mathbb{Z}} \cup \{x^{\pm 2^{n-2}} y^l z^{\pm 2^{n-2}}\}_{l \in \mathbb{Z}} \cup \{x^k y^{\pm 2^{n-2}} z^{\pm 2^{n-2}}\}_{k \in \mathbb{Z}}. \end{aligned}$$

□

Proposition 5.3.10. *The representation φ is $\mathcal{R}_2(G)$-invariant.*

Proof. Let P be one of stubborn subgroups listed in 3.2.8 and let $g \in G$ be such that $g^{-1}Pg \subseteq N_1$. We have to prove that for each $t \in P^\infty$ holds $\chi_\varphi(t) = \chi_\varphi(g^{-1}tg)$. If $t \in T(P^\infty)$, then the conclusion follows from 5.3.8 (since $\mathrm{res}^{N_1^\infty}_{T^\infty} \varphi$ is W_G-invariant). If $t \notin T(P^\infty)$, then by 5.3.9 we have $\chi_\varphi(h) = 0 = \chi_\varphi(g^{-1}hg)$. □

Proposition 5.3.11. *The represenation φ extends to a map $f_G : BG \to BU(m)^\wedge_2$ (where $m = 2^{46}$).*

Proof. We need to check the assumptions of Theorem 4.4.8. In fact, φ contains a subrepresentation

$$\mathrm{ind}^{N_1^\infty}_{T^\infty} (\varrho_{0,0,1} \otimes \varrho_{0,1,0}) = \mathrm{ind}^{N_1^\infty}_{T^\infty} \varrho_{0,1,1}.$$

Hence $\mathrm{res}^{N_1^\infty}_{L_3^\infty} \varphi$ contains a representation $\theta\bar{\otimes}\alpha_1\bar{\otimes}\alpha_1$ and therefore $\mathrm{res}^{N_1^\infty}_{L_1^\infty} \varphi$ contains $\theta\bar{\otimes}\eta_2^5$ (cf. 3.3.26). Then φ satisfies the condition 4.4.8.(c). □

5.4 Adams operations on $BSU(2)^\wedge_2$

Denote $L := SU(2)$ and $\mathcal{J} := \mathcal{R}_2(L)$. Let N and Q be 2-stubborn subgroups of L defined in 4.5.1 and 4.5.2. Let

$$F := (EL \times_L (-))^\wedge_2 : \mathcal{J} \longrightarrow \mathbf{Sp} \tag{5.4.1}$$

be a stubborn decomposition diagram of L. In the present section we construct an action of the group of homotopy self-equivalences of $(BL^\wedge_2)^n$ on a functor which is homotopy equivalent to the stubborn decomposition functor of L^n. Next, we apply this construction to the description of 2-homotopy representations of $H \cong L^3/\{\pm 1\}$ which are extensions of $\mathrm{res}^{N_G^\infty}_{N_H^\infty} \varphi$. These results play a crucial role in the construction of a faithful representation of the 2-compact group $DI(4)$.

Proposition 5.4.2. *For each odd $k \in \mathbb{Z}^\wedge_2$ there exists a unique homotopy self-equivalence ψ_k of $BL^\wedge_2$ (called* the Adams operation*) such that its restriction to the completion of the maximal torus of L is induced by the multiplication by k. Moreover, any self-equivalence of $BL^\wedge_2$ is, up to homotopy, an Adams operation, and ψ_k is homotopic to $\psi_{k'}$ if and only if $k = \pm k'$.*

Proof. It is a consequence of [DW1, 5.4]. $\square$

For any ring R let R^* denote the multiplicative group of its invertible elements and for any space X let $\mathrm{HAut}(X)$ be the group of homomopy classes of self-equivalences of X. Since $\psi_k \circ \psi_{k'} = \psi_{kk'}$, then by 5.4.2 we have

$$\mathrm{HAut}(BL_2^\wedge) \simeq (\mathbb{Z}_2^\wedge)^*/\{\pm 1\} \simeq \{\psi_k : k \in 1 + 4\mathbb{Z}_2^\wedge\}.$$

Before going any further we introduce some definitions. Recall that C_r denotes the cyclic group of order r. Let K_n be the field with 5^{2^n} elements and let $K := \bigcup_n K_n$. Let $S \subseteq K^*$ be the subgroup of all 2^r-th roots of unity and let $S_n := S \cap K_n$. Similarily, let $\Psi := \{4l + 1 \in (\mathbb{Z}_2^\wedge)^*\}$ and let $\Psi_n := \{4l + 1 \in C^*_{2^{n+2}}\}$. Note that there exists a natural isomorphism $(\mathbb{Z}_2^\wedge)^* \to \mathrm{Aut}(S)$ given by $l(x) = x^l$.

Proposition 5.4.3. *The group S_n is cyclic of order 2^{n+2}.*

Proof. By induction we prove that 2^{n+2} is the greatest power of 2 which divides $5^{2^n} - 1$. Since S_n is the 2-Sylow subgroup of $K_n^* \simeq C_{5^{2^n}-1}$, then it is cyclic and has order 2^{n+2}. $\square$

Proposition 5.4.4. *The group Ψ_n is cyclic of order 2^n and 5 is its generator.*

Proof. For $n = 1$ the conclusion is clear, so assume that $n > 1$ and that 5 is a generator of Ψ_{n-1}. The only element of order 2 in Ψ_{n-1} is $2^n + 1$; therefore $5^{2^{n-2}} \equiv 2^n + 1 \mod 2^{n+1}$. But both lifts of $2^n + 1$ to Ψ_n have order 4. It implies that 5 has order 2^n in Ψ_n. $\square$

Proposition 5.4.5. *There exists a homomorphism $\alpha : \Psi \to \mathrm{Aut}(K)$ such that for each $x \in S$ holds $\alpha(l)(x) = x^l$.*

Proof. For each n let $\alpha_n : \Psi_n \to \mathrm{Aut}(K_n)$, where α_n maps the generator 5 onto the Frobenius automorphism. By passing to the limit we obtain the homomorphism $\alpha : \Psi \to \mathrm{Aut}(K)$ which satisfies the required conditions. $\square$

Define the following subgroups of the special linear group $SL_2(K)$:

$$N' = \left\langle \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}_{x \in S}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle \qquad Q' = \left\langle \begin{pmatrix} \zeta_4 & 0 \\ 0 & \zeta_4^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

where ζ_4 is a 4-th root of unity. Let $\mathcal{J}'$ be a full subcategory of the category of $SL_2(K)$-orbits containing only objects $SL_2(K)/N'$ and $SL_2(K)/Q'$. There is an obvious natural equivalence $\mathcal{J}' \cong \mathcal{J}$. Define a functor

$$F' := (ESL_2(K) \times_{SL_2(K)} (-)_2^\wedge : \mathcal{J}' \to \Psi - \mathbf{Sp}, \tag{5.4.6}$$

where Ψ acts on $ESL_2(K)$ in the way given in 5.4.5.

Let $\operatorname{res}^{\mathbf{Sp}}_{\mathbf{HSp}}$ be the forgetful functor from the category of spaces into the homotopy category.

Now our goal is to prove that the functors F and F' are *weakly equivalent*, i.e. that there exists a natural transformation $T : F \to F'$ (in the category of diagrams of spaces) which induces an equivalence $\operatorname{res}^{\mathbf{Sp}}_{\mathbf{HSp}} F \xrightarrow{\cong} \operatorname{res}^{\mathbf{Sp}}_{\mathbf{HSp}} F'$. This kind of problems was considered in [DK]. Let us recall main resuls of this paper:

Definition 5.4.7. A map $f : X \to Y$ is *centric*, if it induces a weak homotopy equivalence $\operatorname{map}(X,X)_{id_X} \to \operatorname{map}(X,Y)_f$. A diagram of spaces is centric, if every map in this diagram is centric.

Remark. It is clear that whether or not a map is centric is a homotopy property. Therefore we can say that a homotopy class of maps (or a diagram in the homotopy category) is centric.

Given a centric diagram $D : \mathcal{C} \to \mathbf{Sp}$ (or $D : \mathcal{C} \to \mathbf{HSp}$) construct functors $\alpha_i(D) : \mathcal{C}^{op} \to \mathbf{Ab}$ by setting $\alpha_i(C) = \pi_i(\operatorname{map}(D(C), D(C))_{id_{D(C)}})$. For each map $c : C_0 \to C_1$ let $\alpha_i(c)$ be a composite

$$\begin{aligned}\pi_i(\operatorname{map}(D(C_1), D(C_1))_{id_{D(C_1)}}) &\xrightarrow{D(c)^*} \pi_i(\operatorname{map}(D(C_0), D(C_1))_{D(c)}) \\ &\xrightarrow{D(c)_*} \pi_i(\operatorname{map}(D(C_0), D(C_0))_{id_{D(c)}})\end{aligned}$$

Theorem 5.4.8. ([DK, 1.1(b)]) *Suppose that $\mathcal{C}$ is a small category and that D, D' are centric diagrams of spaces such that its restrictions to the homotopy category $\operatorname{res}^{\mathbf{Sp}}_{\mathbf{HSp}} D$ and $\operatorname{res}^{\mathbf{Sp}}_{\mathbf{HSp}} D'$ are naturally equivalent. If the groups $H^{i+1}(\mathcal{C}; \alpha_i)$ vanish for $i \geq 1$, then D and D' are weakly equivalent.* □

Proposition 5.4.9. *The diagram $F : \mathcal{J} \to \mathbf{Sp}$ is centric.*

Proof. Let N, Q be the 2-stubborn subgroups of L defined in 4.5. Since each self-homotopy equivalence is centric, it is sufficient to check that the map $BQ \to (BN)_2^\infty$ induced by the inclusion $i : Q \subset N$ is centric. By the Dwyer-Zabrodsky theorem we have a sequence of homotopy equivalences

$$B\{\pm 1\} \cong BZ(Q) \xrightarrow{\cong} (\mathrm{map}(BQ, BQ)_{id})_2^\wedge \\ \xrightarrow{Bi_*} (\mathrm{map}(BQ, BN_2^\wedge)_{Bi})_2^\wedge \xleftarrow{\cong} BC_N(Q) \cong B\{\pm 1\}.$$

The conclusion follows. □

Corollary 5.4.10. The diagrams F and F' are weakly equivalent. In particular, there is a homotopy equivalence $(\mathrm{hocolim}_{\mathcal{J}'} F')_2^\wedge \to BL_2^\wedge$.

Proof. Obviously both F and F' are centric. By 4.3.1 the cohomology groups of $\mathcal{J}$ with any coefficients vanish above dimension 1. The conclusion follows from 5.4.8. □

As a consequence we obtain the following

Theorem 5.4.11. *For each $\psi \in \Psi \simeq \mathrm{HAut}(BL_2^\wedge)$ the following diagram commutes up to homotopy:*

$$\begin{array}{ccc} \mathrm{hocolim}_{\mathcal{J}} F' & \xrightarrow{\alpha} & BL_2^\wedge \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\ \mathrm{hocolim}_{\mathcal{J}} F' & \xrightarrow{\alpha} & BL_2^\wedge \end{array}$$

Proof. By 5.4.5 ψ acts on the completed maximal torus of L by a multiplication. The conclusion follows from 5.4.2. □

By [DW1, 5.5], each homotopy self-equivalence of the spaces $(BL_2^\wedge)^n$ is a composition of a permutation of factors and some Adams operations on its multiplies. More precisely, $\mathrm{HAut}((BL_2^\wedge)^n) \cong \mathrm{HAut}(BL_2^\wedge) \wr \Sigma_n$. Here follow the generalization of Theorem 5.4.11 for the group L^n:

Theorem 5.4.12. *For each $n \geq 1$ there is an action of the group*

$$\mathrm{HAut}((BL_2^\wedge)^n) \cong \mathrm{HAut}(BL_2^\wedge) \wr \Sigma_n$$

on the functor $(F')^n$ (treated as an object of the category $\mathbf{Diag_{Sp}}$, cf. 1.4.10), such that the following diagram commutes:

$$\begin{array}{ccc} \operatorname{hocolim}_{\mathcal{J}^n}(F')^n & \xrightarrow{\alpha} & (BL_2^\wedge)^n \\ \downarrow \psi & & \downarrow \psi \\ \operatorname{hocolim}_{\mathcal{J}^n}(F')^n & \xrightarrow{\alpha} & (BL_2^\wedge)^n \end{array}$$

Proof. Each element $\psi \in \mathrm{HAut}((BL_2^\wedge)^n)$ is a composition

$$BL_2^\wedge \times \cdots \times BL_2^\wedge \ni (x_1, \ldots, x_n) \mapsto (\psi_{l_1}(x_{\sigma(1)}), \ldots, \psi_{l_n}(x_{\sigma(n)})) \in BL_2^\wedge \times \cdots \times BL_2^\wedge,$$

for some $\sigma \in \Sigma_n$ and $\psi_{l_i} \in \mathrm{HAut}(BL_2^\wedge)$. Define an action of ψ on $(F')^n$ by

$$(F')^n \xrightarrow{\sigma} (F')^n \xrightarrow{(\psi_{l_1},\ldots,\psi_{l_n})} (F')^n. \qquad \square$$

What we really need is to prove the analogue of 5.4.12 for the group $L^n/\{\pm 1\}$ where $n > 1$. It can be done, but with some additional effort. Let $\mathbf{n} = \{1, \ldots, n\}$. Define a functor $F''_n : (\mathcal{J}')^n \to (\Psi \wr \Sigma_n) - \mathbf{Sp}$ by putting, for each $A \subseteq \mathbf{n}$

$$F''_n((P')^A \times (Q')^{\mathbf{n}\setminus A}) := (E(SL_2(K)^n/\{\pm 1\})/((P')^A \times (Q')^{\mathbf{n}\setminus A}/\{\pm 1\}) \tag{5.4.13}$$

The action of $\Psi \wr \Sigma_n$ on $SL_2(K)^n/\{\pm 1\}$ defined by the formula

$$(\psi_{l_1}, \ldots, \psi_{l_n}; \sigma)(g_1, \ldots, g_n) = (\psi_{l_1}(g_{\sigma(1)}), \ldots, \psi_{l_n}(g_{\sigma(n)})).$$

induces, for each $A \subseteq \mathbf{n}$ the following map

$$(E(SL_2(K)^n/\{\pm 1\})/((P')^A \times (Q')^{\mathbf{n}\setminus A}) \xrightarrow{(\psi_{l_1},\ldots,\psi_{l_n};\sigma)} (E(SL_2(K)^n/\{\pm 1\})/((P')^{\sigma(A)} \times (Q')^{\mathbf{n}\setminus\sigma(A)}).$$

These maps define an action of $\Psi \wr \Sigma_n$ on the diagram F''_n (treated as an object of $\mathbf{Diag_{Sp}}$; cf. 1.4.10).

Proposition 5.4.14. *The functor F''_n is weakly equivalent to the stubborn decomposition diagram of the group $L^n/\{\pm 1\}$.*

Proof. Similarly to the proof of 5.4.9 we show that F''_n is centric. Then we have to check that the groups $H^{i+1}((\mathcal{J}')^n;\alpha_i)$ vanish. For each object $P \in (J')^n$ we have

$$\mathrm{map}(F''_n(P), F''_n(P))_{id} = B(Z_{SL_2(K)^n/\{\pm 1\}}(P)) = \{\pm 1\}^n/\langle -1,\ldots,-1\rangle.$$

Then the functors α_i are constant and for each P we have

$$\Lambda^j(\mathrm{Aut}_{(\mathcal{J}')^n}(P);\alpha_i(P)) = 0$$

for $j > 0$ (by 4.2.6(b)). Then $H^{i+1}((\mathcal{J}')^n;\alpha_i) = 0$ for all i. Now the conclusion follows from 5.4.8. □

As a consequence, we obtain the following

Theorem 5.4.15. *For each $\psi \in \Psi \wr \Sigma_n \simeq \mathrm{HAut}(B(L^n/\{\pm 1\}))$ the following diagram commutes up to homotopy:*

$$\begin{array}{ccc}
\mathrm{hocolim}_{\mathcal{J}^n}\,(F')^n & \xrightarrow{\alpha} & B(L^n/\{\pm 1\})^\wedge_2 \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\
\mathrm{hocolim}_{\mathcal{J}^n}\,(F')^n & \xrightarrow{\alpha} & B(L^n/\{\pm 1\})^\wedge_2
\end{array}$$

Here follows the technical result which is crucial in the proof of the main theorem:

Theorem 5.4.16. *Let ϱ be an $\mathcal{J}^3$-invariant complex representation of $(N^\infty)^3$ (resp. $(N^\infty)^3/\{\pm 1\}$) and choose any map $\psi \in \mathrm{HAut}((BL^\wedge_2)^3)_{B\varrho}$ (resp. $\psi \in \mathrm{HAut}(B(L^3/\{\pm 1\})^\wedge_2)_{B\varrho}$). Assume that ϱ is ψ-invariant, i.e. that $(B\varrho)^\wedge_2$ is homotopic to the composition $(B\varrho)^\wedge_2 \circ f|_{(BN^\infty)^3}$. Let E_ϱ be a set of homotopy classes of extensions of $B\varrho^\wedge_2$ to a map from $(BL^\wedge_2)^3$ (resp. $(B(L^3/\{\pm 1\})^\wedge_2)$). Then the following diagram commutes:*

$$\begin{array}{ccc}
H^2(\mathcal{J}^3;\Xi^\varrho)\times E_\varrho & \xrightarrow{\mu} & E_\varrho \\
\downarrow{\scriptstyle \psi^*\times\psi^*} & & \downarrow{\scriptstyle \psi^*}, \\
H^2(\mathcal{J}^3;\Xi^\varrho)\times E_\varrho & \xrightarrow{\mu} & E_\varrho
\end{array}$$

where μ is an action defined in 1.4.13. Moreover, E_ϱ is a free and transitive $H^2(\mathcal{R}^3;\Pi^\varrho_2)$-set.

Proof. It is a consequence of 1.4.13(1), 1.4.9 and 5.4.12 (resp. 5.4.15). □

5.5 A homotopy representation of H

Let φ be a representation of N_G^∞ defined in 5.3.1. By 5.3.11 it extends to the map $f_G : BG_2^\wedge \to BU(m)_2^\wedge$ (where $m = 2^{46}$). Let $f_H : BH_2^\wedge \to BU(m)_2^\wedge$ be the composition of f_G with an inclusion $BH_2^\wedge \to BG_2^\wedge$. The main result of this section is that f_H is homotopy $\mathrm{GL}_2(\mathbb{F}_2)$-invariant, where the action of $\mathrm{GL}_2(\mathbb{F}_2)$ comes from the centralizer decomposition diagram of $DI(4)$.

Recall [DW1, Section 6] how the action of $\mathrm{GL}_2(\mathbb{F}_2)$ is defined. Note that $W_H \subset W_G$. It appears that the group $N_{W_D}(W_H)/W_H$ acts on $BH_2^\wedge$. Since $N_{W_D}(W_H)/W_H \simeq \mathrm{GL}_2(\mathbb{F}_2)$, this determines the action of $\mathrm{GL}_2(\mathbb{F}_2)$ on $BH_2^\wedge$.

The group $\mathrm{GL}_2(\mathbb{F}_2)$ is generated by an order 2 element (namely s) which stabilizes the inclusion $BH_2^\wedge \to BG_2^\wedge$ and by some element of order 3, say t. Elementary calculations show that t is represented by the element $r^2sv^2 \in W_D$. The invariance of f_H under the action of s is clear. The main effort will be to show that $f_H \circ t \simeq f_H$.

In the basis $\{e_i''\}$ (cf. 5.1.3) the element t acts on T^∞ by the multiplication by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{5.5.1}$$

(recall that k is an odd 2-adic integer such that $k^2 - k - 2 = 0$).

Denote $N_H := N_H(T)$ and $\omega := \mathrm{res}_{N_H^\infty}^{N_G^\infty} \varphi$

Proposition 5.5.2. *The morphism t stabilizes the representation ω.*

Proof. Strightforward from 5.3.2. □

Corollary 5.5.3. Both maps f_H and $t \simeq f_H$ restrict to the same $\mathcal{R}_2(H)$-invariant representation ω.

By 5.4.16 the set E_ω of extensions of ω to a map from $B(L^3/\{\pm 1\})_2^\wedge$ is a free and transitive $H^2(\mathcal{J}^3; \Xi^\omega)$-set. By 4.5.20 $E_\omega \simeq \bar{\mathbb{Z}}_2^\wedge[Z_\omega^2(3)^\pm/\sim]$, where $Z_\omega^2(3)^\pm$ is the set defined in 4.5.18. For simplicity denote $Z := Z_\omega^2(3)$, $Z^\pm := Z_\omega^2(3)^\pm$. In order to calculate the set Z we need to describe the set of irreducible subrepresentations of ω.

Let $Rt(\omega) \subset (\mathbb{Q}_2^\wedge)^3$ be the set of roots of ω (in dual basis $\{e_i''\}$). Obviously W_D acts of $Rt(\omega)$.

Proposition 5.5.4. *The set $Rt(\omega)$ contains all combinations of roots having the form $w^*(\frac{1}{2}, \frac{1}{2}, 0)$ for $w \in W_D$.*

Proof. Since the sequence $(1, 0, 0)$ (in dual basis $\{e_i\}$) has the form $(\frac{1}{2}, \frac{1}{2}, 0)$ in basis $\{e''_i\}$, then the conclusion follows from the definition of φ (5.3.1) and properties of the tensor product. □

Roots having the form $w^*(\frac{1}{2}, \frac{1}{2}, 0)$ will be called *basic roots*. The set of all basic roots will be denoted by $BRt(\omega)$. Put $c = \frac{1}{2}$, $d = \frac{k-1}{4}$. Note that $c \equiv d \equiv \frac{1}{2} \mod 2$.

Proposition 5.5.5. *We have*

$$\begin{aligned} BRt(\omega) = \{&(\pm c, \pm c, 0), (\pm c, 0, \pm c), (0, \pm c, \pm c), \\ &(\pm 2d, 0, 0), (0, \pm 2d, 0), (0, 0, \pm 2d), \\ &(\pm(c+d), \pm d, \pm d), (\pm d, \pm(c+d), \pm d), (\pm d, \pm d, \pm(c+d))\} \end{aligned}$$

Proof. It is the list (5.3.3) converted into the basis $\{e''_i\}$. □

Directly from the definition (5.3.1) we obtain

Corollary 5.5.6.

$$Rt(\omega) \cong \left\{ \sum_{(r_1,r_2,r_3)\in BRt(\omega)} l_{(r_1,r_2,r_3)}(r_1, r_2, r_3) : 0 \leq l_{(r_1,r_2,r_3)} \leq 16 \right\}.$$

Proposition 5.5.7. *If even integers l_c and l_d satisfy inequalities $|l_c| \leq 16 \cdot 8$, $|l_d| \leq 16 \cdot 14$ and $|l_c - l_d| \leq 16 \cdot 14$, then $(l_c c + l_d d, 0, 0) \in Rt(\omega)$.*

Proof. Introduce a relation $\overset{1}{\sim}$ on $BRt(\omega)$ by

$$(r_1, r_2, r_3) \overset{1}{\sim} (r'_1, r'_2, r'_3) \Leftrightarrow r_1 = r'_1 \wedge r_2 = -r'_2 \wedge r_3 = -r'_3.$$

Let $SRt(\omega) \subseteq Rt(\omega)$ be the set all combinations

$$\sum_{(r_1,r_2,r_3)\in BRt(\omega)} l_{(r_1,r_2,r_3)}(r_1, r_2, r_3)$$

such that the coefficients $l_{(r_1,r_2,r_3)}$ are constant on equivalence classes of the relation $\overset{1}{\sim}$. Then $SRt(\omega)$ contains exactly sequences

$$(m_c(2c) + m_d(2d) + m_{c+d}(2c + 2d), 0, 0),$$

where the integers m_c, m_d and m_{c+d} satisfy inequalities $-2 \cdot 16 \leq m_c \leq 2 \cdot 16$, $-5 \cdot 16 \leq m_d \leq 5 \cdot 16$, $m_{c+d} = -2 \cdot 16 \leq m_{c+d} \leq 2 \cdot 16$. □

Proposition 5.5.8. *If odd integers l_c, l_d satisfy inequalities $|l_c| \leq 8 \cdot 16$, $|l_d| \leq 14 \cdot 16$ and $|l_c - l_d| \leq 14 \cdot 16$, then $(l_c c + l_d d, c + d, c + d) \in Rt(\omega)$.*

Proof. By 5.5.7 we have $((l_c - 1)c + (l_d - 1)d, 0, 0) \in Rt(\omega)$ and it may be written without using basic roots $(c+d, d, d)$ and $(0, c, c)$ (since $l_c \leq 8 \cdot 16 - 2$ and $l_d \leq 14 \cdot 16 - 2$). Then

$$((l_c - 1)c + (l_d - 1)d, 0, 0) + (c+d, d, d) + (0, c, c) = (l_c c + l_d d, c+d, c+d) \in Rt(\omega).$$

□

Elementary arguments lead to the following corollaries

Corollary 5.5.9. If $(l_c^1 c + l_d^1 d, l_c^2 c + l_d^2 d, l_c^3 c + l_d^3 d) \in Rt(\omega)$ is a root with (2-adic) integer coordinates, then for $i = 1, 2, 3$ we have $l_c^i \equiv l_d^i \mod 2$. Moreover, numbers l_c^i and l_d^i satisfy inequalities analogous to the ones in the formulation of 5.5.7.

Corollary 5.5.10.

$$Rt(\omega) \cup (\mathbb{Z}_2^{\wedge})^3 = \{(l_c^1 c + l_d^1 d, l_c^2 c + l_d^2 d, l_c^3 c + l_d^3 d) : l_c^i, l_d^i \in \mathbb{Z}, \\ l_c^i \equiv l_d^i \mod 2 \wedge |l_c^i| \leq 8 \cdot 16 \wedge |l_c^i| \leq 14 \cdot 16 \wedge |l_c^i - l_d^i| \leq 14 \cdot 16\}$$

Proposition 5.5.11.

$$Z^{\pm} \cong \{(l_c c + l_d d, *, *), (*, l_c c + l_d d, *), (*, *, l_c c + l_d d) : l_c, l_d \in \mathbb{Z} \\ \wedge\, l_c \equiv l_d \mod 2 \wedge |l_c| \leq 8 \cdot 16 \wedge |l_d| \leq 14 \cdot 16 \wedge |l_c - l_d| \leq 14 \cdot 16\}$$

Proof. Recall that $Z^{\pm}$ (cf. 4.5.18) is a set of symbols $(\eta_1, *, *)$, $(*, \eta_2, *)$, $(*, *, \eta_3)$ such that $\eta_1 \bar{\otimes} \eta_2 \bar{\otimes} \eta_3 \subseteq \omega$, and $\eta_i \in \{\alpha_{2k+1}\} \cup \{\beta_{2l}\} \cup \{\tau\} = \mathrm{IR}^{\pm}(N^{\infty})$ for each i. Now the conclusion follows from 5.5.10 (note that we identify representations α_{2k+1}, β_{2l}, τ with 2-adic integers $2k+1$, $2l$ and 0 respectively).

□

Proposition 5.5.12. *There are two equivalence classes of relation $\sim$ (cf. 4.5.18) on the set Z:*

$$Z_0^{\pm} = \{(l_c c + l_d d, *, *), (*, l_c c + l_d d, *), (*, *, l_c c + l_d d) : \\ l_c, l_d \in 2\mathbb{Z} \wedge |l_c| \leq 8 \cdot 16 \wedge |l_d| \leq 14 \cdot 16 \wedge |l_c - l_d| \leq 14 \cdot 16\}$$

and

$$Z_1^{\pm} = \{(l_c c + l_d d, *, *), (*, l_c c + l_d d, *), (*, *, l_c c + l_d d) : \\ l_c, l_d \in 1 + 2\mathbb{Z} \wedge |l_c| \leq 8 \cdot 16 \wedge |l_d| \leq 14 \cdot 16 \wedge |l_c - l_d| \leq 14 \cdot 16\}.$$

Proof. For even l_c, l_d, by 5.5.7 $(l_c c+l_d d,0,0) \in Rt(\omega)$ (and therefore $(0,l_c c+l_d d,0),(0,0,l_c c+l_d d) \in Rt(\omega)$). Moreover, obviously $(0,0,0) \in Rt(\omega)$. Then

$$(l_c c+l_d d,*,*) \sim (*,0,*) \sim (0,*,*) \sim (*,l_c c+l_d d,*) \simeq (*,*,l_c c+l_d d)$$

Then all elements of $Z_0^{\pm}$ are in the same equivalence class as $(0,*,*)$. Similarly, if l_c and l_d are odd, then

$$(l_c c+l_d d,c+d,c+d),(c+d,l_c c+l_d d,c+d),\\ (c+d,c+d,l_c c+l_d d),(c+d,c+d,c+d) \in Rt(\omega).$$

Therefore all elements of $Z_1^{\pm}$ are in the same equivalence class as $(c+d,*,*)$. □

Now we are ready to prove the main result of this section:

Theorem 5.5.13. *There is a bijection between the set E_ω of extensions of $\omega \in \mathrm{Rep}(N_H^\infty)$ to a map from $BH_2^\wedge$ and the set $\mathbb{Z}_2^\wedge$. Moreover, an element $t \in \mathrm{HAut}(BH_2^\wedge)$ acts trivially on E_ω.*

Proof. By 5.4.16 and 4.5.20 there is a commutative diagram

$$\begin{array}{ccc} \bar{\mathbb{Z}}_2^\wedge\{Z_0^{\pm},Z_1^{\pm}\} \times E_\omega & \xrightarrow{\mu} & E_\omega \\ \downarrow{\scriptstyle t^*\times t^*} & & \downarrow{\scriptstyle t^*} \\ \bar{\mathbb{Z}}_2^\wedge\{Z_0^{\pm},Z_1^{\pm}\} \times E_\omega & \xrightarrow{\mu} & E_\omega \end{array} .$$

Since the automorphism t of $BH_2^\wedge$ acts on $Rt(\omega)$ by permuting coordinates (cf. 5.5.1), it fixes the equivalence classes $Z_0^{\pm}$ and $Z_1^{\pm}$. Therefore the map

$$t^* : \bar{\mathbb{Z}}_2^\wedge\{Z_0^{\pm},Z_1^{\pm}\} \longrightarrow \bar{\mathbb{Z}}_2^\wedge\{Z_0^{\pm},Z_1^{\pm}\}$$

is an identity. Thus $t^* : E_\omega \to E_\omega$ is a $\bar{\mathbb{Z}}_2^\wedge\{Z_0^{\pm},Z_1^{\pm}\}$-map. Since E_ω is a free and transitive $\bar{\mathbb{Z}}_2^\wedge\{Z_0^{\pm},Z_1^{\pm}\}$-set, then $t_* : E_\omega \to E_\omega$ is a shift by some element. Therefore it is an identity, because $t^* \circ t^* \circ t^* = id$ and the group $\bar{\mathbb{Z}}_2^\wedge\{Z_0^{\pm},Z_1^{\pm}\} \simeq \mathbb{Z}_2^\wedge$ has no elements of finite order. □

As a consequence, we obtain

Proposition 5.5.14. *The map f_G is $\mathcal{A}$-invariant, i.e. it extends to a homotopy compatible family of maps from the centralizer decomposition functor of BD into $BU(m)^\wedge_2$.*

Proof. By 5.5.13 the map $f_H = f_G|_{BH}$ is $\mathrm{GL}_2(\mathbb{F}_2)$-invariant. The map $f_G|_{B(T^3 \rtimes \{\pm 1\})^\wedge_2}$ is, by Dwyer-Zabrodsky theorem, the completion of the classifying map of the representation $\mathrm{res}^{N_1^\infty}_{(T^3 \rtimes \{\pm 1\})^\infty} \varphi$, which is $\mathrm{GL}_3(\mathbb{F}_2)$-invariant (since the action of $\mathrm{GL}_3(\mathbb{F}_2)$ is the restriction of the action of the Weyl group). Since $\mathrm{res}^{N_1^\infty}_{\{\pm 1\}^4} \varphi$ is a sum of regular representations, then the map $f_G|_{B\{\pm 1\}^4}$ is $\mathrm{GL}_4(\mathbb{F}_2)$-invariant. □

5.6 Proof of the main theorem

Let $F : \mathcal{A} \to \mathbf{Sp}$ be the centralizer decomposition diagram of BD and let A_i, $i = 1, \ldots, 4$ be objects of $\mathcal{A}$. We have proven that the map

$$f_G : F(A_1) \simeq BG^\wedge_2 \to BU(m)^\wedge_2$$

is $\mathcal{A}$-invariant. Now we have to check that the obstructions to the existence of an extension $BDI(4) \to BU(m)^\wedge_2$ in groups $H^{i+1}(\mathcal{A}; \Pi_i)$ vanish, where

$$\Pi_i(A_r) := \pi_i(\mathrm{map}(F(A_r), BU(m)^\wedge_2)_{f_G|_{F(A_r)}}).$$

Oliver [O2] provided a powerful tool for calculating this kind of cohomology groups:

Theorem 5.6.1. ([O2, Thm. 1, Prp. 6]) *Let $\mathcal{A} := \mathcal{A}_p(X)$ be the centralizer decomposition category of a p-compact group X. Let $\mathcal{A}_i$ be the set of objects of $\mathcal{A}$ which have the form $B(\mathbb{Z}/p)^i \to BX$. Then for any $\mathbb{Z}^\wedge_p[\mathcal{A}]$-module F there is a isomorphism $H^*(\mathcal{A}; F) \simeq C^*_{St}(F)$, where*

$$C^i_{St}(F) \cong \prod_{A \in \mathcal{A}_{i+1}} \mathrm{Hom}_{\mathrm{Aut}_{\mathcal{A}}(A)}(St_A, F(A))$$

However, this theorem applies only to the case when the coefficient functor has abelian values (obviously $\Pi(A_r)$ is abelian for $i > 1$).

For $r = 3, 4$ the spaces $F(A_r)$ are 2-completed classifying spaces of 2-toral groups. Therefore, by 2.2.4 the spaces $\mathrm{map}(F(A_r), BU(m)^\wedge_2)_{f_G|_{F(A_r)}}$ are classifying spaces of products of unitary groups. Hence $\Pi_1(\mathbb{F}_2^r) = 0$.

For $r = 1, 2$ this argument fails, since the spaces $F(A_1)$ and $F(A_2)$ are not classifying spaces of 2-toral groups.

Proposition 5.6.2. *The fundamental group of* $\mathrm{map}(BH_2^\wedge, BU(m)_2^\wedge)_{f_H}$ *is abelian.*

Proof. Let $\mathcal{J} := \mathcal{R}_2(L)$. Recall that $\mathcal{R}_2(H) \cong \mathcal{J}^3$. By the Dwyer-Zabrodsky Theorem (2.2.4) we have

$$\begin{aligned}\mathrm{map}(BH_2^\wedge, BU(m)_2^\wedge)_{f_H} &\cong \mathrm{map}(\mathrm{hocolim}_{H/P\in\mathcal{J}^3}(EH\times_H /P)_2^\wedge, BU(m)_2^\wedge)_{f_H}\\ &= \mathrm{holim}_{H/P\in\mathcal{J}^3}\,\mathrm{map}((EH\times_H H/P)_2^\wedge, BU(m)_p^\wedge)_{f_H|_{BP_2^\wedge}}\\ &\simeq (\mathrm{holim}_{H/P\in\mathcal{J}^3}\, BC_{U(m)}(\omega(P^\infty))_2^\wedge)_2^\wedge\end{aligned}$$

The second term of the Bousfield spectral sequence [BK, XI,7.1], which converges to the homotopy groups of the homotopy inverse limit above has the second term

$$\begin{aligned}E_2^{p,q} &= H^p(\mathcal{J}^3; \pi_{p+q}(\mathrm{map}(EH\times_H H/(-), BU(n)_2^\wedge)_{f_H|_{B(-)_2^\wedge}})\\ &= H^p(\mathcal{J}^3; \pi_{p+q}(BC_{U(m)}(\varphi((-)^\infty)_2^\wedge)) \Rightarrow \Pi_q(A_2)\end{aligned}$$

The spaces $BC_{U(m)}(\varphi((-)^\infty)_2^\wedge$ are products of 2-completed classifying spaces of unitary groups (cf. Section 2.4), where the number of multiplies is the number of isomorphism classes of irreducible subrepresentations of $\mathrm{res}^{N_G^\infty}_{P^\infty}\varphi$ and the rank of every one is the multiplicity of the corresponding irreducible reprepresentation. Then the groups $\pi_{p+q}(BC_{U(m)}(\varphi((-)^\infty)_2^\wedge)$ are abelian. Moreover, for $p+q=1,3$ they vanish and for $p+q=2,4$ they are isomorphic to Ξ^ω (cf. 2.4.4), where as before $\omega = \mathrm{res}^{N_G^\infty}_{N_H^\infty}\varphi$, since all irreducible subrepresentations of φ appear with multiplicity at least 16. By 4.5.15 we have $H^3(\mathcal{J}^3;\Xi^\omega) = 0$. Thus $E_2^{p,1} = 0$ for $p \neq 1$. Therefore $\Pi_1(A_2)$ is a subquotient of the abelian group $E_2^{1,1}$ and hence is abelian. □

The following proposition is a staightforward consequence of 5.6.1:

Proposition 5.6.3. *For any* $\mathbb{Z}_2^\wedge[\mathcal{A}]$*-module* M *there is an isomorphism*

$$H^{*-1}(\mathcal{A}; M) \cong H^*(\mathrm{Hom}_{\mathrm{GL}_*(\mathbb{F}_2)}(St_{GL_*(\mathbb{F}_2)}, M(A_*))),$$

where St_Γ *is the Steinberg module of the group* Γ*.*

In particular, $H^i(\mathcal{A}; M) = 0$ for $i > 3$.

Proposition 5.6.4. *For each* $i \geq 1$ *holds* $H^{i+1}(\mathcal{A}; \Pi_i) = 0$*.*

Proof. If $i \geq 3$, then the conclusion is obvious, so it is suffiecient to consider cases $i = 1, 2$ only. Note that the full subcategory of $\mathcal{A}$ with objects A_1 and A_2 is isomorphic to $\mathcal{J}$. Since $\Pi_i(A_r)$ for $r = 3, 4$, then by 4.1.6 and 4.3.1 we have $H^2(\mathcal{A}; \Pi_1) \cong H^2(\mathcal{J}; \mathrm{res}^{\mathcal{A}}_{\mathcal{J}} \Pi_1) = 0$.

If $i = 2$, then by 5.6.3 $H^3(\mathcal{A}; \Pi_2)$ is a quotient of the group

$$\mathrm{Hom}_{\mathrm{GL}_4(\mathbb{F}_2)}(St_{GL_4(\mathbb{F}_2)}, \Pi_2(A_4)).$$

A $\mathbb{Z}_2^\wedge$-module $\Pi_2(A_4) = \pi_2(\mathrm{map}(B\{\pm 1\}^4, BU(m)_2^\wedge)_{\mathrm{res}^{N_1^\infty}_{\{\pm 1\}^4} \varphi})$ is free and has dimension not larger than 2^4 (there are 2^4 isomorphism classes of irreduicible representations of $\{\pm 1\}^4$). The Steinberg module $St_{\mathrm{GL}_4(\mathbb{F}_2)}$ is a second homology group of a geometrical realization of a poset of all non-trivial proper subspaces of $\mathbb{F}_2^4$. By an Euler characteristic argument it has dimension 63 and is an irreducible $\mathbb{Z}_2^\wedge[\mathrm{GL}_4(\mathbb{F}_2)]$-module (cf. [St]). Hence there is no non-zero homomorphism $St_{GL_4(\mathbb{F}_2)} \to \Pi_2(A_4)$. Hence $\mathrm{Hom}_{\mathrm{GL}_4(\mathbb{F}_2)}(St_{GL_4(\mathbb{F}_2)}, \Pi_2(A_4)) = 0$ and thus $H^3(\mathcal{A}; \Pi_2) = 0$. □

As a consequence we obtain the main theorem of this paper:

Theorem 5.6.5. *The map $f_G : BG_2^\wedge \to BU(m)_2^\wedge$ extends to a faithful complex representation of the 2-compact group $DI(4)$.*

Proof. By 5.6.4 the map f_G extends to $BDI(4)$. By 1.2.11 the extension is a classifying map of a monomorphism of 2-compact groups. □

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