The Web Monoid and Opetopic Sets

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CT2010
21st June 2010

Goals

- Make M. Makkai's definition of weak ω-categories useable

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- First step: make opetopic sets easier to work with

To build an opetopic set ...

Example ($n = 1$)

$$\bullet \longrightarrow \bullet$$

- ... start with n-cells

To build an opetopic set ...

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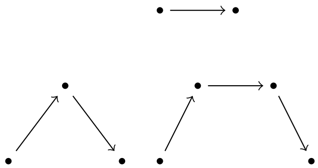

- ... start with n-cells
- Hocus-Pocus – determine the possible $(n+1)$-cells

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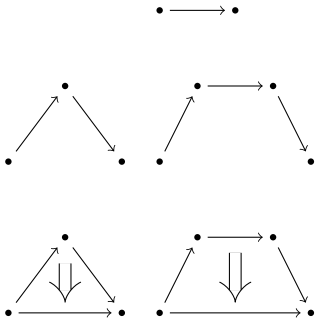

- ...start with n-cells
- Hocus-Pocus – determine the possible $(n + 1)$-cells
- Decide which $(n + 1)$-cells are realized

To build an opetopic set . . .

Example ($n = 2$)

- . . . start with n-cells
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How to do magic

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- Operad for operads (Baez-Dolan). Motivated conceptually.
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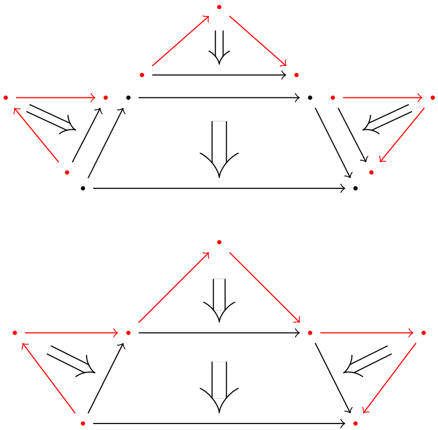
- Web Monoid – seeks the middle ground.

The Geometrical Problem

following Hermida-Makkai-Power

Problem: Composing cells pastes their domains!



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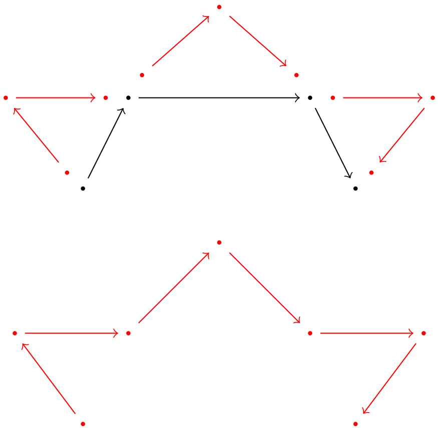

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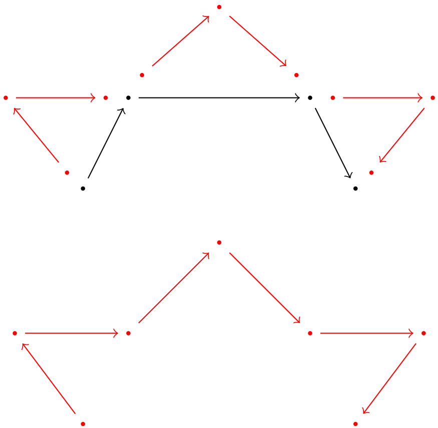
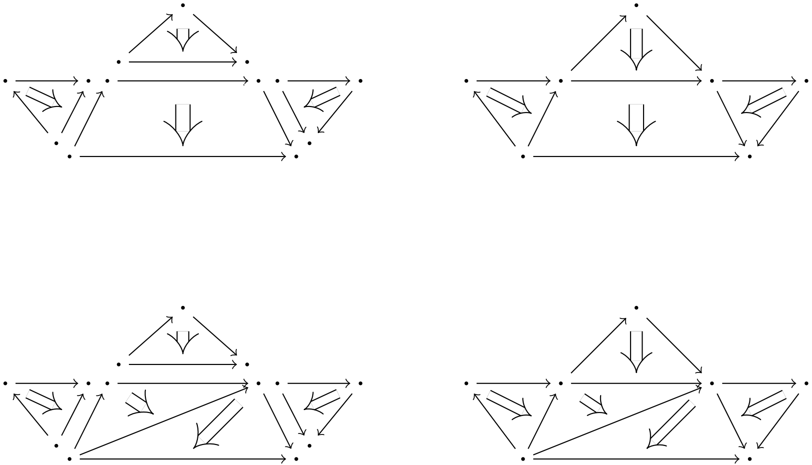


Replace instances of cells (black) with formal composites of cells.

An Insight

This operation commutes with taking formal composites



Lax Monoidal Fibrations

Definition

A lax monoidal fibration over $\mathcal{B}$ is a lax monoid object in the 2-category of fibrations over $\mathcal{B}$, fibered functors and fibered natural transformations.

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- The reindexing functors are lax monoidal
- Our examples: fibers are strong, reindexing – never

Ordinary Signatures

with amalgamation

Category $\mathbf{Sig}_a$

Objects

Sets A together with a typing $\partial^A : A \to O \times O^*$

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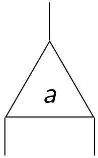

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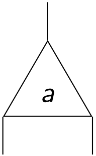

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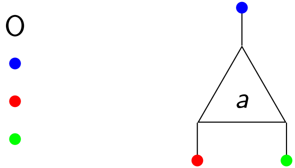

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- $\partial(a)$ lists output and input types of a

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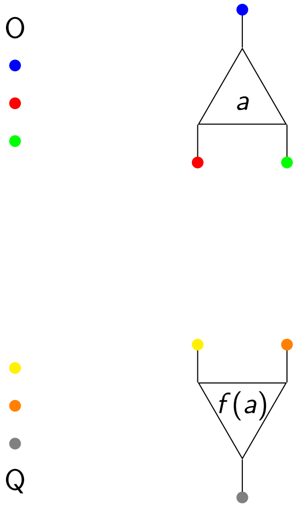

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Morphisms

Triples (f, u, σ):

Ordinary Signatures

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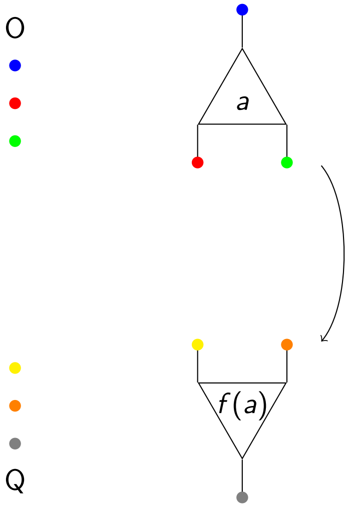

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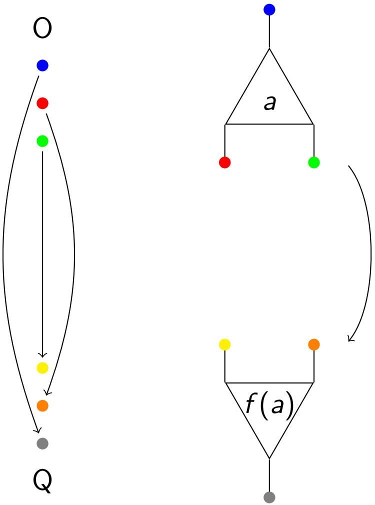

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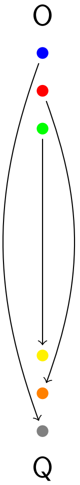

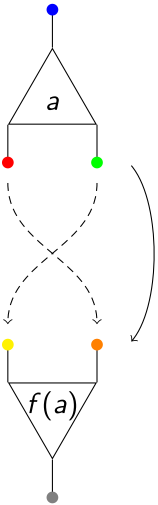

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- σ_a connects inputs of a to those of $f(a)$, respecting types

Lax Monoidal Fibration Structure

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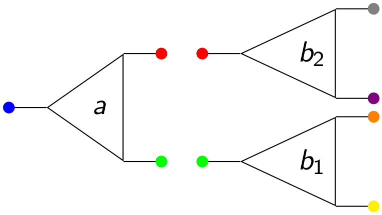

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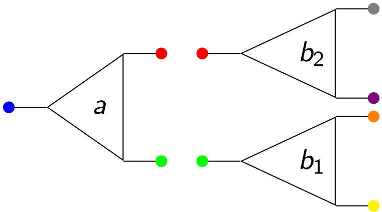

Note the obvious typing

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Set of types becomes a multicategory $M \in Mon(\mathbf{Sig}_a)$ over O.

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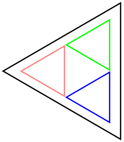
Two Monoidal Structures



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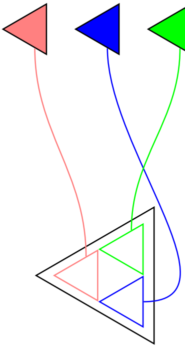
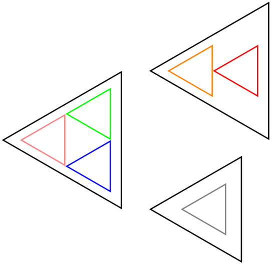


- Obvious one, $A \otimes_M B$, defined using typing in M

Two Monoidal Structures



- New one:
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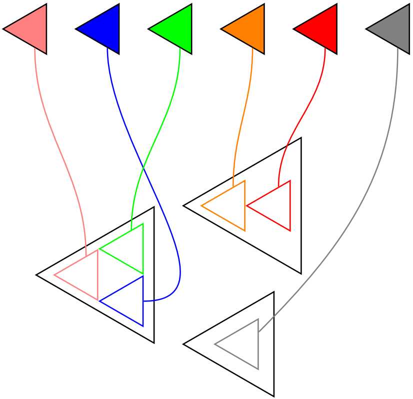
Distributivity



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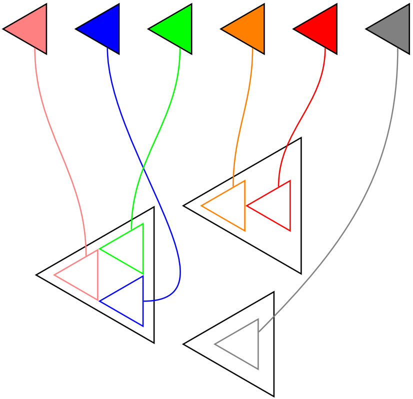

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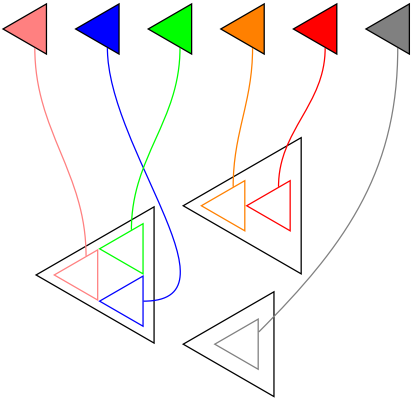

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- A natural isomorphism makes the picture unambiguous

The Three Tensors Theorem

distributivity of monoidal structures

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which satisfy some coherence conditions

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- Each distributes over previous one

The Three Tensors Theorem

statement

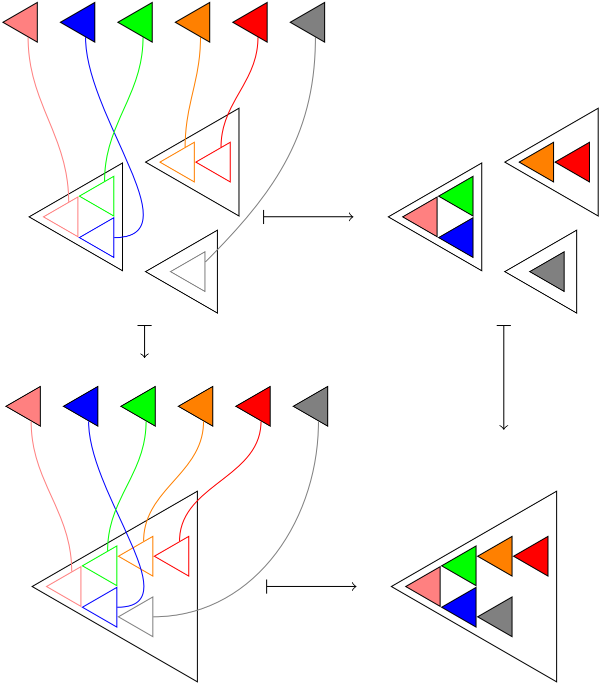
Theorem

There is a unique $\otimes$-monoid structure on $\mathcal{W} := \mathcal{F}_{\odot}(I_{\otimes})$, such that the unit of the adjunction $\eta : I_{\otimes} \to \mathcal{F}_{\odot}(I_{\otimes})$ is the unit of the multiplication $\nu : \mathcal{W} \otimes \mathcal{W} \to \mathcal{W}$, which in turn makes the following main diagram commute:

$$\begin{array}{ccc}
(\mathcal{W} \otimes \mathcal{W}) \odot (\mathcal{W} \otimes \mathcal{W}) & \xrightarrow{\nu \odot \nu} & \mathcal{W} \odot \mathcal{W} \\
\varphi \downarrow & & \\
(\mathcal{W} \odot \mathcal{W}) \otimes \mathcal{W} & & \downarrow \mu \\
\mu \otimes 1 \downarrow & & \\
\mathcal{W} \otimes \mathcal{W} & \xrightarrow{\nu} & \mathcal{W}
\end{array}$$

μ - multiplication in free monoid

Main Diagram



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- The amalgamation permutations of $\mathcal{W}(M)$ cannot be straightened out, in general, even if M is standard.

Opetopic Sets

an inductive definition

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- S_0, the monoid of 0-pasting diagrams, is the pullback of the trivial monoid along $X_0 \to 1$:

$$\begin{array}{ccc} 1 & \longleftarrow & S_0 \\ \downarrow & & \downarrow \partial \\ \{*\}^\dagger & \xleftarrow{(!)^\dagger} & X_0^\dagger \end{array}$$

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  3. $\mathcal{W}(S_n)$ - calculate possible $(n+1)$-pasting diagrams.

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  4. S_{n+1} is the pullback of $\mathcal{W}(S_n)$ along ϑ_{n+1}.

$$\begin{array}{ccc} \mathcal{W}(S_n) & \longleftarrow & S_{n+1} \\ \partial \downarrow & & \downarrow \partial \\ S_n^\dagger & \xleftarrow{(\vartheta_{n+1})^\dagger} & X_{n+1}^\dagger \end{array}$$

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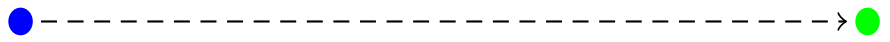
This attaches cell names to the codomain and openings in the domain of every possible diagram.

Opetopic Sets

in pictures



- An element of S_0

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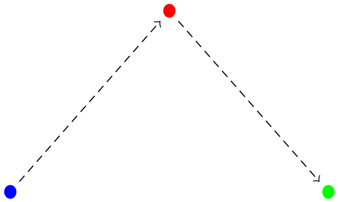

- Multiplication in S_0

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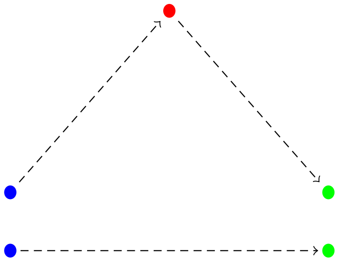

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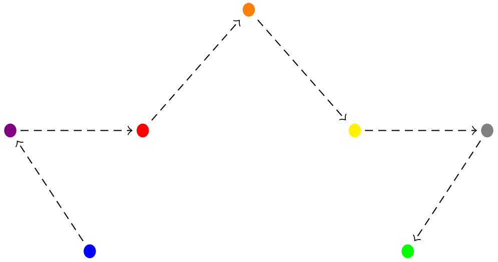

- An element of $\mathcal{W}(S_0)$

Opetopic Sets

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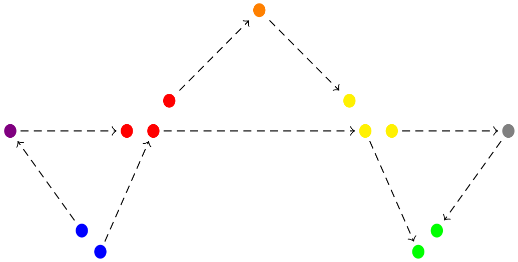

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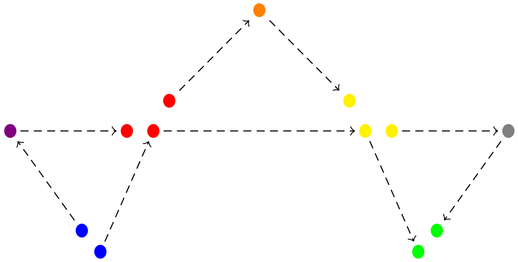

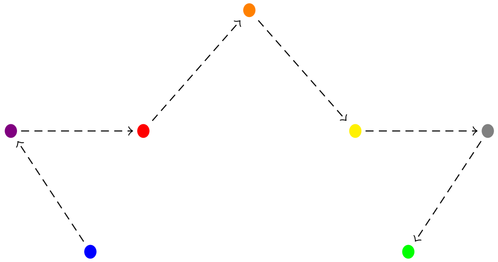

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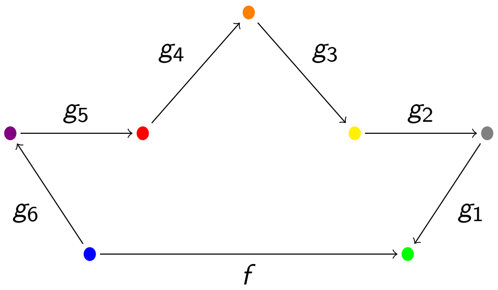

- An element of S_1

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Opetopic Sets

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A morphism of opetopic sets $X \to Y$:

- Maps of cells – functions $f_n : X_n \to Y_n$
- Compatible with forming pasting diagrams, taking domains and codomains.