The coMalcev Monads

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Plan of the talk

- The Kleisli and Eilenberg-Moore objects in 2-categories

- The monoidal structure on Kleisli algebras for a lax monoidal monad

- The PRO for Malcev and coMalcev operations

- The coMaclev monads

- The monoidal structure on Eilenberg-Moore algebras for a lax monoidal coMaclev monad

Let $(\mathcal{C}, \mathcal{S}, \eta, \mu)$ a monad on a 2-category $\mathcal{D}$.

A lax morphism of monads

$$(F, \tau) : (\mathcal{C}, \mathcal{S}, \ldots) \to (\mathcal{C}', \mathcal{S}', \ldots)$$

is a 1-cell and a 2-cell

$$F : \mathcal{C} \to \mathcal{C}', \qquad \tau : \mathcal{S}' \circ F \Rightarrow F \circ \mathcal{S}$$

compatible with η and μ. In oplax morphisms of monads $\tau : F \circ \mathcal{S} \Rightarrow \mathcal{S}' \circ F$ goes the other way.

A transformation of morphisms of monads

$$\xi : (F, \tau) \to (F', \tau')$$

is a 2-cell $\xi : F \to F'$ compatible with τ and τ'.

$\mathbf{Mnd}(\mathcal{D})$ - 2-category of monads, lax morphisms of monads, and transformations

$\mathbf{Mnd}_{op}(\mathcal{D})$ - 2-category of monads, oplax morphisms of monads, and transformations

We have embeddings ι and ι_{op}

$$\mathbf{Mnd}_{op}(\mathcal{D}) \begin{array}{c} \xrightarrow{\mathcal{K}} \\ \xleftarrow[\iota_{op}]{} \\ \xrightarrow[|-|]{} \end{array} \mathcal{D} \begin{array}{c} \xleftarrow{|-|} \\ \xrightarrow[\iota]{} \\ \xleftarrow[EM]{} \end{array} \mathbf{Mnd}(\mathcal{D})$$

$\mathcal{D}$ admits Kleisli objects iff there is a left 2-adjoint $\mathcal{K} \dashv \iota_{op}$.

The unit

$$(F_{\mathcal{S}}, \kappa) : \mathcal{S} \longrightarrow \iota_{op}\mathcal{K}(\mathcal{S}) = 1_{\mathcal{C}_{\mathcal{S}}}$$

$\kappa : F_{\mathcal{S}} \circ \mathcal{S} \Rightarrow F_{\mathcal{S}}$

$\mathcal{D}$ admits EM objects iff there is a right 2-adjoint $\iota \dashv EM$.

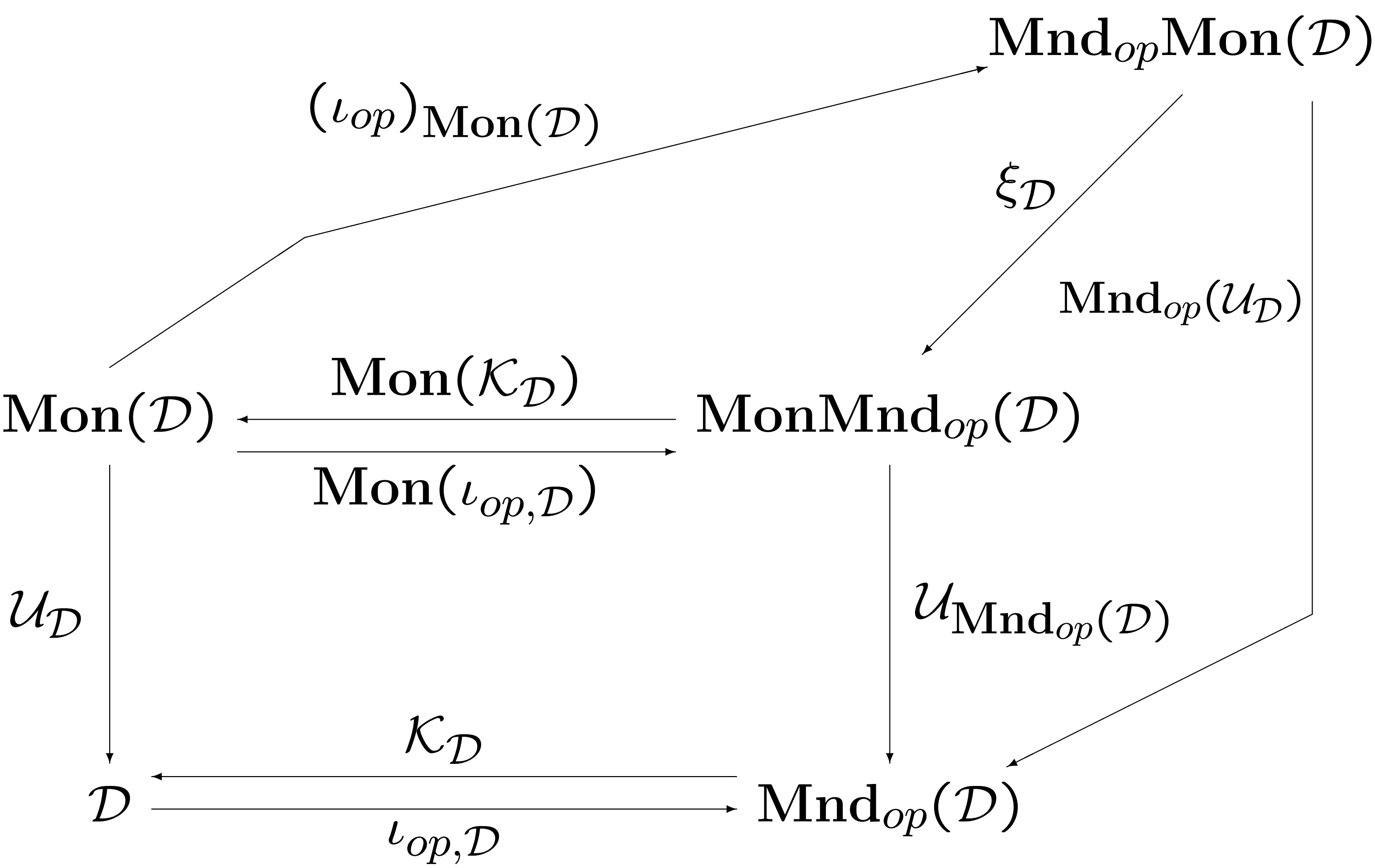
The counit

$$(U^{\mathcal{S}}, \tau) : \iota EM(\mathcal{S}) = 1_{\mathcal{C}^{\mathcal{S}}} \longrightarrow \mathcal{S}$$

$\tau : \mathcal{S} \circ U^{\mathcal{S}} \Rightarrow U^{\mathcal{S}}$

$\mathbf{Mon}(\mathcal{D})$ the 2-category of monoidal categories, lax monoidal functors, and monoidal natural transformations in a 2-category $\mathcal{D}$ with finite products (e.g. $\mathbf{Cat}$).

Kleisli objects in $\mathbf{Mon}(\mathcal{D})$



This argument does not work for EM-objects!

The classical *Malcev operation* $t : X^3 \to X$ such that

$$t(x,x,y) = y, \quad t(x,y,y) = x$$

To formulate these equations we need to know how to duble a variable and how to drop a variable i.e. we need a comonoid structure

$$\delta : X \to X \times X, \;\; f : X \to 1$$

PRO =strict monoidal category on natural numbers, i.e. we replace $(\times, 1)$ by $(\otimes, I)$

The Malcev operation in PRO

We replace $(\times, 1)$ by $(\otimes, I)$ and get a PRO $\mathbf{M}$ whose models are

$(X, \delta : X \to X \otimes X, f : X \to I, \zeta : X \otimes X \otimes X \to X)$

so that (X, δ, f) is a comonoid and the diagram

$$
\begin{array}{ccccc}
X^{\otimes 2} & \xrightarrow{1_X \otimes \delta} & X^{\otimes 3} & \xleftarrow{\delta \otimes 1_X} & X^{\otimes 2} \\
& {\scriptstyle 1_X \otimes f} \searrow & \downarrow {\scriptstyle \zeta} & \swarrow {\scriptstyle f \otimes 1_X} & \\
& & X & &
\end{array}
$$

commutes.

For coMalcev operation we just take $\mathbf{M}^{op}$.

The Malcev monad on $\mathcal{C}$ is a strict model of $\mathbf{M}^{op}$, i.e. $(\mathcal{C}, \mathcal{S}, \eta, \mu, \zeta : \mathcal{S} \to \mathcal{S}^3)$ such that $(\mathcal{C}, \mathcal{S}, \eta, \mu)$ is a monad and

$$\begin{array}{ccccc} S^2 & \xleftarrow{S\mu} & S^3 & \xrightarrow{\mu_S} & S^2 \\ & {\scriptstyle S\eta}\nwarrow & \uparrow{\scriptstyle \zeta} & \nearrow{\scriptstyle \eta_S} & \\ & & S & & \end{array}$$

Lax morphisms of coMalcev monads usual morphisms of monads that additionally preserves ζ

$$\begin{array}{ccc} \mathcal{S}'F & \xrightarrow{\zeta'_F} & \mathcal{S}'^3F \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \tau_{\mathcal{S}^2} \circ \mathcal{S}\tau_{\mathcal{S}} \circ \mathcal{S}^2(\tau)} \\ F\mathcal{S} & \xrightarrow[F(\zeta)]{} & F\mathcal{S}^3 \end{array}$$

Transformations are as usual.

$\mathbf{coMalcev}(\mathcal{D})$ coMalcev monads in $\mathcal{D}$.

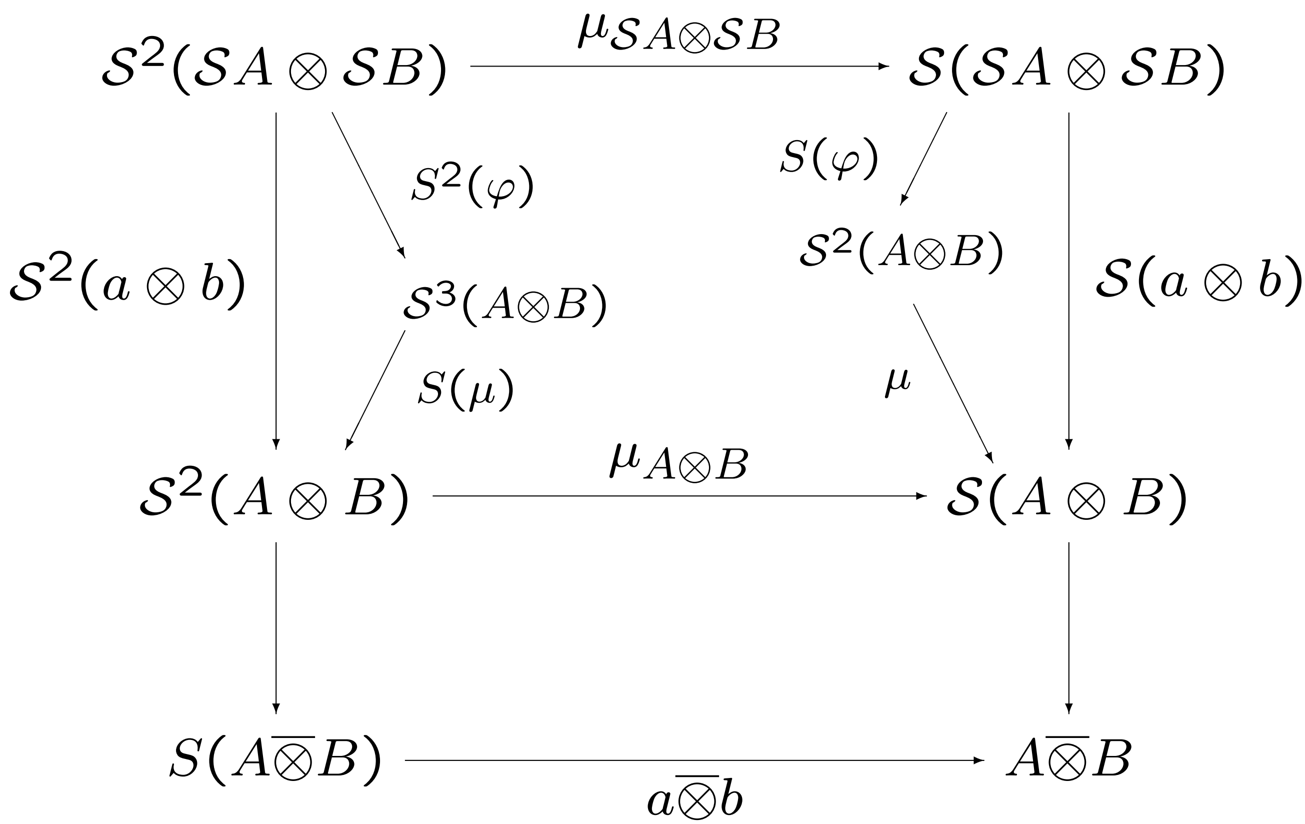
$\mathbf{Cat}_{rc}$ - the 2-category for categories with coequalizers of reflexive pairs, functors preserving such coequalizers, and natural transformations.

Theorem The categories of EM-algebras for coMalcev monoidal monads admit monoidal structure. More precisely the embedding $\bar{\iota}$ has a right 2-adjoint $\overline{EM}$ and it commutes the forgetful functors

$$
\begin{array}{ccc}
\mathbf{MonCat}_{rc} & \underset{\overline{EM}}{\overset{\bar{\iota}}{\rightleftarrows}} & \mathbf{coMalcev}(\mathbf{MonCat}_{rc}) \\
\downarrow & & \downarrow \\
\mathbf{Cat}_{rc} & \underset{EM}{\overset{\iota}{\rightleftarrows}} & \mathbf{Mnd}(\mathbf{Cat}_{rc})
\end{array}
$$

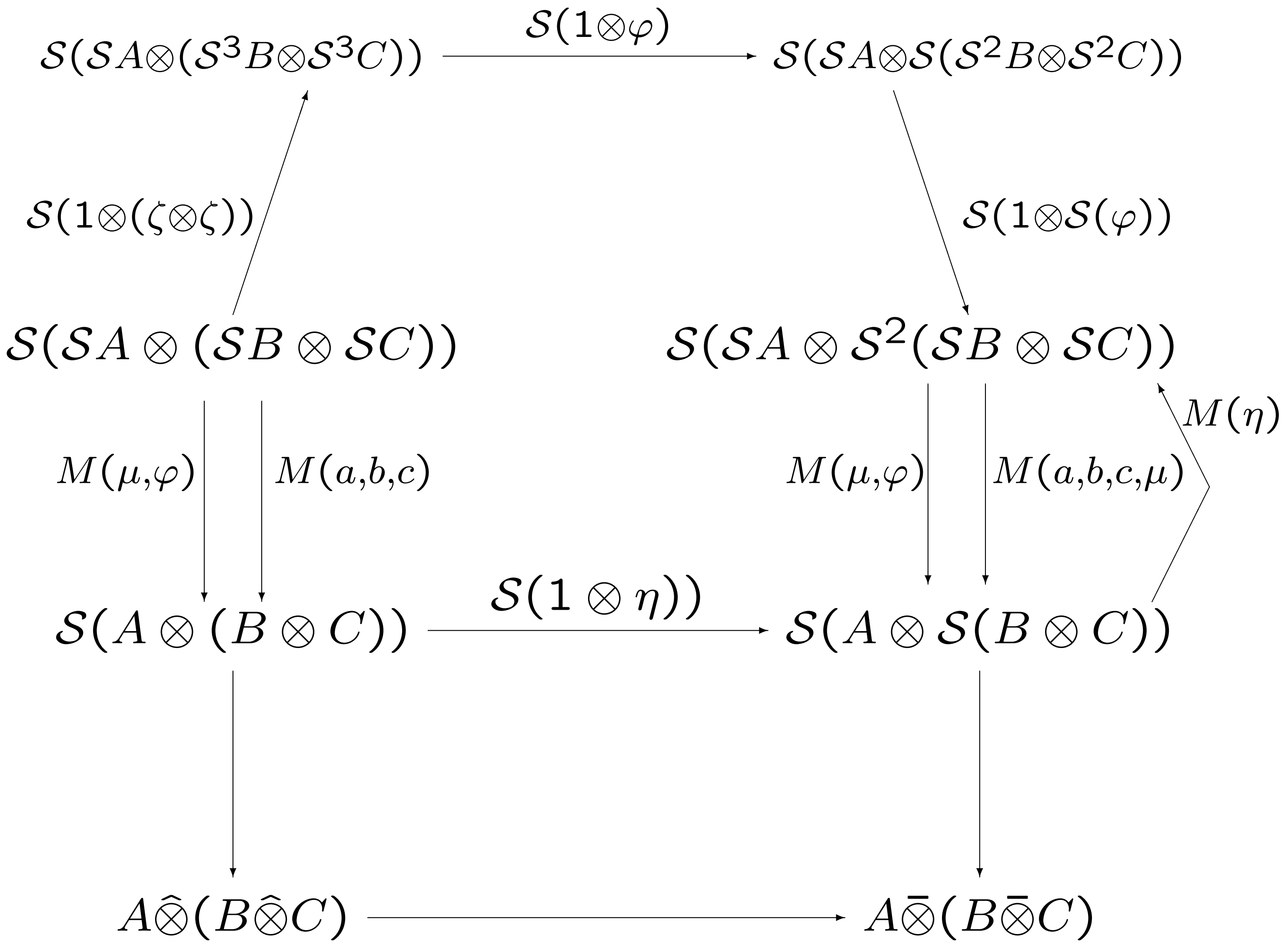
The construction of the tensor.

$((\mathcal{C},\otimes,I,\alpha,\lambda,\rho),(\mathcal{S},\varphi,\overline{\varphi}),\eta,\mu,\zeta)$ - coMalcev monoidal monad



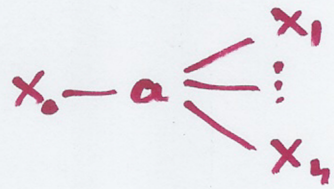
$$(A,a)\ddot{\otimes}(B,b) = (A\overline{\otimes}B, a\overline{\otimes}b)$$

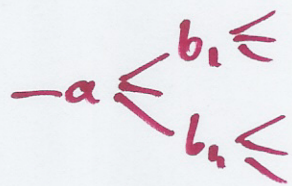
ζ is needed for the associativity of $\ddot{\alpha}$:



$M(\mu,\varphi)$ - morphism depending on μ and η,
$M(a,b,c)$ - morphism depending on a, b and c.

Example Symmetrization monad on multisorted signatutes.

Sig/X $\quad A \quad$ 

$I \quad x - x \qquad A \otimes B$ 

$SA =$ 

$SA \otimes SB \longrightarrow S\, A \otimes B$ (little combing)

$SA \xrightarrow{7} S^3 A$

$(a, \sigma) \longmapsto (a, \sigma, \sigma^{-1}, \sigma)$