3-computads do not form a presheaf category

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This short note is to show that 3-computads do not form a presheaf category.

First recall that S.H. Schanuel made an observation cf. [Carboni-Johnstone] that the category of 2-computads, call it $\mathbf{C_2}$, do form a presheaf category. The category $\mathbf{C_3}$ of 3-computads is a comma category $Set \downarrow T$ for the functor

$$T : \mathbf{C_2} \longrightarrow Set$$

such that $T(A)$ is the set of parallel pairs of 2-cells in the 2-computad A. After [Carboni-Johnstone] we know that $\mathbf{C_3}$ is a presheaf category iff T preserves wide pullbacks. We shall show that in fact T do not preserves even binary pullback (which mean that $\mathbf{C_3}$ is not even an elementary topos).

Let A be a 2-computad with one 0-cell x, one 1-cell id_x the identity on x. Moreover A has as 2-cells all cells generated by two (indeterminate) 2-cells $a_1, a_2 : id_x \to id_x$. Thus any 2-cell in A is of form $a_1^m \circ a_2^n$, for $m, n \in \omega$. Note that if $m = n = 0$ then $a_1^m \circ a_2^n = id_{id_x}$. Let B be a 2-computad isomorphic to A. Let b_1, b_2 be the indeterminate 2-cells in B. Let C be a 2-computad with the same 0- and 1-cells as A and has one indeterminate 2-cell c. Clearly, we have unique maps of 2-computads $\alpha : A \to C$ and $\beta : B \to C$.

Let observe first that the functor

$$T' : \mathbf{C_2} \longrightarrow Set$$

such that $T'(A)$ is the set of 2-cells in the 2-computad A, do not preserve the pullback

$$\begin{array}{ccc} A \times_C B & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow[\alpha]{} & C \end{array}$$

We have that

$$T'(\alpha)(a_1 \circ a_2) = c \circ c = T'(\beta)(b_1 \circ b_2).$$

Moreover we have

$$T'(\pi_A)((a_1, b_1) \circ (a_2, b_2)) = a_1 \circ a_2 = T'(\pi_A)((a_1, b_2) \circ (a_2, b_1))$$

and

$$T'(\pi_B)((a_1, b_1) \circ (a_2, b_2)) = b_1 \circ b_2 = T'(\pi_B)((a_1, b_2) \circ (a_2, b_1)).$$

But the 2-cells $(a_1, b_1) \circ (a_2, b_2)$ and $(a_1, b_2) \circ (a_2, b_1)$ in $A \times_C B$, are different. This shows that the above pullback in not preserved by T'.

As all the pairs of 2-cells in A, B, C are parallel, to apply T to the above pullback is the same thing as to apply $T' \times T'$ to it. So T does not preserves this pullback, as well.