The Malliavin-Stein approach

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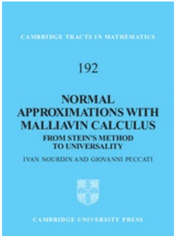
Overview of the lectures

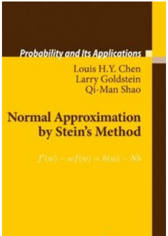
- Introduction to Malliavin calculus: dimension one
- Introduction to Malliavin calculus: any dimension
- Malliavin calculus and absolute continuity: dimension one
- Malliavin calculus and absolute continuity: any dimension
- The Malliavin-Stein approach: dimension one
- The Malliavin-Stein approach: any dimension
- Some applications

Overview

To go further, some references:

- I. Nourdin (2012): Lectures on Gaussian approximations with Malliavin calculus. *Sém. Probab. XLV*, pp. 3-89.
- I. Nourdin and G. Peccati (2012): *Normal Approximations with Malliavin Calculus: from Stein's Method to Universality*. Cambridge Tracts in Mathematics. Cambridge University Press.
- L. Chen, L. Goldstein, Q.-M. Shao (2010): *Normal Approximation by Stein's Method*. Probability and Its Applications. Springer





Introduction to Malliavin calculus: dimension one

Preliminaries on Hermite polynomials

We first recall some useful properties of Hermite polynomials.

- We write $d\gamma(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$, $x \in \mathbb{R}$.
- **Proposition**. The family $(H_p)_{p\in\mathbb{N}} \subset \mathbb{R}[X]$ of Hermite polynomials ($H_0 = 1, H_1 = X, H_2 = X^2 - 1, H_3 = X^3 - 3X$, etc.) has the following properties.
  - (a) $XH_p = H_{p+1} + pH_{p-1}$
  - (b) $H_p' = pH_{p-1}$
  - (c) $H_p(x) = (-1)^p \, e^{\frac{x^2}{2}} \, \frac{d^p}{dx^p} \left\{ e^{-\frac{x^2}{2}} \right\}$
  - (d) $\left(\frac{1}{\sqrt{p!}} H_p \right)_{p\in\mathbb{N}}$ is an orthonormal basis of $L^2(\gamma)$, that is, each $\varphi \in L^2(\gamma)$ can always be expanded as $\varphi = \sum_{p=0}^{\infty} a_p H_p$ with $\sum_{p=0}^{\infty} p! a_p^2 < \infty$ and $\langle H_p, H_q \rangle_{L^2(\gamma)} = p! \delta_{pq}$ (Kronecker symbol).

THE MALLIAVIN DERIVATIVE OPERATOR D

- Let $\varphi \in L^2(\gamma)$.
- We have $\varphi = \sum_{p=0}^{\infty} a_p H_p$ where

$$\boxed{a_p = \frac{1}{p!} \langle \varphi, H_p \rangle_{L^2(\gamma)} = \frac{1}{p!} \mathbb{E}[\varphi(N) H_p(N)],\ N \sim N(0,1)}.$$

- We have $\mathbb{E}[\varphi(N)^2] = \sum_{p=0}^{\infty} a_p^2 p! < \infty$.
- For $k \in \mathbb{N}$, we set $\boxed{\mathbb{D}^{k,2}(\gamma) = \{\varphi \in L^2(\gamma) : \sum_{p=0}^{\infty} p^k p! a_p^2 < \infty\}}$.
- (*Remark*: $\mathbb{D}^{0,2}(\gamma) = L^2(\gamma)$.)

The Malliavin derivative operator D

- For $\varphi = \sum_{p=0}^{\infty} a_p H_p \in \mathbb{D}^{1,2}(\gamma)$, we set

$$\boxed{D\varphi = \sum_{p=0}^{\infty} p\, a_p\, H_{p-1}}.$$

- *Remarks*:
  (i) if $\varphi \in \mathbb{D}^{1,2}(\gamma) \cap C^1(\mathbb{R})$, then $D\varphi = \varphi'$;
  (ii) D can be thought as the **Malliavin derivative operator in dimension 1**.

THE DIVERGENCE OPERATOR δ

- We define Domδ as the set

$$\left\{\varphi \in L^2(\gamma) : \ \exists c > 0, \forall \psi \in C_c^1, \left|\int \varphi\psi' d\gamma\right| \leq c\|\psi\|_{L^2(\gamma)}\right\}.$$

- If $\varphi \in \text{Dom}\delta$, then $\psi \mapsto \int \varphi\psi' d\gamma$ is linear and continuous from C_c^1 (viewed as a *dense* subset of $L^2(\gamma)$) to $\mathbb{R}$.
- As such, it can be extended to a linear form of $L^2(\gamma)$.
- By the Riesz representation theorem, there exists a unique element of $L^2(\gamma)$, written $\delta\varphi$, such that

$$\boxed{\int \varphi\psi' d\gamma = \int (\delta\varphi)\psi d\gamma} \quad \text{for all } \psi \in C_c^1.$$

THE DIVERGENCE OPERATOR δ

- **Definition**. The previous operator $\delta : \mathrm{Dom}\delta \to L^2(\gamma)$ is called the **divergence operator**.

THE DIVERGENCE OPERATOR δ

Proposition.

1. We have $\mathbb{D}^{1,2}(\gamma) \subset \mathrm{Dom}\delta$.
2. Moreover, if $\varphi \in \mathbb{D}^{1,2}(\gamma)$, then

$$\boxed{(\delta\varphi)(x) = x\varphi(x) - (D\varphi)(x)}.$$

Proof.

- If $\varphi = \sum_{p=0}^{\infty} a_p H_p \in \mathbb{D}^{1,2}(\gamma)$, then

$$-D\varphi + x\varphi = \sum_{p=0}^{\infty} \left\{ -p\, a_p\, H_{p-1} + a_p(H_{p+1} + pH_{p-1}) \right\} = \sum_{p=1}^{\infty} a_{p-1} H_p.$$

- Let $\psi \in C_c^1$. We have $\psi = \sum_{p=0}^{\infty} b_p H_p$ and $\psi' = \sum_{p=1}^{\infty} p b_p H_{p-1} = \sum_{p=0}^{\infty} (p+1) b_{p+1} H_p$.

Proposition.

1. We have $\mathbb{D}^{1,2}(\gamma) \subset \mathrm{Dom}\delta$.
2. Moreover, if $\varphi \in \mathbb{D}^{1,2}(\gamma)$, then

$$\boxed{(\delta\varphi)(x) = x\varphi(x) - (D\varphi)(x)}.$$

Proof (continued). Hence

$$\begin{aligned}\langle \varphi, \psi' \rangle_{L^2(\gamma)} &= \sum_{p=0}^{\infty} p!(p+1)a_p b_{p+1} \\ \langle x\varphi - D\varphi, \psi \rangle_{L^2(\gamma)} &= \sum_{p=1}^{\infty} a_{p-1} b_p p! = \sum_{p=0}^{\infty} a_p b_{p+1}(p+1)!.\end{aligned}$$

That is, these two quantities are the same. Moreover,

$$\begin{aligned}\left|\langle \varphi, \psi' \rangle_{L^2(\gamma)}\right| &\leq \sqrt{\sum_{p=0}^{\infty}(p+1)!a_p^2}\sqrt{\sum_{p=0}(p+1)!b_{p+1}^2} \\ &\leq \mathrm{cst}(\varphi) \times \|\psi\|_{L^2(\gamma)}. \quad \square\end{aligned}$$

EXAMPLE

Example.

- We have, for any $p \in \mathbb{N}$:

$$\delta H_p = xH_p - H_p' = xH_p - pH_{p-1} = H_{p+1}.$$

- By induction, $\boxed{H_p = \delta^p 1}$ for all $p \in \mathbb{N}$.

AN EXPRESSION FOR THE ENTRIES

A useful expression for the entries.

- If $\varphi = \sum_{p=0}^{\infty} a_p H_p \in \mathbb{D}^{k,2}(\gamma)$, then

$$\begin{aligned} k! a_k &= \langle \varphi, H_k \rangle_{L^2(\gamma)} = \langle \varphi, \delta H_{k-1} \rangle_{L^2(\gamma)} \\ &= \langle \varphi', H_{k-1} \rangle_{L^2(\gamma)} \quad \text{(duality)} \\ &= \dots \\ &= \langle \varphi^{(k)}, 1 \rangle_{L^2(\gamma)}. \end{aligned}$$

  That is,

$$\boxed{a_k = \frac{1}{k!} \mathbb{E}[\varphi^{(k)}(N)]}, \quad N \sim N(0,1).$$

- In particular, $a_0 = \mathbb{E}[\varphi(N)]$.
- Moreover, $\mathbb{E}[H_k(N)] = 0$ for all $k \geq 1$.

An application

- We have

$$
\begin{aligned}
e^{cx} &= \sum_{k=0}^{\infty} \frac{1}{k!} \mathbb{E}\left(\frac{d^k}{dx^k} e^{cx} \bigg|_{x=N} \right) H_k(x) \\
&= \sum_{k=0}^{\infty} \frac{c^k}{k!} \mathbb{E}[e^{cN}] H_k(x) = e^{\frac{c^2}{2}} \sum_{k=0}^{\infty} \frac{c^k}{k!} H_k(x).
\end{aligned}
$$

(Compare with $e^{cx} = \sum_{k=0}^{\infty} \frac{c^k}{k!} x^k$.)

- **Corollary**. If $U, V \sim N(0,1)$ are jointly Gaussian and if $k, l \in \mathbb{N}$ then

$$
\boxed{\mathbb{E}[H_k(U) H_l(V)] = \begin{cases} k! \mathbb{E}[UV]^k & \text{if } k = l \\ 0 & \text{otherwise.} \end{cases}}
$$

Proof of the corollary

Proof.

- We have, on one hand

$$\mathbb{E}[e^{xU+yV}] = e^{\frac{x^2+y^2}{2}} \sum_{k,l=0}^{\infty} \frac{x^k y^l}{k!l!} \mathbb{E}[H_k(U)H_l(V)].$$

- On the other hand,

$$\begin{aligned} \mathbb{E}[e^{xU+yV}] &= e^{\frac{1}{2}\mathrm{Var}(xU+yV)} = e^{\frac{1}{2}\{x^2+y^2+2xy\mathbb{E}[UV]\}} \\ &= e^{\frac{x^2+y^2}{2}} e^{xy\,\mathbb{E}[UV]} = e^{\frac{x^2+y^2}{2}} \sum_{k=0}^{\infty} \frac{x^k y^k}{k!} \mathbb{E}[UV]^k. \end{aligned}$$

- By identification,

$$\mathbb{E}[H_k(U)H_l(V)] = \begin{cases} k!\mathbb{E}[UV]^k & \text{if } k = l \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

THE ORNSTEIN-UHLENBECK SEMIGROUP $(P_t)_{t\geq 0}$

- **Definition**. For $t \geq 0$ and $\varphi = \sum_{p=0}^{\infty} a_p H_p \in L^2(\gamma)$, we set

$$\boxed{P_t\varphi = \sum_{p=0}^{\infty} e^{-pt} a_p H_p}.$$

This defines the **Ornstein-Uhlenbeck semigroup**.

Proposition.

(a) $P_sP_t = P_{s+t}$

(b) P_0 is the identity operator, that is, $P_0\varphi = \varphi$

(c) P_∞ is the expectation operator, that is, $P_\infty\varphi = \mathbb{E}[\varphi(N)]$

(d) [*contractivity*][1] $\|P_t\varphi\|_{L^2(\gamma)} \leq \|\varphi\|_{L^2(\gamma)}$ for any $\varphi \in L^2(\gamma)$.

(e) [*Mehler formula*] one has

$$\boxed{(P_t\varphi)(x) = \mathbb{E}[\varphi(e^{-t}x + \sqrt{1-e^{-2t}}N)]},$$

for $N \sim N(0,1)$ and any $\varphi \in L^2(\gamma)$.

(f) $DP_t\varphi = e^{-t}P_t\varphi'$ for any $\varphi \in \mathbb{D}^{1,2}(\gamma)$.

[1]We have actually much better: $\|P_t\varphi\|_{L^{1+e^{2t}}(\gamma)} \leq \|\varphi\|_{L^2(\gamma)}$.

Exercise

Exercise. Prove the points (a) to (f) of the previous proposition.

THE GENERATOR L OF $(P_t)_{t\geq 0}$

- For any $\varphi \in \mathbb{D}^{2,2}(\gamma)$, we can write

$$\begin{aligned}
\frac{d}{dt}(P_t\varphi)(x) &= -xe^{-t}\mathbb{E}[\varphi'(e^{-t}x + \sqrt{1-e^{-2t}}N)] \\
&\quad + \frac{e^{-2t}}{\sqrt{1-e^{-2t}}}\mathbb{E}[\varphi'(e^{-t}x + \sqrt{1-e^{-2t}}N)N] \\
&= -xe^{-t}P_t\varphi'(x) + e^{-2t}P_t\varphi''(x),
\end{aligned}$$

  where in the last line we have used that $\mathbb{E}[Ng(N)] = \mathbb{E}[g'(N)]$.
- Now, set $\boxed{L = -\delta D}$ and let us compute $LP_t\varphi$.

THE GENERATOR L OF $(P_t)_{t\geq 0}$

- We can write

$$\begin{aligned}(LP_t\varphi)(x) &= -\delta(DP_t\varphi)(x) = -e^{-t}\delta(P_t\varphi')(x)\\ &= -xe^{-t}P_t\varphi'(x) + e^{-2t}P_t\varphi''(x)\\ &= \frac{d}{dt}(P_t\varphi)(x).\end{aligned}$$

- That is, L is **the generator** of $(P_t)_{t\geq 0}$.

Expansion of the variance

- Let us show how, using the previously introduced operators, one can derive useful expansions for the variance in $L^2(\gamma)$.
- Let $\varphi \in \mathbb{D}^{\infty,2}(\gamma)$ of the form $\varphi = \sum_{p=0}^{\infty} a_p H_p$.
- We have

$$\mathbb{E}[\varphi(N)^2] = \sum_{p=0}^{\infty} p! a_p^2 = \mathbb{E}[\varphi(N)]^2 + \sum_{p=1}^{\infty} \frac{1}{p!} \mathbb{E}[\varphi^{(p)}(N)]^2.$$

- That is,

$$\boxed{\mathbf{Var}(\varphi(N)) = \sum_{p=1}^{\infty} \frac{1}{p!} \mathbb{E}[\varphi^{(p)}(N)]^2}.$$

Expansion of the variance

- Now, let us introduce, for $0 < t \leq 1$,

$$g(t) = \mathbb{E}\left[\left(P_{\log \frac{1}{\sqrt{t}}}\varphi(N)\right)^2\right].$$

- We have $g(1) = \mathbb{E}[\varphi(N)^2]$ and $g(0) = \mathbb{E}[\varphi(N)]^2$, so that

$$\mathbf{Var}(\varphi(N)) = g(1) - g(0) = \int_0^1 g'(t)dt.$$

Expansion of the variance

- We compute

$$
\begin{aligned}
g'(t) &= -\frac{1}{t}\,\mathbb{E}\left[P_{\log\frac{1}{\sqrt{t}}}\varphi(N)\times LP_{\log\frac{1}{\sqrt{t}}}\varphi(N)\right] \\
&= \frac{1}{t}\,\mathbb{E}\left[\left(DP_{\log\frac{1}{\sqrt{t}}}\varphi(N)\right)^2\right] \\
&= \mathbb{E}\left[\left(P_{\log\frac{1}{\sqrt{t}}}\varphi'(N)\right)^2\right].
\end{aligned}
$$

- Clearly, by iterating:

$$
g^{(k)}(t) = \mathbb{E}\left[\left(P_{\log\frac{1}{\sqrt{t}}}\varphi^{(k)}(N)\right)^2\right].
$$

Expansion of the variance

- Now, we use Taylor:

$$g(0) = g(1) + \sum_{k=1}^{m} g^{(k)}(1)\frac{(-1)^k}{k!} + \frac{1}{m!}\int_1^0 (-t)^m g^{(m+1)}(t)dt.$$

- We deduce

$$\boxed{\mathbf{Var}(\varphi(N)) = \sum_{k=1}^{m} \frac{(-1)^{k+1}}{k!}\mathbb{E}[\varphi^{(k)}(N)^2] + \frac{(-1)^m}{m!}\int_0^1 t^m g^{(m+1)}(t)dt},$$

with $\int_0^1 t^m g^{(m+1)}(t)dt \geq 0$.

Expansion of the variance

- If $m = 1$, we recover the classical **Poincaré inequality**:

$$\boxed{\mathbf{Var}(\varphi(N)) \leq \mathbb{E}[\varphi'(N)^2]}.$$

- If $m = 2$, one obtains

$$\boxed{\mathbf{Var}(\varphi(N)) \geq \mathbb{E}[\varphi'(N)^2] - \frac{1}{2}\mathbb{E}[\varphi''(N)^2]}.$$

- If $m = 3$, one obtains

$$\boxed{\mathbf{Var}(\varphi(N)) \leq \mathbb{E}[\varphi'(N)^2] - \frac{1}{2}\mathbb{E}[\varphi''(N)^2] + \frac{1}{6}\mathbb{E}[\varphi'''(N)^2]}.$$

- Etc.

Introduction to Malliavin calculus: any dimension

PREAMBLE

- For the sake of simplicity and to avoid technicalities, in this series of lectures we will only consider the case where the underlying Gaussian process is a classical Brownian motion $B = (B_t)_{t \geq 0}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- It will also be always **implicitely assumed** that the σ-field $\mathcal{F}$ is generated by B, that is, $\mathcal{F} = \sigma\{B_t : t \geq 0\}$.
- That is, each time we speak about a random variable, it is implicit that it is measurable with respect to B.

Chaotic expansion

- **Theorem**. Any $F \in L^2(\Omega)$ can be uniquely expanded as

$$\boxed{F = \mathbb{E}[F] + \sum_{p=1}^{\infty} I_p(f_p)}, \tag{1}$$

where each $f_p : \mathbb{R}_+^p \to \mathbb{R}$ is symmetric[2] and square integrable, and where

$$\boxed{I_p(f_p) = p! \int_0^{\infty} dB_{t_1} \dots \int_0^{t_{p-2}} dB_{t_{p-1}} \int_0^{t_{p-1}} dB_{t_p} f_p(t_1, \dots, t_p)}.$$

- (1) is called the **chaotic expansion of** F.

[2] that is, for all $\sigma \in \mathfrak{S}_p$ and all $x_1, \dots, x_p \in \mathbb{R}_+$, one has $f_p(x_{\sigma(1)}, \dots, x_{\sigma(p)}) = f_p(x_1, \dots, x_p)$

LINK WITH DIMENSION 1

Theorem. If $h : \mathbb{R}_+ \to \mathbb{R}$ is such that $\int_0^\infty h^2(t)dt = 1$ then, for any integer $p \geq 1$:

$$\boxed{H_p\left(\int_0^\infty h(t)dB_t\right) = I_p(h^{\otimes p})},$$

where $h^{\otimes p}(t_1, \ldots, t_p) = h(t_1)\ldots h(t_p)$ is symmetric and square integrable.

Proof. We make use of Itô's formula.

- Let $t \in \mathbb{R}$ and, for any $x \in \mathbb{R}$ and $a \geq 0$, set

$$\widetilde{H}_p(x,a) = \begin{cases} a^{p/2}H_p(x/\sqrt{a}) & \text{if } a \neq 0 \\ x^p & \text{if } a = 0 \end{cases}.$$

- Using the properties of Hermite polynomials, it is readily checked that $\left(\frac{1}{2}\frac{\partial^2}{\partial^2 x} - \frac{\partial}{\partial a}\right)\widetilde{H}_p = 0$ and $\frac{\partial}{\partial x}\widetilde{H}_p = p\widetilde{H}_{p-1}$.

Link with dimension 1

- *Proof* (continued). Itô's formula implies

$$
\begin{aligned}
& \widetilde{H}_p \left(\int_0^t h(u) dB_u, \int_0^t h^2(u) du \right) \\
= \; & p \int_0^t dB_{t_1} \, h(t_1) \widetilde{H}_{p-1} \left(\int_0^{t_1} h(u) dB_u, \int_0^{t_1} h^2(u) du \right) \\
= \; & \dots \\
= \; & p! \int_0^t dB_{t_1} \, h(t_1) \int_0^{t_1} dB_{t_2} \, h(t_2) \dots \int_0^{t_{p-2}} dB_{t_{p-1}} \, h(t_{p-1}) \\
& \qquad \times \widetilde{H}_1 \left(\int_0^{t_{p-1}} h(u) dB_u, \int_0^{t_{p-1}} h^2(u) du \right) \\
= \; & p! \int_0^t dB_{t_1} \, h(t_1) \int_0^{t_1} dB_{t_2} \, h(t_2) \dots \int_0^{t_{p-1}} dB_{t_p} \, h(t_p).
\end{aligned}
$$

- The conclusion follows by letting $t \to \infty$ and by observing that $\widetilde{H}_p(x, 1) = H_p(x)$. □

Continuous quadratic variation of Brownian motion

Example: *continuous quadratic variation of Brownian motion*

- Let $F = \int_0^T (B_{u+1} - B_u)^2 du$ be the continuous quadratic variation of the Brownian motion B over the time interval $[0, T]$.
- We have, since $B_{u+1} - B_u = \int_0^\infty \mathbf{1}_{[u,u+1]}(t) dB_t \sim N(0, 1)$,

$$\begin{aligned} F &= \mathbb{E}[F] + \int_0^T H_2(B_{u+1} - B_u) du \\ &= \mathbb{E}[F] + \int_0^T I_2(\mathbf{1}_{[u,u+1]^2}) du \\ &= \mathbb{E}[F] + I_2(f_2), \end{aligned}$$

  where $f_2(s,t) = \int_0^T \mathbf{1}_{[u,u+1]^2}(s,t) du$.
- This is the chaotic expansion of F.

Exercise. Let $T > 0$. For each of the following expressions of F, compute its chaotic expansion.

1. $F = (B_T)^n$ with $n \in \mathbb{N}^*$.
2. $F = e^{B_T}$.
3. $F = \int_0^T B_u du$.
4. $F = \int_0^T (B_{u+1} - B_u)^3 du$.

Isometry-orthogonality for multiple integrals

- **Theorem**. For any $p, q \geq 1$ and any $f \in L^2_s(\mathbb{R}^p_+)$ and $g \in L^2_s(\mathbb{R}^q_+)$:

$$\boxed{\mathbb{E}[I_p(f) I_q(g)] = \begin{cases} 0 & \text{if } p \neq q \\ p! \langle f, g \rangle_{L^2(\mathbb{R}^p_+)} & \text{if } p = q \end{cases}}$$

Isometry-orthogonality for multiple integrals

- Let $U, V \sim N(0,1)$ be jointly Gaussian. Without loss of generality, we can assume that $U = \int_0^\infty u(t)dB_t$ and $V = \int_0^\infty v(t)dB_t$ with $\|u\|_{L^2(\mathbb{R}_+)} = \|v\|_{L^2(\mathbb{R}_+)} = 1$ and $\langle u, v\rangle_{L^2(\mathbb{R}_+)} = \mathbb{E}[UV]$.
- If $p, q \geq 1$, we can write

$$\mathbb{E}[H_p(U)H_q(V)] = \mathbb{E}[I_p(u^{\otimes p})I_q(v^{\otimes q})].$$

- As a result, if $p \neq q$ then $\mathbb{E}[H_p(U)H_q(V)] = 0$.
- If $p = q$, then

$$\begin{aligned} \mathbb{E}[H_p(U)H_q(V)] &= p!\langle u^{\otimes p}, v^{\otimes p}\rangle_{L^2(\mathbb{R}^p)} = p!\langle u, v\rangle^p_{L^2(\mathbb{R})} \\ &= p!\mathbb{E}[UV]^p. \end{aligned}$$

MULTIPLICATION FORMULA FOR MULTIPLE INTEGRALS

- **Theorem** (*Multiplication formula*): If $f \in L^2_s(\mathbb{R}^p_+)$ and $g \in L^2_s(\mathbb{R}^q_+)$ then

$$\boxed{I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} r!\binom{p}{r}\binom{q}{r} I_{p+q-2r}(f\widetilde{\otimes}_r g)},$$

where

$$\begin{aligned} & f \otimes_r g(x_1, \ldots, x_{p+q-2r}) \\ = & \int f(x_1, \ldots, x_{p-r}, u_1, \ldots, u_r) g(x_{p-r+1}, \ldots, x_{p+q-2r}, u_1, \ldots, u_r) \\ & \qquad du_1 \ldots du_r \end{aligned}$$

and $\widetilde{}$ stands for symetrization:

$$\widetilde{h}(x_1, \ldots, x_a) = \frac{1}{a!} \sum_{\sigma \in \mathfrak{S}_a} h(x_{\sigma(1)}, \ldots, x_{\sigma(a)}).$$

Malliavin derivative

- If $F = \sum_{p=0}^{\infty} I_p(f_p) \in L^2(\Omega)$ then

$$\mathbb{E}[F^2] = \sum_{p=0}^{\infty} p! \|f_p\|^2 < \infty.$$

- **Definition**. We set

$$\boxed{\mathbb{D}^{k,2}(\Omega) = \left\{ F \in L^2(\Omega) : \sum_{p=0}^{\infty} p^k p! \|f_p\|^2 < \infty \right\}}.$$

- **Definition** (Malliavin derivative). If $F \in \mathbb{D}^{1,2}(\Omega)$, we set

$$\boxed{D_x F = \sum_{p=1}^{\infty} p I_{p-1}\big(f_p(\cdot, x)\big), \quad x \in \mathbb{R}_+}.$$

Malliavin derivative

- As a particular case, $\boxed{D_x\left(\int_0^\infty h(t)dB_t\right) = h(x)}$.
- The process $DF = (D_xF)_{x\geq 0}$ belongs to $L^2(\Omega \times \mathbb{R}_+)$:

$$
\begin{aligned}
\mathbb{E}[\|DF\|^2_{L^2(\mathbb{R}_+)}] &= \sum_{p=1}^{\infty} p^2 \int_0^\infty \mathbb{E}\left[I_{p-1}(f_p(\cdot,x))^2\right] dx \\
&= \sum_{p=1}^{\infty} p^2 (p-1)! \int_0^\infty \|f_p(\cdot,x)\|^2 dx \\
&= \sum_{p=1}^{\infty} p p! \|f_p\|^2 < \infty.
\end{aligned}
$$

Malliavin derivative: chain rule

- **Theorem** (*chain rule for D*). If $\phi : \mathbb{R}^d \to \mathbb{R}$ is C^1 and Lipschitz and if $F_1, \ldots, F_d \in \mathbb{D}^{1,2}(\Omega)$, then $\phi(F_1, \ldots, F_d)$ belongs to $\mathbb{D}^{1,2}(\Omega)$ with

$$\boxed{D_x \phi(F_1, \ldots, F_d) = \sum_{k=1}^{d} \frac{\partial \phi}{\partial x_k}(F_1, \ldots, F_d) D_x F_k}.$$

- *Particularly important case*:

$$\boxed{D_x \phi(F) = \phi'(F) D_x F}$$

if $F \in \mathbb{D}^{1,2}(\Omega)$ and $\phi \in C^1 \cap \mathrm{Lip}$.

Exercise

Exercise. Let $T > 0$. For each of the following expressions of F, compute its Malliavin derivative.

1. $F = B_T^n$ with $n \in \mathbb{N}^*$.
2. $F = e^{B_T}$.
3. $F = \int_0^T B_u du$.
4. $F = \int_0^T (B_{u+1} - B_u)^n du$ with $n \in \mathbb{N}^*$.

Exercise. Let $x_0 \in \mathbb{R}$ and let $\sigma, b : \mathbb{R} \to \mathbb{R}$ be C^1 and (globally) Lipschitz. Consider the strong solution $X = (X_t)_{t \geq 0}$ of the stochastic differential equation (or, more correctly, stochastic integral equation):

$$X_t = x_0 + \int_0^t b(X_u)du + \int_0^t \sigma(X_u)dB_u.$$

The goal of this exercise is to compute the Malliavin derivative of X_t when $t > 0$ is fixed.

EXERCISE (CONTINUED)

1. Let $z = (z_u)_{u\in[0,T]}$ be a simple adapted process, that is of the form
$$z_u = \sum_{i=1}^{k} \xi_i \mathbf{1}_{(t_i,t_{i+1}]}(u),$$
for an integer k, a finite sequence $t_0 = 0 < t_1 < \ldots < t_{k+1} = T$, and random variables $\xi_1, \ldots, \xi_k$ such that ξ_i is $\mathcal{F}_{t_i}$-measurable. Assume further that $\xi_i \in \mathbb{D}^{1,2}(\Omega)$ for each i. For any $s \in [0,T]$, show that
$$D_s\Big(\int_0^T z_u du\Big) = \int_0^T D_s z_u du \tag{2}$$
$$D_s\Big(\int_0^T z_u dB_u\Big) = z_s + \int_0^T D_s z_u dB_u. \tag{3}$$
By approximation, one can show that (2)-(3) extend to any adapted process z (not necessarily simple) such that $z_u \in \mathbb{D}^{1,2}(\Omega)$ for $u \in [0,T]$ and $\int_0^T \mathbb{E}[(D_s z_u)^2]du < \infty$.

2. For any $s, t > 0$, show that $D_s X_t = 0$ if $s > t$ whereas, for $s \leq t$,

$$\boxed{D_s X_t = \sigma(X_s) \exp\left\{ \int_s^t \left[b'(X_u) - \frac{1}{2}\sigma'^2(X_u) \right] du + \int_s^t \sigma'(X_u) dB_u \right\}}.$$

DIVERGENCE OPERATOR

- **Definition** (*divergence operator* δ). We have

$$\operatorname{Dom}\delta = \Big\{ u \in L^2(\mathbb{R}_+ \times \Omega) :\ \exists c > 0, \\ \big|\mathbb{E}\langle DF, u\rangle_{L^2(\mathbb{R}_+)}\big| \leq c\|F\|_{L^2(\Omega)}\ \forall F \in \mathbb{D}^{1,2}(\Omega) \Big\}.$$

- If $u \in \operatorname{Dom}\delta$ then $\delta(u)$ is characterized by

$$\boxed{\mathbb{E}[F\delta(u)] = \mathbb{E}\big(\langle DF, u\rangle_{L^2(\mathbb{R}_+)}\big) \quad \forall F \in \mathbb{D}^{1,2}(\Omega)}.$$

ORNSTEIN-UHLENBECK SEMIGROUP

- **Definition** (*Ornstein-Uhlenbeck semigroup*). If $F = \sum_{p=0}^{\infty} I_p(f_p) \in L^2(\Omega)$ and $t \geq 0$, we set

$$\boxed{P_t F = \sum_{p=0}^{\infty} e^{-pt} I_p(f_p)}.$$

- **Definition** (generator). If $F = \sum_{p=0}^{\infty} I_p(f_p) \in \mathbb{D}^{2,2}(\Omega)$, we set

$$\boxed{LF = -\sum_{p=0}^{\infty} p\, I_p(f_p)}.$$

- **Proposition**: $\boxed{L = \frac{d}{dt}_{|t=0} P_t}$ and $\boxed{L = -\delta D}$.

Ornstein-Uhlenbeck semigroup

- **Definition** (*pseudo-inverse of the generator*). If $F = \sum_{p=0}^{\infty} I_p(f_p) \in L^2(\Omega)$, we set

$$\boxed{L^{-1}F = -\sum_{p=1}^{\infty} \frac{1}{p} I_p(f_p)}.$$

- **Theorem**: for all $F \in L^2(\Omega)$, we have

$$\boxed{F = \mathbb{E}[F] - \delta D L^{-1} F}.$$

- *Proof.* $F = \mathbb{E}[F] + LL^{-1}F = \mathbb{E}[F] - \delta D L^{-1} F$. □

Exercise. Let $F \in \mathbb{D}^{1,2}(\Omega)$. The goal of this exercise is to check that

$$\boxed{\mathbf{Var}(F) = \int_0^\infty e^{-t}\, \mathbb{E}[\langle DF, P_t(DF)\rangle_{L^2(\mathbb{R}_+)} dt}.$$

We recall that $P_t F = \sum_{p=0}^\infty e^{-pt} I_p(f_p)$ if $F = \sum_{p=0}^\infty I_p(f_p)$ is the chaotic expansion of F, that $\frac{d}{dt} P_t = L P_t$, and that $L = -\delta D$ (as operators).

1. Show that $\mathbf{Var}(F) = \mathbb{E}\left[F(P_0 F - P_\infty F)\right]$.
2. Deduce that $\mathbf{Var}(F) = \int_0^\infty \mathbb{E}[F \times \delta(D P_t F)] dt$.
3. Show that $D_x P_t F = e^{-t} P_t D_x F$ for all $x, t \geq 0$.
4. Conclude.

Exercise. For any $T > 0$ and $v \in \mathbb{R}$, we set $F_T = \int_0^T (B_{u+1} - B_u)^2 du$ and $\rho(v) = (1 - |v|)_+$ (with $y_+ = \max(y, 0)$ the positive part of y).

1. Compute $\mathbb{E}[F_T]$.
2. Show that $\mathbb{E}[(B_{u+1} - B_u)(B_{v+1} - B_v)] = \rho(u - v)$ for all $u, v \geq 0$.
3. Show that $\int_0^\infty \mathbf{1}_{[u,u+1]}(x)\mathbf{1}_{[v,v+1]}(x)dx = \rho(u - v)$ for all $u, v \geq 0$.
4. Show that $F_T - \mathbb{E}[F_T]$ belongs to the second Wiener chaos.
5. Show that $\mathbf{Var}(F_T) = 2\int_{-T}^T \rho(y)^2 (T - |y|)dy$.
   Hint: Use that $\mathbf{Var}(F) = \mathbb{E}[(F - \mathbb{E}[F])^2]$ and that $\mathbb{E}[H_2(U)H_2(V)] = 2(\mathbb{E}[UV])^2$ if $U, V \sim N(0, 1)$ are jointly Gaussian.
6. Deduce that $\mathbf{Var}(F_T) \sim 4T/3$ as $T \to \infty$.

Malliavin calculus and absolute continuity: dimension one

- **Theorem**. Let $F \in \mathbb{D}^{2,2}(\Omega)$ be such that

$$\boxed{\|DF\|^2 = \int_0^\infty (D_x F)^2 dx > 0 \text{ almost surely}}.$$

  Then F has a density.
- Before proving this theorem, let us first see a nice application.
- **Shigekawa's theorem**. Let F have the form $F = I_p(f)$, with $f \neq 0$. Then F has a density.

Proof of Shigekawa's theorem. We will proceed by induction on p.

- $\underline{p=1}$: one has $F = I_1(f) = \int_0^\infty f(t)dB_t \sim N(0, \int_0^\infty f^2(t)dt)$ with $\int_0^\infty f^2(t)dt > 0 \to$ this is then OK!
- $\underline{p-1 \to p}$: Let $F = I_p(f)$ with $f \neq 0$. We need to check that $\int_0^\infty (D_x F)^2 dx > 0$ almost surely.
  - We have $D_x F = p I_{p-1}(f(\cdot, x))$.
  - Since $f \neq 0$, there exists $h \in L^2(\mathbb{R}_+)$ such that $\mathbf{y} \in \mathbb{R}_+^{p-1} \mapsto \int_0^\infty f(\mathbf{y}, x)h(x)dx$ is a *non-zero* element of $L_s^2(\mathbb{R}_+^{p-1})$.
  - We then have that $\int_0^\infty D_x F h(x)dx = pI_{p-1}\left(\int_0^\infty f(\cdot, x)h(x)dx\right)$ has a density (induction assumption).
  - As a result, using first Cauchy-Schwarz,

$$\mathbb{P}\left(\int_0^\infty (D_x F)^2 dx = 0\right) \leq \mathbb{P}\left(\int_0^\infty D_x F\, h(x)dx = 0\right) = 0,$$

  - That is, $\int_0^\infty (D_x F)^2 dx > 0$ a.s. and F has a density. □

Proof of the absolute continuity theorem.

- *Goal*: According to the Radon-Nikodym criterion, we must show that if, $A \in \mathcal{B}(\mathbb{R})$ satisfies $\lambda(A) = 0$ (with λ the Lebesgue measure) then $\mathbb{P}(F \in A) = 0$.
- Let $B \in \mathcal{B}(\mathbb{R})$ be a *bounded* Borel set. We claim that

$$\mathbb{E}\left[\mathbf{1}_{\{F\in B\}} \|DF\|^2\right] = \mathbb{E}\left[\int_{-\infty}^{F} \mathbf{1}_B(x)dx \times (-LF)\right].$$

Proof of the absolute continuity theorem (continued).

- Indeed, let $h : \mathbb{R} \to [0,1]$ be continuous with compact support.
- Then $x \mapsto \int_{-\infty}^{x} h(t)dt$ is C^1 and Lipschitz.
- We deduce, using $L = -\delta D$ and the duality formula (first equality) as well as the chain rule for D (second equality),

$$\begin{aligned}\mathbb{E}\left[\int_{-\infty}^{F} h(x)dx \times (-LF)\right] &= \mathbb{E}\left[\left\langle D\left(\int_{-\infty}^{F} h(x)dx\right), DF\right\rangle\right] \\ &= \mathbb{E}\left[h(F)\|DF\|^2\right].\end{aligned}$$

- Thus, the claim is satisfied with h instead of $\mathbf{1}_B$.
- We deduce the claim by approximation (Lusin's theorem and dominated convergence).

Proof of the absolute continuity theorem (continued).

- We now apply the claim to $B = A \cap [-n, n]$, where $n \in \mathbb{N}$ and $A \in \mathcal{B}(\mathbb{R})$ satisfies $\lambda(A) = 0$:

$$\mathbb{E}\left[\mathbf{1}_{\{F \in A \cap [-n,n]\}} \|DF\|^2\right] = \mathbb{E}\left[\int_{-\infty}^{F} \mathbf{1}_{A \cap [-n,n]}(x)dx \times (-LF)\right].$$

- Since $\int_{-\infty}^{\cdot} \mathbf{1}_{A \cap [-n,n]}(x)dx = 0$ a.e., one obtains that

$$\mathbb{E}\left[\mathbf{1}_{\{F \in A \cap [-n,n]\}} \|DF\|^2\right] = 0$$

  for all $n \in \mathbb{N}$.

- By monotone convergence ($n \to \infty$), it comes that

$$\mathbb{E}\left[\mathbf{1}_{\{F \in A\}} \|DF\|^2\right] = 0.$$

- The desired conclusion follows since $\|DF\|^2 > 0$ a.s. $\square$

Exercise.

- Let $x_0 \in \mathbb{R}$ and let $\sigma, b : \mathbb{R} \to \mathbb{R}$ be C^1 and (globally) Lipschitz.
- Consider the strong solution $X = (X_t)_{t\geq 0}$ of the stochastic differential equation (or, more correctly, stochastic integral equation):

$$X_t = x_0 + \int_0^t b(X_u)du + \int_0^t \sigma(X_u)dB_u.$$

- If $\sigma(x_0) \neq 0$, show that X_t has a density for any $t > 0$.

Exercise

Exercise. Let $F \in \mathbb{D}^{1,2}(\Omega)$ be such that $\mathbb{E}[F] = 0$, and let us consider the function $g_F : \mathbb{R} \to \mathbb{R}$ defined through the following identity:

$$g_F(F) = \mathbb{E}\big[\langle DF, -DL^{-1}F\rangle_{L^2(\mathbb{R}_+)} \,|\, F\big].$$

1. Let C be a Borel set of $\mathbb{R}$, and set $\phi_C(x) = \int_0^x \mathbf{1}_C(t)dt$ (with the usual convention $\int_0^x = -\int_x^0$ for negative x).
   1.1 Show that $x\phi_C(x) \geq 0$ for all $x \in \mathbb{R}$.
   1.2 Deduce that $\mathbb{E}[g_F(F)\mathbf{1}_{\{F \in C\}}] \geq 0$.
   1.3 Conclude that $g_F(F) \geq 0$ a.s.
2. If $g_F(F) > 0$ a.s., show that F has a density.
3. Assume conversely that F has a density, say ρ. Show that $g_F(F) = \frac{\int_F^\infty y\rho(y)dy}{\rho(F)}$ and deduce that $g_F(F) > 0$ a.s.

Malliavin calculus and absolute continuity: any dimension

Absolute continuity in any dimension

Theorem (Malliavin)

- Let $F = (F_1, \ldots, F_d)$ be such that $F_i \in \mathbb{D}^\infty(\Omega)$ for all $i = 1, \ldots, d$.
- Let $\Gamma = \big(\langle DF_i, DF_j\rangle\big)_{1\le i,j\le d}$ be the Malliavin matrix of F.
- If $\det\Gamma > 0$ almost surely, then F has a density.

Three remarks:

(a) If $d = 1$, then Γ reduces to $\|DF\|^2$, and one recovers the result in dimension one.

(b) The Malliavin matrix is a *Gram matrix*; as such, it is symmetric and positive, meaning that $\det\Gamma \geq 0$ a.s..

(c) The imposed regularity assumption (namely, $F_i \in \mathbb{D}^\infty$) is too much demanding (actually: $F_i \in \mathbb{D}^{1,2}$ is enough).

PROOF OF MALLIAVIN'S THEOREM

- *Goal*: According to the Radon-Nikodym criterion, and like in dimension 1, we must and will show that $\mathbb{P}(F \in A) = 0$ for each Borelian set $A \in \mathcal{B}(\mathbb{R}^d)$ of Lebesgue measure zero.
- Fix $i = 1, \ldots, d$ as well as a test function $\phi \in C_c^\infty(\mathbb{R}^d)$.
- Using the chain rule, one can write

$$\begin{pmatrix} \langle D\phi(F), DF_1 \rangle \\ \vdots \\ \langle D\phi(F), DF_d \rangle \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^d \frac{\partial \phi}{\partial x_k}(F) \langle DF_k, DF_1 \rangle \\ \vdots \\ \sum_{k=1}^d \frac{\partial \phi}{\partial x_k}(F) \langle DF_k, DF_d \rangle \end{pmatrix} = \Gamma \begin{pmatrix} \frac{\partial \phi}{\partial x_1}(F) \\ \vdots \\ \frac{\partial \phi}{\partial x_d}(F) \end{pmatrix}.$$

Proof of Malliavin's theorem

- Thanks to the identity $\mathrm{Adj}\Gamma \times \Gamma = \det\Gamma\,\mathrm{Id}$ (where $\mathrm{Adj}\,\Gamma$ refers to the adjugate of Γ), one deduces from

$$\begin{pmatrix} \langle D\phi(F), DF_1\rangle \\ \vdots \\ \langle D\phi(F), DF_d\rangle \end{pmatrix} = \Gamma \begin{pmatrix} \frac{\partial\phi}{\partial x_1}(F) \\ \vdots \\ \frac{\partial\phi}{\partial x_d}(F) \end{pmatrix}.$$

that

$$\boxed{\det\Gamma\,\frac{\partial\phi}{\partial x_i}(F) = \sum_{j=1}^{d}(\mathrm{Adj}\,\Gamma)_{j,i}\langle D\phi(F), DF_j\rangle}.$$

Proof of Malliavin's theorem

- For any $\phi \in C_c(\mathbb{R})$, one has

$$\phi(x) = \int_{-\infty}^{x} \phi'(y)dy = \int_{-\infty}^{\infty} \phi'(y)\mathbf{1}_{[0,\infty)}(x-y)dy = (\phi' * \mathbf{1}_{[0,\infty)})(x).$$

- Assume $d \geq 2$ and let $\phi \in C_c^\infty(\mathbb{R}^d)$ be a test function.
- A multivariate extension of the identity $\phi = \phi' * \mathbf{1}_{[0,\infty)}$ is

$$\boxed{\phi = \sum_{i=1}^{d} \frac{\partial \phi}{\partial x_i} * \frac{\partial Q_d}{\partial x_i}},$$

where Q_d denotes the Poisson kernel on $\mathbb{R}^d$, defined as

$$Q_d(x) = c_d \begin{cases} \mathbf{1}_{[0,\infty)}(x_1) & \text{if } d = 1 \\ \log(x_1^2 + x_2^2) & \text{if } d = 2 \\ (x_1^2 + \ldots + x_d^2)^{\frac{d}{2}-1} & \text{if } d \geq 3, \end{cases}$$

with c_d a universal constant whose exact value is useless here.

- One can write

$$
\begin{aligned}
& \mathbb{E}[\det\Gamma \times \phi(F)] \\
= & \sum_{i=1}^{d} \int_{\mathbb{R}^d} \frac{\partial Q_d}{\partial x_i}(y) \mathbb{E}\left[\det\Gamma \frac{\partial \phi}{\partial x_i}(F-y)\right] dy \\
= & \sum_{i=1}^{d} \int_{\mathbb{R}^d} \frac{\partial Q_d}{\partial x_i}(y) \mathbb{E}\left[\sum_{j=1}^{d} (\mathrm{Adj}\,\Gamma)_{j,i} \langle D\phi(F-y), DF_j\rangle\right] dy \\
= & \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \frac{\partial Q_d}{\partial x_i}(y) \mathbb{E}\left[\delta((\mathrm{Adj}\,\Gamma)_{j,i} DF_j)\phi(F-y)\right] dy \\
= & \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \phi(y) \mathbb{E}\left[\delta((\mathrm{Adj}\,\Gamma)_{j,i} DF_j)\frac{\partial Q_d}{\partial x_i}(F-y)\right] dy.
\end{aligned}
$$

Proof of Malliavin's theorem

- Let B be a *bounded* Borel set.
- By a standard approximation argument (e.g. based on Lusin's theorem), one can extend the previous formula, a priori only valid for smooth ϕ, to $\phi = \mathbf{1}_B$:

$$\boxed{\mathbb{E}[\det \Gamma \, \mathbf{1}_B(F)] = \sum_{i,j=1}^{d} \int_B \mathbb{E}\left[\delta((\mathrm{Adj}\,\Gamma)_{j,i}\, DF_j) \frac{\partial Q_d}{\partial x_i}(F-y)\right] dy.}$$

- Now, let A be a Borel set of Lebesgue measure zero.
- From the framed formula with $B = A \cap [-n, n]$, one deduces

$$\mathbb{E}\left[\det \Gamma \times \mathbf{1}_{A\cap[-n,n]}(F)\right] = 0 \quad \text{for all } n \geq 1.$$

- The desired conclusion follows by letting $n \to \infty$ and because $\det \Gamma > 0$ a.s. $\square$

The Malliavin-Stein approach: dimension one

Malliavin-Stein approach in dimension one

- **Theorem** (Charles Stein).
  - Let $N \sim N(0,1)$.
  - Let F be any random variable such that $\mathbb{E}[F^2] < \infty$.
  - Then

$$\boxed{d_{TV}(F,N) \leq \sup_{\substack{\phi \in C^1 \\ \|\phi'\|_\infty \leq 2}} \left| \mathbb{E}[\phi'(F)] - \mathbb{E}[F\phi(F)] \right|}.$$

- We recall that the *total variation distance* between (the laws of) F and N means the following quantity:

$$d_{TV}(F,N) = \sup_{A \in \mathcal{B}(\mathbb{R})} \left| P(F \in A) - P(N \in A) \right|.$$

MALLIAVIN-STEIN APPROACH IN DIMENSION ONE

- *Proof of Stein's theorem*. First, one observes that

$$\begin{aligned} d_{TV}(F,N) &\leq \sup_{h:\mathbb{R}\to[0,1]} \big|\mathbb{E}[h(F)] - \mathbb{E}[h(N)]\big| \\ &= \sup_{\substack{h:\mathbb{R}\to[0,1] \\ h\in C^0}} \big|\mathbb{E}[h(F)] - \mathbb{E}[h(N)]\big| \quad \text{(by Lusin).} \end{aligned}$$

- Now, fix $h:\mathbb{R}\to[0,1]$ *continuous*, and set

$$\boxed{\phi(x) = e^{\frac{x^2}{2}} \int_{-\infty}^{x} \big(h(a) - \mathbb{E}[h(N)]\big) e^{-\frac{a^2}{2}} da}.$$

- One easily observes that, equivalently:

$$\boxed{\phi(x) = -e^{\frac{x^2}{2}} \int_{x}^{\infty} \big(h(a) - \mathbb{E}[h(N)]\big) e^{-\frac{a^2}{2}} da}.$$

- Since $h : \mathbb{R} \to [0,1]$ is continuous, it is immediate that

$$\begin{aligned}\phi(x) &= e^{\frac{x^2}{2}} \int_{-\infty}^{x} \big(h(a) - \mathbb{E}[h(N)]\big) e^{-\frac{a^2}{2}} da \\ &= -e^{\frac{x^2}{2}} \int_{x}^{\infty} \big(h(a) - \mathbb{E}[h(N)]\big) e^{-\frac{a^2}{2}} da\end{aligned}$$

is $\boxed{C^1}$, and satisfies

$$\boxed{\phi'(x) = x\phi(x) + h(x) - \mathbb{E}[h(N)]}.$$

- Moreover, we claim that $\boxed{|\phi'(x)| \leq 2}$ for all $x \in \mathbb{R}$.

- Since $\phi'(x) = x\phi(x) + h(x) - \mathbb{E}[h(N)]$, the claim will be checked if we show that $\big|x\phi(x)\big| \leq 1$.
- If $x \geq 0$, using that $\phi(x) = -e^{\frac{x^2}{2}} \int_x^\infty \big(h(a) - \mathbb{E}[h(N)]\big)e^{-\frac{a^2}{2}}da$:

$$|x\phi(x)| \leq xe^{\frac{x^2}{2}} \int_x^\infty e^{-\frac{a^2}{2}}da \leq e^{\frac{x^2}{2}} \int_x^\infty ae^{-\frac{a^2}{2}}da = 1.$$

- If $x < 0$, this time with $\phi(x) = e^{\frac{x^2}{2}} \int_{-\infty}^x \big(h(a) - \mathbb{E}[h(N)]\big)e^{-\frac{a^2}{2}}da$:

$$|x\phi(x)| \leq |x|e^{\frac{x^2}{2}} \int_{-\infty}^x e^{-\frac{a^2}{2}}da \leq e^{\frac{x^2}{2}} \int_{-\infty}^x |a|e^{-\frac{a^2}{2}}da = 1.$$

Malliavin-Stein approach in dimension one

- To conclude the proof of Stein's theorem, we fix a continuous $h : \mathbb{R} \to [0,1]$, and we let ϕ be defined as before.
- Since $\phi'(x) = x\phi(x) + h(x) - \mathbb{E}[h(N)]$, we have

$$\big|\mathbb{E}[h(F)] - \mathbb{E}[h(N)]\big| = \big|\mathbb{E}[\phi'(F)] - \mathbb{E}[F\phi(F)]\big|.$$

- Since ϕ belongs to C^1, and is such that $\big|\phi'(x)\big| \leq 2$ for all $x \in \mathbb{R}$, we deduce that

$$\big|\mathbb{E}[h(F)] - \mathbb{E}[h(N)]\big| \leq \sup_{\substack{\phi \in C^1 \\ \|\phi'\|_\infty \leq 2}} \big|\mathbb{E}[\phi'(F)] - \mathbb{E}[F\phi(F)]\big|,$$

from which the desired conclusion follows. □

Malliavin-Stein approach in dimension one

Theorem (Nourdin-Peccati).

- Let $F \in \mathbb{D}^{1,2}(\Omega)$ with $\mathbb{E}[F] = 0$.
- Let $N \sim N(0,1)$.
- Then

$$\boxed{d_{TV}(F,N) \leq 2\,\mathbb{E}\Big|1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mathbb{R}_+)}\Big|}.$$

MALLIAVIN-STEIN APPROACH IN DIMENSION ONE

Proof.

- We use Stein's theorem to bound $d_{TV}(F, N)$ by

$$\sup_{\substack{\phi \in C^1 \\ \|\phi'\|_\infty \leq 2}} \Big| \mathbb{E}[\phi'(F)] - \mathbb{E}[F\phi(F)] \Big|.$$

- Now, let $\phi \in C^1$ be such that $\|\phi'\|_\infty \leq 2$.
- We have, using $F = LL^{-1}F$ (since $\mathbb{E}[F] = 0$) and $L = -\delta D$,

$$\mathbb{E}[F\phi(F)] = \mathbb{E}\big[\delta(-DL^{-1}F)\,\phi(F)\big].$$

- By duality, one deduces $\mathbb{E}[F\phi(F)] = \mathbb{E}\big[\langle D\phi(F), -DL^{-1}F\rangle\big]$.
- Eventually, using the chain rule:

$$\mathbb{E}[F\phi(F)] = \mathbb{E}\big[\phi'(F)\langle DF, -DL^{-1}F\rangle\big].$$

Proof (continued).

- By plugging into Stein's bound, we get

$$\begin{aligned} & \sup_{\substack{\phi\in C^1\\ \|\phi'\|_\infty\leq 2}} \Big|\mathbb{E}[\phi'(F)] - \mathbb{E}[F\phi(F)]\Big| \\ = & \sup_{\substack{\phi\in C^1\\ \|\phi'\|_\infty\leq 2}} \Big|\mathbb{E}[\phi'(F)](1-\langle DF, -DL^{-1}F\rangle)\Big| \\ \leq & \; 2\,\mathbb{E}\Big|1-\langle DF, -DL^{-1}F\rangle_{L^2(\mathbb{R}_+)}\Big|. \end{aligned}$$

□

Fourth moment theorem

Fourth moment theorem

- The goal of this section is to prove the following result:
- **Theorem** (Nourdin-Peccati).
  - Let $F = I_p(f)$ with $f \in L^2_s(\mathbb{R}_+)$ such that $\mathbb{E}[F^2] = p!\|f\|^2 = 1$.
  - Then
  $$\boxed{d_{TV}(F,N) \leq \frac{2}{\sqrt{3}}\sqrt{\left|\mathbb{E}[F^4] - 3\right|}}.$$
- We recover the celebrated and surprising **fourth moment theorem** of Nualart and Peccati (2005): if $F_n = I_p(f_n)$ is a sequence of pth multiple Wiener-Itô integrals normalized so that $\mathbb{E}[I_p(f_n)^2] \to 1$, then $I_p(f_n) \to N(0,1)$ *if* and only if $\mathbb{E}[I_p(f_n)^4] \to 3$.

Fourth moment theorem

The proof relies on the followin lemma.
Lemma. If $F = I_p(f)$ with $\mathbb{E}[F^2] = 1$, then

$$\boxed{\mathbb{E}\left[\left(1 - \frac{1}{p}\|DF\|^2\right)^2\right] \leq \frac{1}{3}\left(\mathbb{E}[F^4] - 3\right)}.$$

Proof of the FMT. Let $F = I_p(f)$ with $\mathbb{E}[F^2] = 1$.

- We have $d_{TV}(F, N) \leq 2\,\mathbb{E}\left|1 - \langle DF, -DL^{-1}F\rangle\right|$.
- But $\langle DF, -DL^{-1}F\rangle = \frac{1}{p}\|DF\|^2$ since $L^{-1}F = -\frac{1}{p}F$.
- Hence $\mathbb{E}\left|1 - \langle DF, -DL^{-1}F\rangle\right| \leq \sqrt{\mathbb{E}\left[\left(1 - \frac{1}{p}\|DF\|^2\right)^2\right]}$.
- We conclude thanks to Lemma B. $\square$

Proof of the lemma. Let $F = I_p(f)$ with $\mathbb{E}[F^2] = 1$.

- *First step*. Since $D_xF = pI_{p-1}(f(\cdot, x))$,

$$
\begin{aligned}
\frac{1}{p}\|DF\|^2 &= \frac{1}{p}\int_0^\infty (D_xF)^2 dx = p\int_0^\infty I_{p-1}(f(\cdot,x))^2 dx \\
&= p\sum_{r=0}^{p-1} r!\binom{p-1}{r}^2 I_{2p-2-2r}\left(\int_0^\infty f(\cdot,x)\widetilde{\otimes}_r f(\cdot,x)dx\right) \\
&= p\sum_{r=1}^{p}(r-1)!\binom{p-1}{r-1}^2 I_{2p-2r}(f\widetilde{\otimes}_r f).
\end{aligned}
$$

- Using that $(r-1)!\binom{p-1}{r-1}^2 = \frac{rr!}{p^2}\binom{p}{r}^2$, we deduce

$$
\frac{1}{p}\|DF\|^2 = 1 + \sum_{r=1}^{p-1}\frac{rr!}{p}\binom{p}{r}^2 I_{2p-2r}(f\widetilde{\otimes}_r f).
$$

Proof of Lemma B (continued).

- As a result,

$$\boxed{\mathbb{E}\left[\left(\frac{1}{p}\|DF\|^2-1\right)^2\right]=\sum_{r=1}^{p-1}\frac{r^2r!^2}{p^2}\binom{p}{r}^4(2p-2r)!\|f\widetilde{\otimes}_rf\|^2}.$$

- *Second step*. One has

$$\begin{aligned}\mathbb{E}[F^4] &= \mathbb{E}[F\times F^3]=\mathbb{E}[LL^{-1}F\times F^3]\\ &= \frac{1}{p}\,\mathbb{E}[\delta(DF)\times F^3]\quad \text{(using } L^{-1}F=-\tfrac{1}{p}F \text{ and } L=-\delta D\text{)}\\ &= \frac{1}{p}\,\mathbb{E}[\langle DF,D(F^3)\rangle]=\frac{3}{p}\,\mathbb{E}[F^2\|DF\|^2]\quad \text{(by the chain rule}\\ &= 3\,\mathbb{E}[F^2(\frac{1}{p}\|DF\|^2-1)]+3.\end{aligned}$$

Proof of Lemma B (continued).

- But $F^2 = \sum_{r=0}^{p} r! \binom{p}{r}^2 I_{2p-2r}(f \widetilde{\otimes}_r f)$ by the multiplication formula, whereas

$$\frac{1}{p}\|DF\|^2 - 1 = \sum_{r=1}^{p-1} \frac{rr!}{p} \binom{p}{r}^2 I_{2p-2r}(f \widetilde{\otimes}_r f)$$

  as shown in the first step.

- As a result, using that $\mathbb{E}[F^4] - 3 = 3\,\mathbb{E}[F^2(\frac{1}{p}\|DF\|^2 - 1)]$

$$\boxed{\mathbb{E}[F^4] - 3 = 3 \sum_{r=1}^{p-1} \frac{r}{p} r!^2 \binom{p}{r}^4 (2p-2r)!\, \|f \widetilde{\otimes}_r f\|^2.}$$

Proof of Lemma B (continued).

- Comparing the two formulas

$$\mathbb{E}\left[\left(\frac{1}{p}\|DF\|^2 - 1\right)^2\right] = \sum_{r=1}^{p-1} \frac{r^2 r!^2}{p^2} \binom{p}{r}^4 (2p-2r)! \|f \widetilde{\otimes}_r f\|^2.$$

and

$$\mathbb{E}[F^4] - 3 = 3 \sum_{r=1}^{p-1} \frac{r}{p} r!^2 \binom{p}{r}^4 (2p-2r)! \, \|f \widetilde{\otimes}_r f\|^2.$$

we deduce

$$\mathbb{E}[F^4] - 3 \geq 3\, \mathbb{E}\left[\left(1 - \frac{1}{p}\|DF\|^2\right)^2\right].$$

□

Application to fractional Brownian motion

Application to fractional Brownian motion

- Let B^H be a fractional Brownian motion of index $H \in (0, 1)$.
- That is, B^H is a centered Gaussian process with covariance

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

- Set

$$\boxed{F_n = \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \left[\left(B_{k+1}^H - B_k^H \right)^2 - 1 \right]},$$

where $\sigma_n > 0$ is chosen so that $\mathbb{E}[F_n^2] = 1$.

Application to fractional Brownian motion

Theorem.

- If $0 < H \leq \frac{3}{4}$, then (Breuer-Major '83):

$$\boxed{F_n \overset{\text{law}}{\to} F_\infty \sim N(0,1)}.$$

- If $\frac{3}{4} < H < 1$, then (Taqqu '75):

$$\boxed{F_n \overset{\text{law}}{\to} F_\infty \sim \text{Rosenblatt}}.$$

- More precisely,

$$d_{TV}(F_n, F_\infty) = 0 \begin{cases} n^{-\frac{1}{2}} & \text{if } 0 < H < \frac{5}{8} \\ n^{-\frac{1}{2}}(\log n)^{\frac{3}{2}} & \text{if } H = \frac{5}{8} \\ n^{4H-3} & \text{if } \frac{5}{8} < H < \frac{3}{4} \\ (\log n)^{-1} & \text{if } H = \frac{3}{4} \\ n^{\frac{3}{2}-2H} & \text{if } \frac{3}{4} < H < 1 \end{cases}.$$

Proof of the normal approximation.

- *First step*. We let $\mathcal{H} = \overline{\text{span}\{B_k^H : k \in \mathbb{N}\}}^{L^2(\Omega)} \subset L^2(\Omega)$.
- $\mathcal{H}$ is a real separable Hilbert space, so there exists an isometric bijection $\phi : \mathcal{H} \to L^2(\mathbb{R}_+)$.
- Set $e_k = \phi(B_{k+1}^H - B_k^H)$.
- **Claim**: $\boxed{\{I_1(e_k) : k \in \mathbb{N}\} \overset{\text{law}}{=} \{B_{k+1}^H - B_k^H : k \in \mathbb{N}\}}$.
- Hence

$$\begin{aligned} F_n &\overset{\text{law}}{=} \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \left(I_1(e_k)^2 - 1 \right) \\ &= \frac{1}{\sigma_n} \sum_{k=0}^{n-1} I_2(e_k \otimes e_k) \\ &= I_2(f_n), \quad \text{with } f_n = \tfrac{1}{\sigma_n} \textstyle\sum_{k=0}^{n-1} e_k \otimes e_k. \end{aligned}$$

Proof of the normal approximation (continued).

- *Second step*. We set

$$\rho(r) = \frac{1}{2}\big(|r+1|^{2H} + |r-1|^{2H} - 2|r|^{2H}\big), \quad r \in \mathbb{Z}.$$

- **Exercise**:

$$\boxed{\sigma_n^2 = 2\sum_{k,l=0}^{n-1} \rho^2(k-l) = 2n \sum_{|r|<n} \rho^2(r)\big(1 - \frac{|r|}{n}\big)}.$$

Application to fractional Brownian motion

Proof of the normal approximation (continued).

Exercise:

1. If $H < \frac{3}{4}$ then $\sum_{r\in\mathbb{Z}} \rho^2(r) < \infty$ and

$$\boxed{\sigma_n \sim \sqrt{2\sum_{r\in\mathbb{Z}} \rho^2(r)}\sqrt{n}}.$$

2. If $H = \frac{3}{4}$ then $\sum_{r\in\mathbb{Z}} \rho^2(r) = \infty$ and

$$\boxed{\sigma_n \sim \frac{3}{4}\sqrt{n\log n}}.$$

Proof of the normal approximation (continued).

Exercise:

1. We have $\mathbb{E}\left[\left(1-\frac{1}{2}\|DF_n\|^2\right)^2\right] = 8\|f_n \otimes_1 f_n\|^2$.
2. Using Young inequality[3], shows that

$$\|f_n \otimes_1 f_n\|^2 \leq \frac{n}{\sigma_n^4}\left(\sum_{|k|<n} |\rho(k)|^{\frac{4}{3}}\right)^3.$$

3. Conclude by using that $d_{TV}(F_n, N) \leq 2\,\mathbb{E}\left|1-\frac{1}{2}\|DF_n\|^2\right|$.

□

[3] Young inequality: if $s, p, q \geq 1$ are such that $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{s}$, then

$$\|u * v\|_{\ell^s(\mathbb{Z})} \leq \|u\|_{\ell^p(\mathbb{Z})}\|v\|_{\ell^q(\mathbb{Z})}$$

The Malliavin-Stein approach: any dimension

Fourth Moment Theorem (Nualart-Peccati). Fix an integer $p \geq 2$, and let $\{f_n\}_{n\geq 1} \subset L^2_s(\mathbb{R}^p_+)$. Assume further that $\mathbb{E}[I_p(f_n)^2] \to \sigma^2$ as $n \to \infty$ for some $\sigma > 0$. Then, the following three assertions are equivalent as $n \to \infty$:

(1) $I_p(f_n) \overset{\text{law}}{\to} N(0, \sigma^2)$;

(2) $\mathbb{E}[I_p(f_n)^4] \to 3\sigma^4$;

(3) $\|f_n \otimes_r f_n\| \to 0$ for each $r = 1, \dots, p-1$.

Multivariate case

- **Theorem** (Peccati-Tudor). Consider d integers $p_1, \dots, p_d \geq 1$, with $d \geq 2$. Assume that all the p_i's are pairwise different[4]. For each $i = 1, \dots, d$, let $\{f_n^i\}_{n \geq 1} \subset L_s^2(\mathbb{R}_+^{p_i})$ satisfying $\mathbb{E}[I_{p_i}(f_n^i)^2] \to \sigma_i^2$ as $n \to \infty$ for some $\sigma_i > 0$. Then, the following two assertions are equivalent as $n \to \infty$:
  1. $I_{p_i}(f_n^i) \overset{\text{law}}{\to} N(0, \sigma_i^2)$ for all $i = 1, \dots, d$;
  2. $\big(I_{p_1}(f_n^1), \dots, I_{p_d}(f_n^d)\big) \overset{\text{law}}{\to} N_d\big(0, \text{diag}(\sigma_1^2, \dots, \sigma_d^2)\big)$.
- In other words, for sequences of vectors of multiple Wiener-Itô integrals, **componentwise convergence** to Gaussian always implies **joint convergence**.
- Using multivariate Stein's method, one can associate a rate to this convergence.

[4] We also know what happens when such an assumption is not satisfied

Two situations

- Very often, second order results for $\boxed{F_n = \mathbb{E}[F_n] + \sum_{p=1}^{\infty} I_p(f_{p,n})}$ can be deduced from the behaviour of its **chaotic projections** (in case of asymptotic gaussianity or not).

- **Situation 1**: F_n is **dominated by one of its projection**, and it inherits the rigid asymptotic structure of sequences inside a Wiener chaos (see next slide).

- **Situation 2**: **no single projection dominates**, and interactions have to be dealt with (see Breuer-Major theorem).

A RIGID STRUCTURE

Fix $p \geq 2$, and let $F_n = I_p(f_n)$, $n \geq 1$ with variance 1 (say).

- *Nourdin and Poly (2013)*: If $F_n \stackrel{\text{law}}{\to} Z$, then Z has a density.
- *Nualart and Peccati (2005)*: $F_n \stackrel{\text{law}}{\to} N(0,1)$ iff $\mathbb{E}F_n^4 \to 3 (= \mathbb{E}Z^4)$.
- *Peccati and Tudor (2005)*: componentwise convergence towards Gaussian implies joint convergence.
- *Nourdin and Peccati (2009)*: $F_n \stackrel{\text{law}}{\to} (Z^2-1)/\sqrt{2}$ iff $\mathbb{E}F_n^4 - 12\mathbb{E}F_n^3 \to -36$.
- *Nourdin and Rosiński (2014)*: if $H_n = I_q(g_n)$ (with variance 1), then F_n, H_n are asymptotically independent iff $\mathbf{Cov}(H_n^2, F_n^2) \to 0$.

Illustration: a modern proof of the Breuer-Major theorem

Theorem (Breuer-Major, 1983).

- Let $\{X_k\}_{k\geq 1}$ be a centered stationary Gaussian family such that $\mathbb{E}[X_k X_l] = \rho(k-l), k, l \geq 1$. Assume further that $\rho(0) = 1$, that is, each X_k is $\mathcal{N}(0,1)$ distributed.
- Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function of $L^2(\gamma)$ and let us expand it in terms of Hermite polynomials as $\varphi = \sum_{p=0}^{\infty} a_p H_p$.
- Assume $a_0 = \mathbb{E}[\varphi(X_1)] = 0$ and $\boxed{\sum_{k\in\mathbb{Z}} |\rho(k)|^r < \infty}$, where r is the Hermite rank of φ, that is, $r = \inf\{p : \, a_p \neq 0\}$.
- Then, as $n \to \infty$,

$$\boxed{V_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \varphi(X_k) \overset{\text{law}}{\to} N(0, \sigma^2)},$$

with σ^2 given by $\sigma^2 = \sum_{p=r}^{\infty} p! a_p^2 \sum_{k\in\mathbb{Z}} \rho(k)^p \in [0, \infty)$. (The fact that $\sigma^2 \in [0,\infty)$ is part of the conclusion.)

A couple of remarks about Breuer-Major

- The original proof consisted to show that **all the moments** of V_n converge to those of the Gaussian law $N(0, \sigma^2)$. As anyone might guess, this required a high ability and a lot of combinatorics.
- Assume $r \geq 2$ and $\rho(k) \sim |k|^{-D}$ as $|k| \to \infty$ for some $D \in (0, \frac{1}{r})$. In this case, one can show that

$$\boxed{n^{dD/2-1} \sum_{k=1}^{n} \varphi(X_k) \overset{\text{law}}{\to} \text{non-Gaussian}}$$

  This shows that the limit is usually **non-Gaussian** when $\rho \notin \ell^r(\mathbb{Z})$.
- There exists a **functional version** of Breuer-Major, in which the sum $\sum_{k=1}^{n}$ is replaced by $\sum_{k=1}^{[nt]}$ for $t \geq 0$. It is actually not that much harder to deal with and, unsurprisingly, the limiting process is then the standard Brownian motion multiplied by σ.

Proof of Breuer-Major

- We first compute the **limit variance**, which will justify the formula we have claimed for σ^2.
- We can write

$$\mathbb{E}[V_n^2] = \frac{1}{n}\,\mathbb{E}\left[\left(\sum_{p=r}^{\infty} a_p \sum_{k=1}^{n} H_p(X_k)\right)^2\right]$$

$$= \frac{1}{n}\sum_{p,q=r}^{\infty} a_p a_q \sum_{k,l=1}^{n} \mathbb{E}[H_p(X_k)H_q(X_l)]$$

$$= \frac{1}{n}\sum_{p=r}^{\infty} p! a_p^2 \sum_{k,l=1}^{n} \rho(k-l)^p = \sum_{p=r}^{\infty} p! a_p^2 \sum_{k\in\mathbb{Z}} \rho(k)^r \big(1-\frac{|k|}{n}\big)\mathbf{1}_{\{|k|<n\}}.$$

- By dominated convergence theorem, we can prove that $\mathbb{E}[V_n^2] \to \sigma^2$, with $\sigma^2 \in [0,\infty)$ like in the statement of Breuer-Major.

Proof of Breuer-Major (continued)

- We now check the **gaussianity**.
- We shall do it in three steps of increasing generality (but of decreasing complexity!):
  (i) when $\varphi = H_p$ has the form of a Hermite polynomial (for some $p \geq 1$);
  (ii) when $\varphi = P \in \mathbb{R}[X]$ is a real polynomial;
  (iii) in the general case, that is, when $\varphi \in L^2(\gamma)$.

Proof of Breuer-Major (continued)

Case where $\varphi = H_p$ is the pth Hermite polynomial.

- The space $\mathcal{H} := \overline{\text{span}\{X_1, X_2, \ldots\}}^{L^2(\Omega)}$ is a real separable Hilbert space.
- Let $\Phi : \mathcal{H} \to L^2(\mathbb{R}_+)$ be an isometry. Set $e_k = \Phi(X_k)$ for each $k \geq 1$.
- We have $\rho(k-l) = \mathbb{E}[X_k X_l] = \int_0^\infty e_k(x) e_l(x) dx$, $k, l \geq 1$.
- If $B = (B_t)_{t \geq 0}$ denotes a standard Brownian motion, we deduce that

$$\boxed{\{X_k\}_{k\geq 1} \overset{\text{law}}{=} \left\{ \int_0^\infty e_k(t) dB_t \right\}_{k \geq 1}},$$

these two families being indeed centered, Gaussian and having the same covariance structure (by construction of the e_k's).

Proof of Breuer-Major (continued)

Case where $\varphi = H_p$ is the pth Hermite polynomial.

- We deduce that $V_n = I_p(f_n)$, with

$$f_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} e_k^{\otimes p}.$$

- We already showed that $\mathbb{E}[V_n^2] \to \sigma^2$ as $n \to \infty$.
- So, according to Fourth Moment Theorem, to get that $V_n \to N(0, \sigma^2)$ it remains to check that $\|f_n \otimes_a f_n\| \to 0$ for any $a = 1, \ldots, p-1$.
- We have

$$f_n \otimes_a f_n = \frac{1}{n} \sum_{k,l=1}^{n} \rho(k-l)^a \, e_k^{\otimes p-a} \otimes e_l^{\otimes p-a},$$

implying in turn

$$\|f_n \otimes_a f_n\|^2 = \frac{1}{n^2} \sum_{i,j,k,l=1}^{n} \rho(i-j)^a \rho(k-l)^a \rho(i-k)^{p-a} \rho(j-l)^{p-a}.$$

PROOF OF BREUER-MAJOR (CONTINUED)

Case where φ is any polynomial.

- One has $\varphi = \sum_{p=r}^{N} a_p H_p$ for some *finite* integer $N \geq r$.
- Peccati-Tudor theorem and the previous case yield that

$$\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} H_{r}(X_k), \ldots, \frac{1}{\sqrt{n}} \sum_{k=1}^{n} H_{N}(X_k) \right) \overset{\text{law}}{\to} N\big(0, \text{diag}(\sigma_d^2, \ldots, \sigma_N^2)\big),$$

  where $\sigma_p^2 = p! \sum_{k \in \mathbb{Z}} \rho(k)^p$, $p = r, \ldots, N$.
- We deduce that

$$V_n = \frac{1}{\sqrt{n}} \sum_{p=r}^{N} a_p \sum_{k=1}^{n} H_p(X_k) \overset{\text{law}}{\to} N\left(0, \sum_{p=r}^{N} a_p^2 p! \sum_{k \in \mathbb{Z}} \rho(k)^p \right).$$