Convex Poincaré inequality
(and weak transportation inequalities)

Based on joint work with Radosław Adamczak.

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Warsaw, July 5, 2017

Setting

Standing assumption:

μ satisfies the *convex Poincaré* inequality, i.e. $\forall$ convex $f : \mathbb{R}^n \to \mathbb{R}$,

$$\mathrm{Var}_\mu(f) \leq \frac{1}{\lambda} \mathbb{E}_\mu |\nabla f|^2.$$

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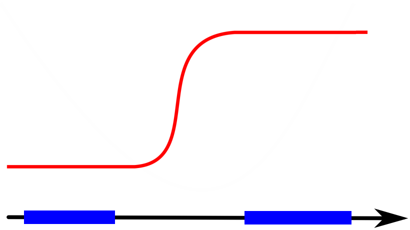

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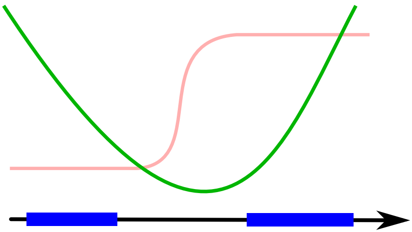

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Goal: $\forall$ convex or concave $f : \mathbb{R}^n \to \mathbb{R}$ with $|\nabla f| \le c$,

$$\mathbb{E}_\mu fe^f - \mathbb{E}_\mu e^f \ln(\mathbb{E}_\mu e^f) =: \mathrm{Ent}_\mu(e^f) \le C\, \mathbb{E}_\mu |\nabla f|^2 e^f.$$

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- Implies two-level concentration:

$$\mathbb{P}_{\mu^{\otimes N}}(|f - \mathbb{E}_{\mu^{\otimes N}} f| \geq t) \leq \exp\Big(-C_1 \frac{t^2}{L_2^2(f)} \wedge \frac{t}{L_1(f)}\Big).$$

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- Convex case, $n = 1$: Feldheim-Marsiglietti-Nayar-Wang '15, Gozlan-Roberto-Samson-Shu-Tetali '15.

Main result

Theorem (Adamczak-St. '17)

$\forall$ convex or concave $f : \mathbb{R}^n \to \mathbb{R}$ with $|\nabla f| \leq c < \sqrt{2\lambda}/e$,

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where $C = C(\lambda, c, n)$.

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$$F(1) = \mathbb{E}_\mu f^2 e^f \lesssim \mathbb{E}_\mu |\nabla f|^2 e^f.$$

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Question

If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and 1-Lipschitz, then, for $t \ge 0$,

$$\mathbb{P}_\mu(f \ge \mathrm{Med}_\mu f + t) \le 2\exp(-C(\lambda)t).$$

Do we also have

$$\mathbb{P}_\mu(f \le \mathrm{Med}_\mu f - t) \overset{?}{\le} 2\exp(-C(\lambda)t) \ ?$$