Comparison of weak and strong moments for vectors with independent coordinates

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Log-concave vectors

Definition

We say that a random vector X in $\mathbb{R}^n$ is log-concave if for any compact subsets A, B of $\mathbb{R}^n$ and any $\lambda \in (0,1)$ we have

$$\mathbb{P}(X \in A)^{\lambda} \mathbb{P}(X \in B)^{1-\lambda} \leqslant \mathbb{P}(X \in \lambda A + (1-\lambda) B).$$

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Proposition

If X is a log-concave vector and $\|\cdot\|$ is a seminorm on $\mathbb{R}^n$, then for every $1 \leqslant p \leqslant q$ we have

$$(\mathbb{E}\|X\|^p)^{1/p} \geqslant C\frac{p}{q}(\mathbb{E}\|X\|^q)^{1/q}.$$

The Paouris inequality

Theorem [Paouris, 2006]

For a log-concave vector X in $\mathbb{R}^n$ and any $p \geqslant 1$ we have

$$\left(\mathbb{E}\|X\|_2^p\right)^{1/p} \leqslant C\left(\mathbb{E}\|X\|_2 + \sigma_X(p)\right),$$

where $\sigma_X(p)$ is the p-th weak moment of X defined by

$$\sigma_X(p) := \sup_{\|t\|_2=1}\left(\mathbb{E}\left|\langle t, X\rangle\right|^p\right)^{1/p}.$$

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In other words,

$$\left(\mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^n t_i X_i\Big|^p\right)^{1/p} \leqslant C\left[\mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^n t_i X_i\Big| + \sup_{t\in T}\left(\mathbb{E}\Big|\sum_{i=1}^n t_i X_i\Big|^p\right)^{1/p}\right]$$

for $T = B_2^n$ (Euclidean ball of radius 1, with the centre at the origin).

Comparison of weak and strong moments – general case

Question

For which (reasonable) class of vectors in $\mathbb{R}^n$ holds the following: for any X of this class, any set $T \subset \mathbb{R}^n$, and $p \geqslant 1$ we have

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- Consider $T = B_2^n$ only.

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- Consider all T and assume X has independent coordinates. Then the comparison holds under the assumption that the norms of these coordinates grow α-regularly (C_1, C_2 depend on α) – Latała, Tkocz 2015. **Can we relax this assumption?**

The main results

Comparison of weak and strong moments (definition)

$$\Big(\mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^{n} t_i X_i\Big|^p\Big)^{1/p} \leqslant C_1 \mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^{n} t_i X_i\Big| + C_2 \sup_{t\in T}\Big(\mathbb{E}\Big|\sum_{i=1}^{n} t_i X_i\Big|^p\Big)^{1/p} \quad (1)$$

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Theorem [Latała, S., 2016]

If $X_1, \ldots, X_n$ are independent random vectors such that for any $q \geqslant 2$:

$$\|X_i\|_{2q} \leqslant \alpha\|X_i\|_q, \quad (2)$$

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Assume $X_1, X_2, \ldots$ are i.i.d. and satisfy (1) for any $n \geqslant 1$, $p \geqslant 1$ and $T = \{\pm e_1,\ldots,\pm e_n\}$. Then X_1 satisfies (2) with α depending on C_1, C_2.

Corollaries

Main theorem

$$\|X_i\|_{2q} \leqslant \alpha \|X_i\|_q, \quad \text{for all } q \geqslant 2 \tag{3}$$

implies that for all $p \geqslant 1$ and $T \subset \mathbb{R}^n$ we have

$$\Big(\mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^n t_i X_i\Big|^p\Big)^{1/p} \leqslant C_1(\alpha)\mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^n t_i X_i\Big| + C_2(\alpha)\sup_{t\in T}\Big(\mathbb{E}\Big|\sum_{i=1}^n t_i X_i\Big|^p\Big)^{1/p}$$

Corollary 1 – tail estimate

(3) implies that for all $u \geqslant 0$ we have

$$\mathbb{P}\Big(\sup_{t\in T}\Big|\sum_{i=1}^n t_i X_i\Big| \geqslant C_3(\alpha)\Big[u + \mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^n t_i X_i\Big|\Big]\Big) \leqslant C_4(\alpha)\sup_{t\in T}\mathbb{P}\Big(\Big|\sum_{i=1}^n t_i X_i\Big| \geqslant u\Big),$$

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Corollary 2 – Khintchine-Kahane-type inequalities

(4) implies that for all $p \geqslant q \geqslant 2$ and any non-empty set T in $\mathbb{R}^n$ we have,

$$\Big(\mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^n t_i X_i\Big|^p\Big)^{1/p} \leqslant C_5(\alpha)\Big(\frac{p}{q}\Big)^{\max\{1/2, \log_2 \alpha\}}\Big(\mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^n t_i X_i\Big|^q\Big)^{1/q}$$

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The exponent $\max\{1/2, \log_2 \alpha\}$ is optimal.

Further questions

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- What happens if the coordinates of X are not independent?
- Does the comparison hold for all log-concave X for the general T?
- Assume again $X_1, \ldots, X_n$ are independent. When does the comparison hold with $C_1 = 1$?