Two-sided moment estimates for random chaoses

Rafał Meller

University of Warsaw

Warsaw school in probability, Warsaw July 2017

Definitions and notation

C is a uniwersal constant, which may differ at each occurance. $C(\alpha)$ is a constant which may depend on α. We write $a \sim b$ if exists $C > 0$ such that $a/C \leq b \leq aC$. We write $a \sim^{\alpha} b$ if $a/C(\alpha) \leq b \leq aC(\alpha)$.
If X is a random variable then we write

$$N_X(t) = -\ln \mathbb{P}\left(|X| \geq t\right)$$
$$\hat{N}_X(t) = \begin{cases} t^2 & |t| \leq 1 \\ N_X(|t|) & |t| > 1 \end{cases}.$$

We say that X has log-concave tails if the function N_X is convex.
We say that X belogns to $\mathbb{S}(\alpha)$ class if for every $p \geq 1$
$\|X\|_{2p} \leq \alpha \|X\|_p$.
Remark $\mathbb{S}(12)$ contains all variables with the log-concave tails.

Formulation of the problem

Let $(X_j^i)_{i,j\in\mathbb{N}}$ be independent random variables, $a_{i_1,\ldots,i_d} \in \mathbb{R}$. Then

$$S = \sum a_{i_1,\ldots,i_d} X_{i_1}^1 \cdot \ldots \cdot X_{i_d}^d$$

is called a random chaos. In particular if $d = 1$ then $S = \sum a_i X_i$.
Problem Can we find tight bounds on $\|S\|_p$ under additional assumption on the random variables?

Results for $d = 1$

The case of $d = 1$ may be considered closed due to the following results of *Latała*.

Theorem (Latała 1997)

Let

$$|||(X_i)|||_p = \inf\{t > 0 \ : \ \sum \ln\left(\Phi_p\left(\frac{X_i}{t}\right)\right) < p\},$$

where $\Phi_p(X) = \mathbb{E}|1 + X|^p$. *If* $X_1, X_2, \ldots$ *are independent and either nonnegative,* $p \geq 1$ *or symmetric and* $p \geq 2$ *then*

$$\frac{e-1}{e^2}|||(X_i)|||_p \leq \left\|\sum X_i\right\|_p \leq e|||(X_i)|||_p. \tag{1}$$

Before that result the estimate for $\|\sum_i a_i X_i\|_p$ were known in the radamacher case [Hitczenko, Montgomery-Smith 1994], log-concave tails case [Gluskin, Kwapień 1995] and log-convex tails case [Hitczenko, Montgomery-Smith, Oleszkiewicz 1997].

Case $d \geq 2$ and positive random variables

Theorem (Meller 2016)

Assume that $(X_i^j)_{i \leq n, j \leq d}$ are independent, nonnegative, $\mathbb{E}X_j^i = 1$ and belog to $\mathbb{S}(\alpha)$ class. Then

$$\|S\|_p \sim^{d,\alpha} \sup\{\sum a_{i_1,\ldots,i_d}(1 + b_{i_1}^1) \cdot \ldots \cdot (1 + b_{i_d}^d)\},$$

where the supremum is taken over $(b_i^j)_{i \leq n, j \leq d}$ such that for any j $\sum_i N_{X_i^j}(b_i^j) \leq p$.

The proof relies on the result of Latała and Łochowski (2003) for nonnegative variables with the log concave tails.

Example

Let $(X_i^1), (X_j^2)$ be independent, nonnegative, from $\mathbb{S}(\alpha)$ class. If $d = 1$ then theorem reeds as

$$\left\| \sum a_i X_i^1 \right\|_p \sim^{\alpha} \sup\{\sum a_i(1 + b_i) \ : \ \sum N_{X_i}(b_i) \leq p\}.$$

If $d = 2$, then

$$\left\| \sum a_{i,j} X_i^1 X_j^2 \right\|_p \sim^{\alpha} \sup\{\sum a_{i,j}(1 + b_i)(1 + c_j)\},$$

where the supremum is taken over b_i, c_j such that $\sum N_{X_i^1}(b_i) \leq p$, $\sum N_{X_j^2}(c_j) \leq p$.

What we know in the symmetric case and $d \geq 2$

Not much

- arbitraly d, independent normal variables [Latała 2006]
- arbitraly d, independent variables with log convex tails [Kolesko Latała 2015]
- $d = 2, 3$, independent variables with log concave tails [Adamczak, Latała 1999, 2012]
- $d = 2$, independent variables from the $\mathbb{S}(\alpha)$ class [Meller 2017]

The main reason that why we lack results is difficulty in bounding

$$\mathbb{E} \sup_{t \in T} \sum_i t_i X_i,$$

where $T \subset \mathbb{R}^n$ is not nice set.

Case d=2 and symmetric random variables

The following is true

Theorem (Meller 2017)

If (X_i^1), (X_i^2) *are indepednent, belong to* $\mathbb{S}(\alpha)$ *class,* $\|X_i^1\|_2 = \|X_i^2\|_2 = 1$ *then*

$$\|S'\|_p \sim^\alpha \sup\{\sum a_{i,j} b_i c_j \ : \ \sum \hat{N}_{X_i^1}(b_i) \le p, \ \sum \hat{N}_{X_j^2}(c_j) \le p\} + \sup\{\sum_i \sqrt{\sum_j a_{i,j}^2} b_i \ : \ \sum \hat{N}_{X_i^1}(b_i) \le p\}. \quad (2)$$

It is a generealization of Latała's Theorem (1999). The proof uses mainly idea's from the paper of Adamczak and Latała.
A proposal from (2) is a one dimenssional chaos moment estimate

$$\left\|\sum a_i X_i^1\right\|_p \sim^\alpha \sup\{\sum a_i b_i \ : \ \sum \hat{N}_{X_i^1}(|b_i|) \le p\}. \quad (3)$$

An usefool lemma

A following lemma play an important role in bounding moments in the symmetric $\mathbb{S}(\alpha)$, $d = 2$ case.

Lemma (Meller 2017)

If $X_1, X_2, \ldots$ are independent, symmetric from the $\mathbb{S}(\alpha)$ class then

$$\mathbb{E} \sup_{t \in \bigcup_{k=1}^m T_k} \sum t_i X_i \leq C(\alpha) \max_k \mathbb{E} \sup_{t \in T_k} \sum t_i X_i$$
$$+ C(\alpha) \sup_{t,s \in T} \{ \sum (t_i - s_i) x_i \ : \ \sum \hat{N}_{X_i}(|x_i|) \leq C(\alpha) \ln m \}.$$

The above is a consequence of the moment estimate for $d = 1$ and the following theorem

Theorem (Latała, Strzelecka 2016)

If $X_1, X_2, \ldots$ are in symmetric $\mathbb{S}(\alpha)$ class and $T \subset \mathbb{R}^n$ then

$$\|\sup_{t \in T} \sum t_i X_i\|_p \leq C(\alpha) \mathbb{E} \sup_{t \in T} \sum |t_i X_i| + C(\alpha) \sup_{t \in T} \|\sum t_i X_i\|_p.$$