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Diffusion approximations via Stein's method and time changes

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Overview

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Functional limit results

Theorem (Donsker 1951)

Let $X_1, X_2, \ldots$ be i.i.d. random variables, each with mean 0 and variance 1. Then, for:

$$Y_n(t) = n^{-1/2} \sum_{i=1}^{\lfloor nt \rfloor} X_i,$$

$(Y_n(t), t \in [0,1]) \Rightarrow (B(t), t \in [0,1])$, *with respect to the uniform topology, where B is a standard Brownian Motion.*

Notable extension (Stroock and Varadhan 1969):

Weak convergence of scaled (continuous or discrete-time) Markov chains to diffusions (solutions of SDEs).

Functional limit results

Motivation

Look at the discrete model *from a distance* so that:

- it is easier to study (e.g. we can use stochastic analysis)
- it is more robust to changes in the local details

Speed of convergence in Donsker's Theorem (Barbour 1990)

For any $g : D([0,1],\mathbb{R}) \to \mathbb{R}$, such that:

$$\|g\|_M = \sup_{w\in D}\frac{|g(w)|}{1+\|w\|^3} + \sup_{w\in D}\frac{\|Dg(w)\|}{1+\|w\|^2} + \sup_{w\in D}\frac{\|D^2g(w)\|}{1+\|w\|} + \sup_{w,h\in D}\frac{\|D^2g(w+h)-D^2g(w)\|}{\|h\|} < \infty,$$

where $\|\cdot\|$ is the sup norm, there exists a constant C such that

$$|\mathbb{E}g(Y_n) - \mathbb{E}g(B)| \leq Cn^{-1/2}\|g\|_M(\sqrt{\log n} + \mathbb{E}|X_1|^3).$$

Extension: time-changed random walk (K. 2017+)

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Theorem

Let $X_1, X_2, \ldots$ be i.i.d. with mean 0, variance 1 and finite third moment. Let $s : [0,1] \to [0,\infty)$ be a strictly increasing, continuous function with $s(0) = 0$. Define:

$$Y_n(t) = n^{-1/2} \sum_{i=1}^{\lfloor ns(t) \rfloor} X_i, \quad t \in [0,1]$$

and let $(Z(t), t \in [0,1]) = (B(s(t)), t \in [0,1])$, where B is a standard Brownian Motion. Suppose that $g \in M$. Then:

$$|\mathbb{E}g(Y_n) - \mathbb{E}g(Z)| \leq \|g\|_M \left\{ (2133 + 63s(1)) \frac{\sqrt{\log(2s(1)n)}}{\sqrt{n}} \right.$$
$$\left. + \left(s(1) + 3s(1)^{5/2} \right) \mathbb{E}|X_1|^3 n^{-1/2} \right\}.$$

Time-changed Poisson process (K. 2017+)

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Theorem

Let P be a Poisson process with rate 1 and $s^{(n)}:[0,1]\to[0,\infty)$ and $s:[0,1]\to[0,\infty)$ increasing, continuous and 0 at the origin. Let $Z(t)=B(s(t)), t\in[0,1]$ where B is a standard Brownian Motion and $\tilde{Y}_n(t)=\frac{P\left(ns^{(n)}(t)\right)-ns^{(n)}(t)}{\sqrt{n}}$ for $t\in[0,1]$. Then, for all $g\in M$:

$$|\mathbb{E}g(\tilde{Y}_n)-\mathbb{E}g(Z)|\leq \|g\|_M\left\{(2+11s(1))\sqrt{\|s-s^{(n)}\|}\right.$$
$$\left.+\frac{27\sqrt{2}}{2\sqrt{\pi}}\|s-s^{(n)}\|^{3/2}+n^{-1/2}\left[C_1^{(n)}\sqrt{\log(2s(1)n)}+C_2^{(n)}\right]\right\}$$

for explicitly computable $C_1^{(n)}, C_2^{(n)}$ depending only on $s_n(1)$ and $s(1)$.

Moran model and W.-F. diffusion (K. 2017+)

Setup

Moran model with mutation:

- n individuals, two genes: (A and a).
- At exponential rate $\binom{n}{2}$ select two individuals uniformly at random: one randomly selected dies, the other one splits in two.
- In addition, every individual of type A changes its type independently at rate ν_2 and every individual of type a changes its type independently at rate ν_1.
- Let $M_n(t)$ be the proportion of type a genes in the population at time $t \in [0,1]$ under this model. Let $(M(t), t \in [0,1])$ denote the Wright-Fisher diffusion given by:

$$dM(t) = (\nu_2 - (\nu_1 + \nu_2)M(t))dt + \sqrt{M(t)(1 - M(t))}dB_t.$$

Moran model and W.-F. diffusion (K. 2017+)

Theorem

Given the setup above, for any $g \in M$:

$$|\mathbb{E}g(M_n) - \mathbb{E}g(M)| \leq \|g\|_M \Big[C_1(\nu_1,\nu_2)n^{-1/4}$$

$$+ C_2(\nu_1,\nu_2)n^{-1}\sqrt{\log\left(n^2/4+\nu_2 n\right)} + C_3(\nu_1,\nu_2)n^{-1}\sqrt{\log\left(n^2/4+\nu_1 n\right)}\Big],$$

where C_1, C_2, C_3 are explicitly computable.

Method of proof

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Aim

Approximate the distance between a scaled time-changed random walk $Y_n(\cdot) = n^{-1/2} \sum_{i=1}^{\lfloor ns(\cdot) \rfloor} X_i$, (where X_i's are i.i.d. with mean 0, variance 1 and finite third moment) and time-changed Brownian Motion $B(s(t))$.

Idea

First approximate Y_n by $A_n(\cdot) = n^{-1/2} \sum_{i=1}^{\lfloor ns(\cdot) \rfloor} Z_i$, where Z_i's are i.i.d. $\mathcal{N}(0, 1)$, using Stein's method. Then bound the distance between A_n and $B \circ s$ using results about the modulus of continuity of Brownian Motion.

Methodology: Stein's method

- Step 1: Find a (Stein) operator $\mathcal{A}$ acting on a class of real-valued functions, such that:

$$(\forall f \in \text{Domain}(\mathcal{A}) \quad \mathbb{E}_\nu \mathcal{A} f = 0) \Longleftrightarrow \nu = \mathcal{L}(A_n).$$

  For example, find a Markov process whose stationary law is $\mathcal{L}(A_n)$ and let $\mathcal{A}$ be its infinitesimal generator.
- Step 2: For a given function $g \in M$, find $f = f_g$, such that:

$$\mathcal{A} f = g - \mathbb{E} g(A_n).$$

- Step 3: Study the properties of f_g and estimate $|\mathbb{E}\mathcal{A} f_g(Y_n)|$ using Taylor's expansions.

Application to the Poisson process case

Step 1: Bound the distance between $\tilde{Y}_n(\cdot) = \frac{P\left(ns^{(n)}(\cdot)\right) - ns^{(n)}(\cdot)}{\sqrt{n}}$ and $\tilde{A}_n(\cdot) = n^{-1/2}\sum_{i=1}^{\lfloor ns^{(n)}(\cdot)\rfloor} Z_i$, where $Z_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$, using Stein's method, as in the random-walk case.

Step 2: Bound the distance between $\tilde{A}_n(\cdot) = n^{-1/2}\sum_{i=1}^{\lfloor ns^{(n)}(\cdot)\rfloor} Z_i$ and $A_n(\cdot) = n^{-1/2}\sum_{i=1}^{\lfloor ns(\cdot)\rfloor} Z_i$, where $Z_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$, using the Mean Value Theorem and properties of the space M.

Step 3: Bound the distance between $A_n(\cdot) = n^{-1/2}\sum_{i=1}^{\lfloor ns(\cdot)\rfloor} Z_i$ and a time-changed Brownian Motion $B \circ s$, using Brownian modulus of continuity results.

Application to Moran model and W.-F. diffusion

Step 1: Note that M_n jumps up by $\frac{1}{n}$ with intensity $\frac{1}{2}n^2M_n(t)(1-M_n(t))+n\nu_2(1-M_n(t))$ and down by $\frac{1}{n}$ with intensity $\frac{1}{2}n^2M_n(t)(1-M_n(t))+n\nu_1M_n(t)$.

Step 2: Use an idea from [Kur12] and write:

$$M_n(t) = \frac{P_1\left(n^2R_1^n(t)\right)-n^2R_1^n(t)}{n} - \frac{P_{-1}\left(n^2R_{-1}^n(t)\right)-n^2R_{-1}^n(t)}{n} + \int_0^t \left(\nu_2-(\nu_1+\nu_2)M_n(s)\right)ds,$$

where P_1, P_{-1} are i.i.d. Poisson processes with rate 1 and

$$\begin{cases} R_1^n(t) := \int_0^t \left(\frac{1}{2}M_n(s)+\frac{\nu_2}{n}\right)(1-M_n(s))ds \\ R_{-1}^n(t) := \int_0^t \left(\frac{1}{2}(1-M_n(s))+\frac{\nu_1}{n}\right)M_n(s)ds \end{cases} \quad \text{for } t\in[0,1].$$

Application to Moran model and W.-F. diffusion

Step 3: Use the method applied before to time-changed Poisson processes and a coupling between the Moran model and W.-F. diffusion to bound:

- distance between $\frac{P_1\left(n^2 R_1^n(t)\right) - n^2 R_1^n(t)}{n}$ and $B_1 \circ R_1$
- distance between $\frac{P_{-1}\left(n^2 R_{-1}^n(t)\right) - n^2 R_{-1}^n(t)}{n}$ and $B_{-1} \circ R_{-1}$,

where B_1 and B_{-1} are i.i.d. Brownian Motions and $R_1(\cdot) = R_{-1}(\cdot) = \int_0^\cdot \frac{1}{2} M(s)(1 - M(s)) ds$.

Step 4: Use MVT and coupling to bound the distance between $\int_0^\cdot (\nu_2 - (\nu_1 + \nu_2) M_n(s))\, ds$ and $\int_0^\cdot (\nu_2 - (\nu_1 + \nu_2) M(s))\, ds$.

Step 5: Obtain the final bound, upon noting that the diffusive part of the W.-F. diffusion (i.e. $\int_0^\cdot \sqrt{M(s)(1 - M(s)} dB_s$) can be written as a time-changed BM: $B\left(\int_0^\cdot M(s)(1 - M(s) ds\right)$.

Conclusions

- Stein's method can be used to put bounds on distances between inifite-dimensional distributions.
- Starting from bounds on the distance between a time-changed scaled random-walk and a time-changed Brownian Motion, we can bound the distance between various continuous-time Markov chains and diffusions.
- The method only works if the continuous-time Markov chain makes jumps of sizes coming from a finite set and if we have a way of coupling the chain and the diffusion at a fixed time point.
- There is no obvious way of extending this approach to scaled discrete-time Markov chains converging to diffusions or to multidimensional processes.
- The bounds in the diffusion approximation are most likely not sharp. Their order comes from a comparison of the time changes applied to the Poisson processes and the time changes applied to Brownian Motions.

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