Scenery Reconstruction and Locally Biased Functions on the Hypercube

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Warsaw Summer School in Probability 2017

Weizmann Institute of Science
Joint work with Renan Gross

Scenery Reconstruction

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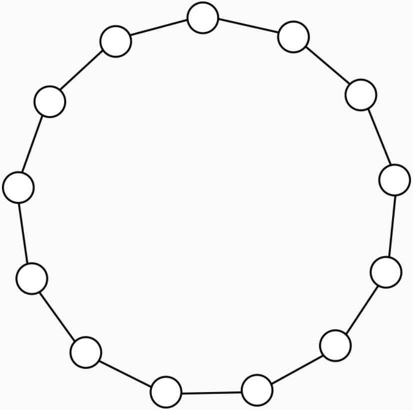
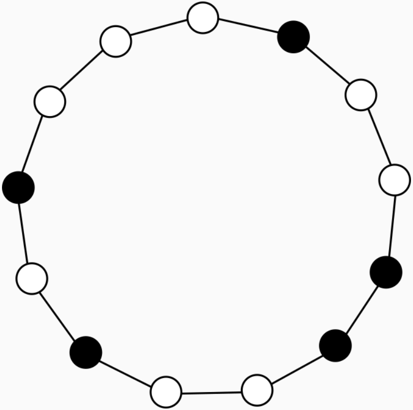


Scenery Reconstruction

Suppose we are given a graph G, colored by some function $f(x)$.



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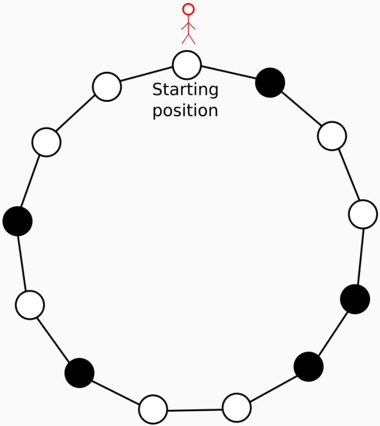

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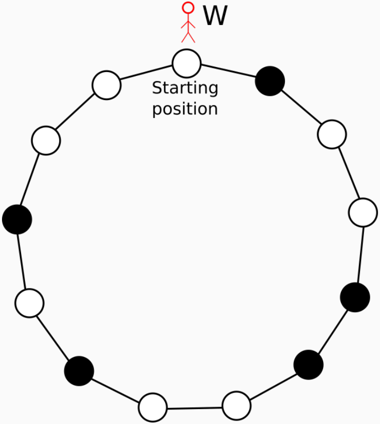

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Reported scenery:

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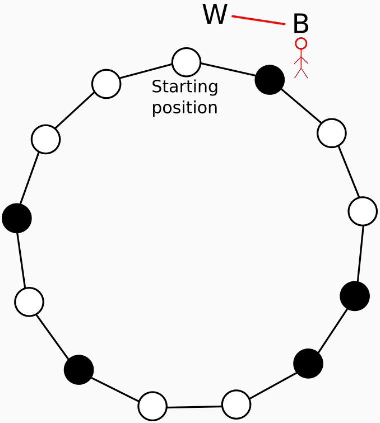

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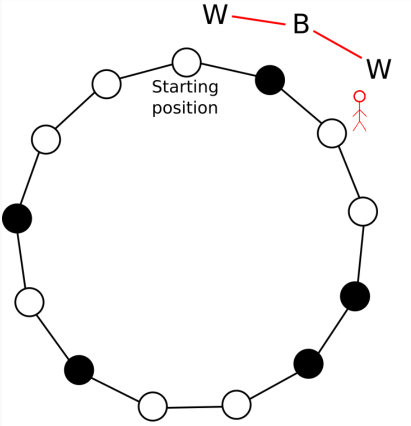

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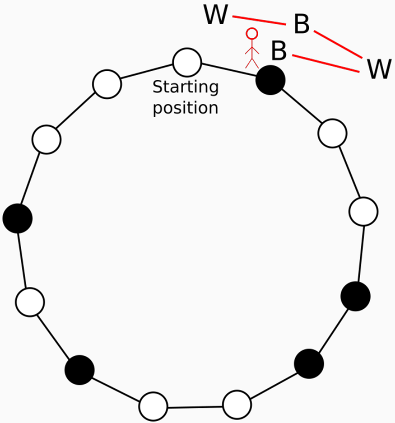

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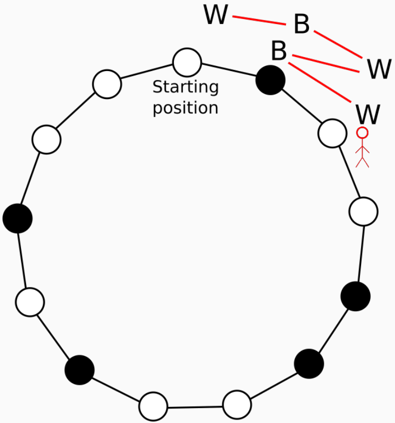

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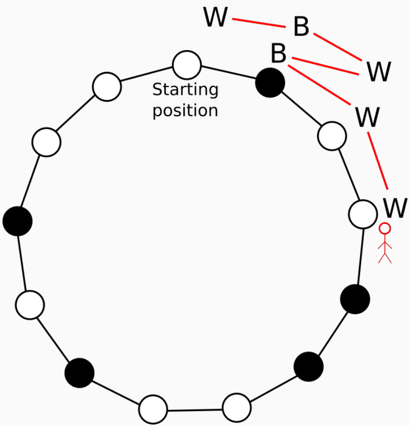

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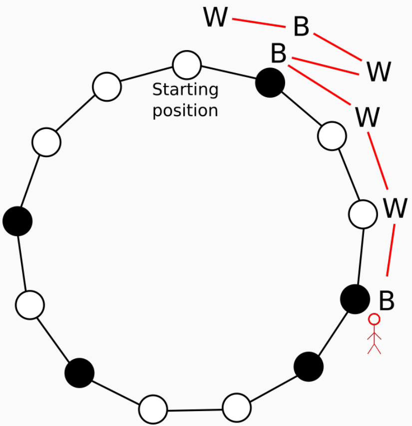

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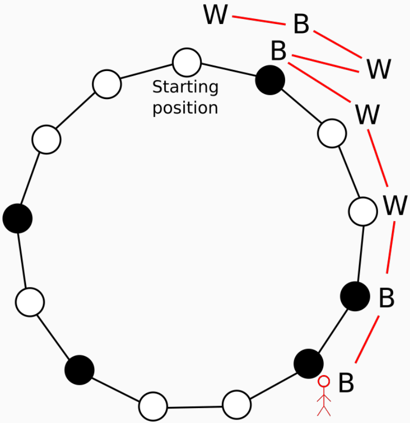

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Suppose we know G. Can we reconstruct the coloring f based only on the trace of the random walk $f(S_t)$?

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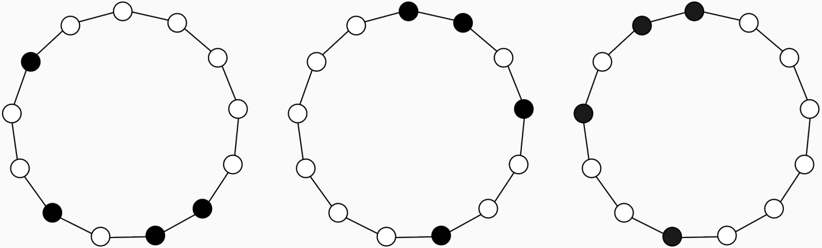
- Up to isomorphisms of the graph.



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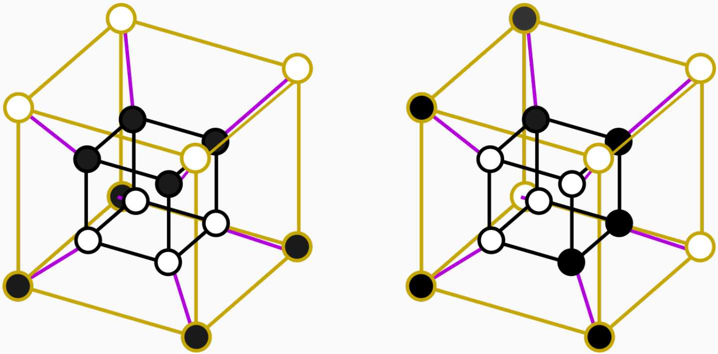

The process $f(S_t)$ is Bernoulli IID with success probability 1/2!

Locally Biased Functions

So we defined locally biased functions, and started investigating them in their own right.

Definition

Let G be a graph. A Boolean function $f : G \to \{-1, 1\}$ is called *locally p-biased*, if for every vertex $x \in G$ we have

$$\frac{|\{y \sim x;\ f(y) = 1\}|}{deg(x)} = p.$$

That is, f is locally p-biased if for every vertex x, f takes the value 1 on exactly a p-fraction of x's neighbors.

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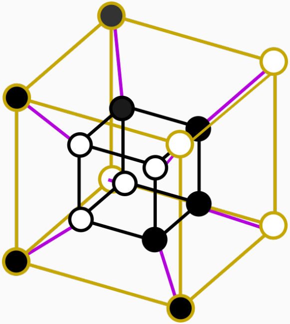
Existence of two non-isomorphic locally biased functions implies that the scenery reconstruction problem cannot be solved.

Locally Biased Functions

Locally biased functions can be defined for any graph.



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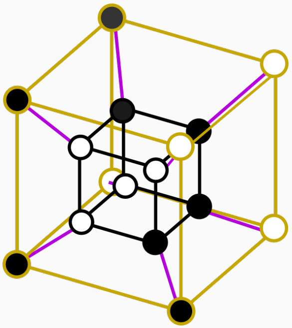

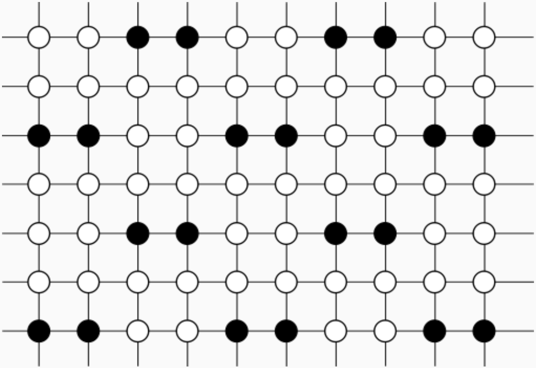

Theorem (characterization)

Let $n \in \mathbb{N}$ be a natural number and $p \in [0, 1]$. There exists a locally p-biased function $f : \{-1, 1\}^n \to \{-1, 1\}$ if and only if $p = b/2^k$ for some integers $b \geq 0, k \geq 0$, and 2^k divides n.

Theorem (size)

Let $n \geq 4$ be even. Let $B^n_{1/2}$ be a maximal class of non-isomorphic locally $1/2$-biased functions, i.e every two functions in $B^n_{1/2}$ are non-isomorphic to each other. Then $\left|B^n_{1/2}\right| \geq C2^{\sqrt{n}}/n^{1/4}$, where $C > 0$ is a universal constant.

We also have lower bounds for some other values of p, and other classes of functions.

Proof of Theorem 1 (Necessary Condition)

Double counting argument:

- Let x be a uniformly random element of the cube. Then $f(x) = 1$ with probability $\ell/2^n$, where $\ell = |\{x \in \{-1,1\}^n;\ f(x) = 1\}|$
- Let y be a uniformly random neighbor of x. Then $f(y) = 1$ with probability p by definition.
- Since both x and y are uniform random vertices,

$$\mathbb{P}(f(x) = 1) = \mathbb{P}(f(y) = 1)$$

- Denoting $p = m/n$ for some $m \in \{0, 1, \ldots, n\}$, this gives

$$p = \frac{\ell}{2^n} = \frac{m}{n}.$$

- Writing $n = c2^k$ gives the desired result.

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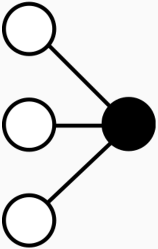

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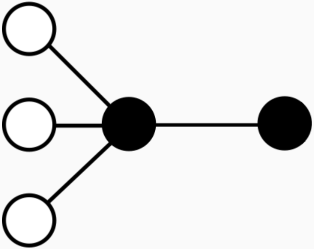

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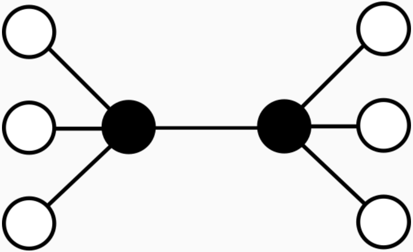

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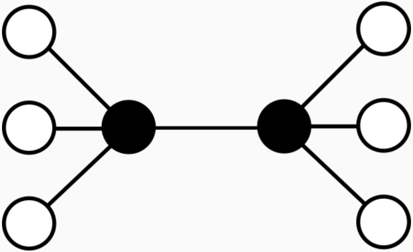

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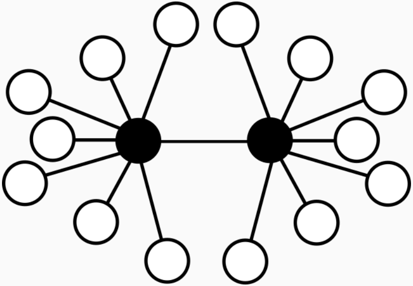

A tile for $n = 8$.

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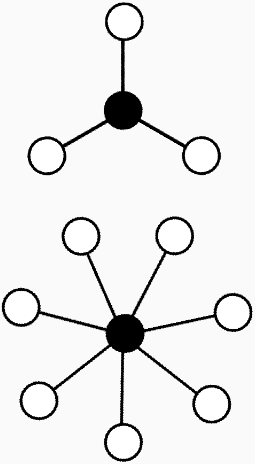

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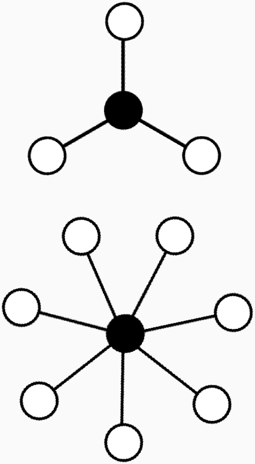

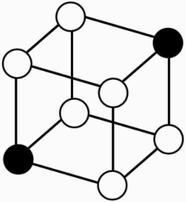

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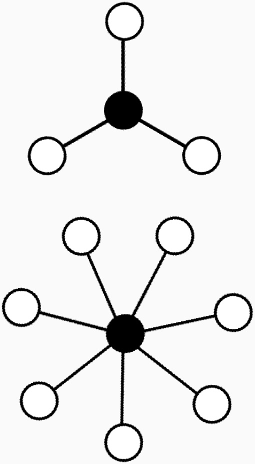

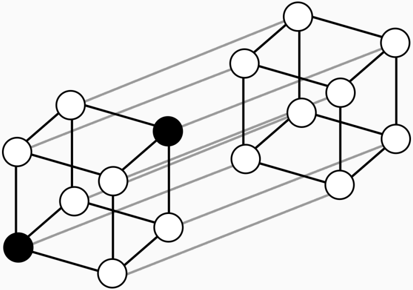

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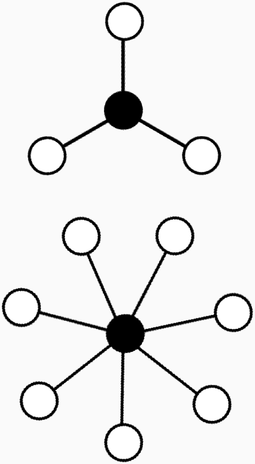

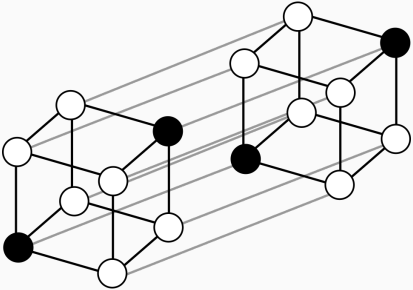

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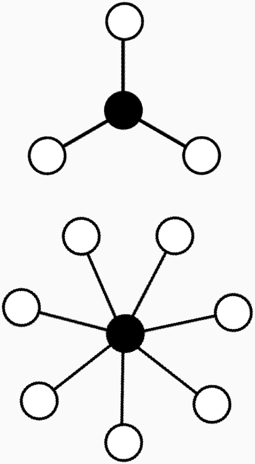
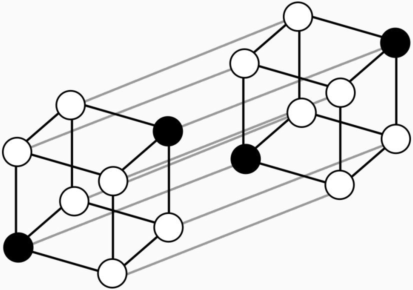




We have a $1/n$ tiling!

Proof of Theorem 1 (Sufficient Condition)

In order to find half tilings we use Hamming perfect codes.

- Let x be in our code (a node we will color).
- The coordinates with index 2^ℓ are the parity coordinates.
- In the other coordinates, we go over all possible values.
- We define:

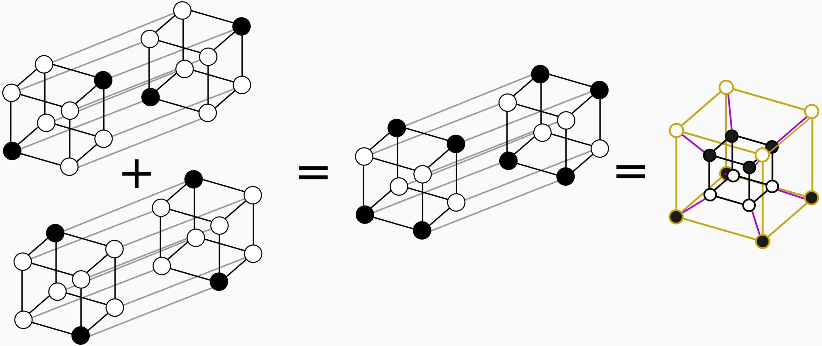
$$x_i = \bigoplus_{j : i \wedge j \neq 0} x_j \qquad \forall i = 2^\ell, \ell \geq 0$$

Where $\oplus$=xor and $\wedge$=and.

Remark: a translation of this will produce disjoint tilings.

Proof of Theorem 1 (Sufficient Condition)

To get m/n instead of $1/n$, combine several disjoint tilings.



Proof of Theorem 1 (Sufficient Condition)

Remark: for $n = 2^m - 1$ there are exponentially many perfect codes (Krotov and Avgustinovich 2008). This gives us exponentially many distinct $1/2^k$-locally biased function. Unfortunately, we do not know how to translate them into disjoint copies.

Proof of Theorem 1 (Sufficient Condition)

We need to generalize for any $n \geq 4$ and $p = m/2^k$.

Proposition

Let $g : \{-1,1\}^k \to \{-1,1\}$ be an $m/2^k$-biased function. Then

$$f(x) = g\left(\prod_{j=0}^{c-1} x_{1+jn}, \ldots, \prod_{j=0}^{c-1} x_{n+jn}\right)$$

is an $m/2^k$-biased function on $\{-1,1\}^{ck}$.

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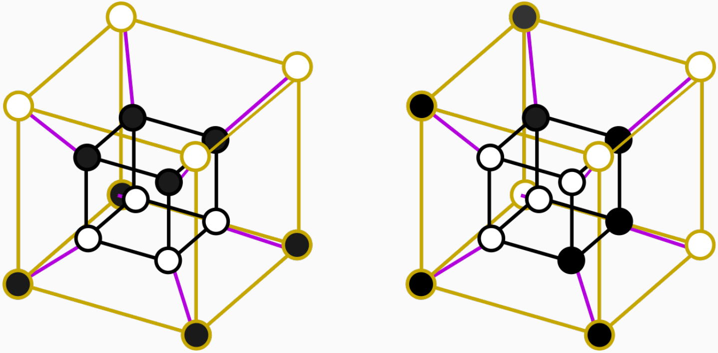

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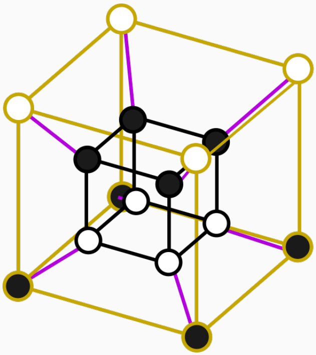

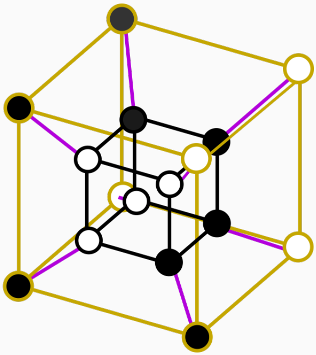

$g(x_1, x_2, x_3, x_4) = x_1 x_2$

$h(x_1, x_2, x_3, x_4) =$
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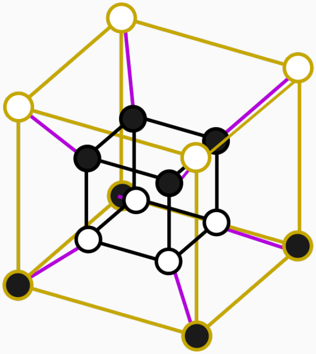

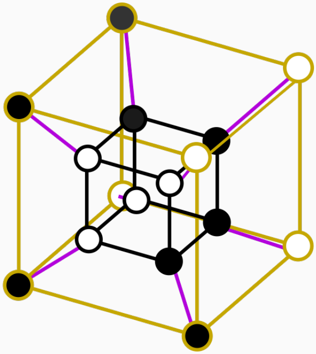

$$g(x_1, x_2, x_3, x_4) = x_1 x_2$$

$$g_n(x_1, \ldots, x_n) = x_1 \cdots x_{n/2}$$

$$h(x_1, x_2, x_3, x_4) =$$
$$\frac{1}{2}\left(x_1x_2 + x_2x_3 - x_3x_4 + x_1x_4\right)$$
$$h_k =$$
$$h\left(\prod_{i=0}^{k-1} x_{1+4i}, \ldots, \prod_{i=0}^{k-1} x_{4+4i}\right)$$

Proof of Theorem 2

Observation

Let $f_i : \{-1,1\}^{n_i} \to \{-1,1\}$ *be locally* $1/2$*-biased functions for* $i = 1,2$ *where* $n_1 + n_2 = n$*. Then*

$$f(x) = f_1(x_1, ..., x_{n_1}) f_2(x_{n_1+1}, ..., x_n)$$

is a locally $1/2$*-biased function on* $\{-1,1\}^n$*.*

We can then build up locally $1/2$-biased functions from the building blocks $h_0, h_1, \ldots$ and $g_0, g_2, g_4, \ldots$.

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We can then build up locally $1/2$-biased functions from the building blocks $h_0, h_1, \ldots$ and $g_0, g_2, g_4, \ldots$.

- Every time we pick h_0, we use 4 bits.
- Every time we pick h_1, we use 8 bits.
- Every time we pick h_2, we use 12 bits.
- $\ldots$
- The rest of the bits are filled with g_k.

Proof of Theorem 2

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The # of different combinations is the same as the # of solutions to:

$$4a_1 + 8a_2 + \cdots + 4ka_k \leq n$$

which is at least

$$C \cdot 2^{\sqrt{n}}/n^{1/4}$$

for some constant C.

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- By uniqueness of the Fourier structure, the functions we constructed are not isomorphic.

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Remarks:

- We used the results for the hypercube to analyze $\mathbb{Z}^n$, but it does not cover all the options.
- Using locally p-biased functions we showed another class of functions that produce indistinguishable sceneries.

Thank You!