A second order equation for Schrödinger Bridges and applications

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Moving towards an "unexpected configuration"

Back in 1932, Schrödinger described the following experiment

- At time $t = 0$ consider N **Brownian particles on the sphere** $(X^i_t)_{t \leq 1, i \leq N}$ arranged according to μ,

$$\mu = \frac{1}{N} \sum_{i=1}^{N} \delta_{X^i_0}.$$

- Let each particle move independently for a unit of time and look at their configuration ν at time 1,

$$\nu = \frac{1}{N} \sum_{i=1}^{N} \delta_{X^i_1}$$

- Assume that ν is an **unexpected configuration**, i.e. $\nu \not\approx$ **vol**.

What was the **most likely** evolution of the particle system towards ν?

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Schrödinger problem: Large Deviations formulation

- $\mathbf{P}$ = stationary process for generator $\mathscr{L}$

$$\mathscr{L} = \frac{1}{2}\Delta - \nabla U \cdot \nabla$$

- μ, ν initial and final configurations
- $\mathscr{H}(\cdot|\mathbf{P})$ = Relative entropy on path space

Schrödinger problem (SP)

$$\min \mathscr{H}(\mathbf{Q}|\mathbf{P}), \quad X_0\#\mathbf{Q} = \mu, X_1\#\mathbf{Q} = \nu$$

- The optimal value $\mathcal{T}_{\mathscr{H}}(\mu, \nu)$ of the Schrödinger problem is called **entropic transportation cost**.
- The **optimal solution** of SP is the **Schrödinger bridge (SB)** between μ and ν.

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SB dynamics: sketch

The SB models a particle system that

- has to **minimize entropy** $\Rightarrow$ particles are willing to arrange theirselves according to the equilibrium configuration $\mathbf{m}$.
- has to reach un **unexpected** final configuration, which looks very different from $\mathbf{m}$

Thus we expect the dynamics to be divided into two phases.

1. Entropy minimization dominates: SB relaxes to $\mathbf{m}$
2. the influence of the final configuration prevails: particles start arranging according to ν and drifts away from $\mathbf{m}$.

At any given time t, how far is μ_t from $\mathbf{m}$?

Goal of the talk: answer this question.

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Finding an equation for the Schrödinger bridge

We call (μ_t) the **marginal flow** of SB, and (v_t) its "speed"

- Quickest way to minimize entropy: **gradient flow of the entropy**

$$v_t = -\nabla \mathscr{H}(\mu_t)$$

- But if particles go along the gradient flow they don't reach the "unexpected configuration" ν
- **Cheap trick**: differentiate in time the gradient flows

$$\frac{\mathbf{D}}{dt} v_t = -\nabla^2 \mathscr{H}(\mu_t) v_t = \nabla^2 \mathscr{H}(\mu_t) \nabla \mathscr{H}(\mu_t)$$

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The Riemannian manifold of Optimal transport

- To make sense of the calculations above we need to choose some **metric structure** on the space of probability measures
- The theory of **optimal transport** can be used to construct one such structure.
- The **continuity equation** gives the notion of speed for a curve

$$\partial_t \mu + \nabla \cdot (\mu_t v_t) = 0$$

- The **convective derivative** can be used to make sense of the **covariant derivative**.

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A second order equation

Our heuristic can be rigorously justified within this structure!

Theorem (C.' 17)

The marginal flow (μ_t) of the Schrödinger bridge solves

$$\frac{\mathbf{D}}{dt} v_t = \frac{1}{8} \nabla^{\mathcal{W}} \mathscr{I}_U(\mu_t)$$

where

- $\frac{\mathbf{D}}{dt}$ *is the* **covariant derivative**
- v_t *is the* **velocity field** *of* (μ_t)
- $\mathscr{I}_U$ *is the* **modified Fisher information**.
- $\nabla^{\mathcal{W}}$ *is the* **gradient** *in the OT sense.*

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A bound for the marginal entropy

At any given time t, how far μ_t from $\mathbf{m}$?

Theorem (C. '17)

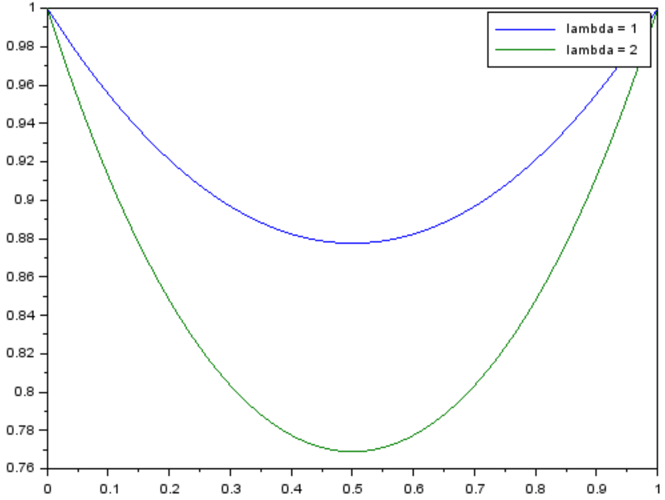
Let M be compact and (mu_t) be the entropic interpolation between μ and ν. The Bakry Émery condition

$$\mathfrak{Ric} + \nabla\nabla U \geq \lambda$$

implies the following bound for all $\mu, \nu \in \mathcal{P}(M)$

$$\mathscr{H}_U(\mu_t) \leq \frac{1-\exp(-\alpha(1-t))}{1-\exp(-\alpha)} \mathscr{H}_U(\mu) + \frac{1-\exp(-\alpha t)}{1-\exp(-\alpha)} \mathscr{H}_U(\nu) - \frac{\cosh(\frac{\alpha}{2}) - \cosh(\alpha(t-\frac{1}{2}))}{\sinh(\frac{\alpha}{2})} \mathcal{T}_{\mathscr{H}}(\mu,\nu)$$

Evolution of the marginal entropy



A new functional inequality

Theorem (C.'17)

Let m be the invariant measure associated with $\mathscr{L}$. Then

$$\mathcal{T}_{\mathscr{H}_U}(\mu, \mathbf{m}) \leq \frac{1}{1 - \exp(-\frac{\alpha}{2})} \mathscr{H}_U(\mu)$$

- A priori bound on the entropic transportation cost
- The entropic cost grows at most linearly with the marginal entropies
- Upper bound for the distance between the Schrödinger bridge and the target law $\mathbf{P}$
- Partial converse to the fact that joint entropies dominate marginal entropies

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Connections with OT

If we consider the SB constructed with

$$\mathscr{L}^{\varepsilon} = \frac{\varepsilon}{2}\Delta$$

Theorem (Mikami '04, Léonard '12)

The following limits hold as $\varepsilon \downarrow 0$.

$$\textit{Schrödinger bridge} \rightarrow \textit{displacement interpolation}$$

and

$$\varepsilon \, \mathcal{T}^{\varepsilon}_{\mathscr{H}_U}(\mu, \nu) \rightarrow \frac{1}{2} W_2^2(\mu, \nu)$$

Using this, we can recover **displacement convexity of the entropy Talagrand's entropy transportation inequality** from our results

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Fisher information and reciprocal characteristic

Reciprocal characteristic

For a given potential U, it is the vector field

$$\nabla \mathscr{U}, \quad \text{where } \mathscr{U} = \frac{1}{2}|\nabla U|^2 - \frac{1}{2}\Delta U$$

In the flat case $M = \mathbb{R}^d$ we obtained the following

Theorem (C.'17)

Let (μ_t) be the marginal flow of (SB). If $\mathscr{U}$ is convex and μ_t is a log-concave law for all t, then

$$t \mapsto \mathscr{I}_U(\mu_t) \quad \textit{is convex}$$

- Convexity points of $\mathscr{I}_U$=log concave measures?

Grazie!

beginframeSchrödinger '33

" *Imaginez que vous observez un* **système de particules en diffusion**, *qui soient en équilibre thermodynamique. Admettons qu' à un instant donné* 0 *vous les ayez trouvées en* **répartition à peu près uniforme** *et que à* 1 *vous ayez trouvé un* **écart spontané et considérable par rapport à cette uniformité**. *On vous demande de quelle manière cet écart sest produit. Quelle en est la* **manière la plus probable ?** "

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