Martingale transport and pricing-hedging duality for American options

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based on joint work with

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MOT set-up

- Price process $S = (S_t)_{t=1}^T$ on $\Omega = \{\omega_0\} \times \mathbb{R}_+^T$ is

$$S_t : \Omega \to \mathbb{R} \quad S_t(\omega_0, ..., \omega_T) = \omega_t \quad t = 1, ..., T$$

  and $\mathbb{F} := (\mathcal{F}_t)_{t=0,1,...,T}$ is the natural filtration of S
- From the prices of $(S_t - K)^+$ for each $K \in \mathbb{R}_+$ and each $t \in \{1, ..., T\}$ we deduce $\mu = (\mu_1, ..., \mu_T) \in \mathfrak{P}_1(\mathbb{R}_+)^T$ such that $S_t \sim \mu_t$
- Looking for the bounds on the price of $g(S_1, ..., S_T)$

$$\inf_{\mathbb{Q}\in\mathcal{M}_\mu} \mathbb{E}^{\mathbb{Q}}[g] \quad \text{and} \quad \sup_{\mathbb{Q}\in\mathcal{M}_\mu} \mathbb{E}^{\mathbb{Q}}[g]$$

$$\mathcal{M}_\mu := \left\{\mathbb{Q} \in \mathfrak{P}(\Omega) \ : \mathcal{L}_{\mathbb{Q}}(S_t) = \mu_t \ \forall t \ \text{ and } S \text{ is an } (\mathbb{Q}, \mathbb{F})\text{-martingale}\right\}$$

MOT set up

- Denote by Λ_1 the set of Lipschitz functions $\lambda : \mathbb{R} \to \mathbb{R}$ and define $\Lambda = \Lambda_1^T$ and by $\mathcal{H}$ the set of all $\mathbb{F}$-predictable $\mathbb{R}$-valued processes.
- Then the final payoff of semi-static trading strategy $(H, \lambda) \in (\mathcal{H}, \Lambda)$

$$(H \circ S)_T + \lambda(S) = \sum_{t=1}^{T} H_t \Delta S_t + \sum_{t=1}^{T} \lambda_t(S_t)$$

- The superhedging price of an option which pays off g at time T:

$$\pi_\mu^E(g) := \inf\{\mu(\lambda) : \exists\ (H, \lambda) \ \text{ s.t. } \ (H \circ S)_T + \lambda(S) \geq g \ \text{ on } \Omega\}$$

where $\mu(\lambda) = \sum_{t=1}^{T} \mu_t(\lambda_t)$

Duality for European option

Theorem (Beiglböck, Henry-Labordère, Penkner (2013))

Suppose that $g : \Omega \to \mathbb{R}$ is bounded from above and upper semicontinuous. Then there exists an optimal measure $\mathbb{Q}^* \in \mathcal{M}_\mu$ and the pricing duality holds:

$$\pi_\mu^E(g) = \sup_{\mathbb{Q} \in \mathcal{M}_\mu} \mathbb{E}^{\mathbb{Q}}[g] = \mathbb{E}^{\mathbb{Q}^*}[g].$$

Weak duality: for $\mathbb{Q} \in \mathcal{M}_\mu$ and $(H, \lambda) \in (\mathcal{H}, \Lambda)$ which superhedges g we have

$$\mathbb{E}^{\mathbb{Q}}[g] \leq \mathbb{E}^{\mathbb{Q}}[(H \circ S)_T + \lambda(S)] = \mu(\lambda)$$

Classical vs robust approach

- Model-specific approach: the price process of the underlying assets $(S_t)_{t\leq T}$ is modelled by a parametric family of stochastic processes.
- Model-independent/robust approach: many possible models, weaker economic assumptions
  - Quasi-sure approach considers a large set of measures $\mathcal{Q}$ (Super-)hedging is required $\mathcal{Q}$-q.s. ($\mathbb{Q}$-a.s. $\forall \mathbb{Q} \in \mathcal{Q}$)
  - Pathwise approach requires hedging property to hold for each path ω in the path space Ω

Acciaio, Bayraktar, Beiglböck, Biagini, Bouchard, Brown, Burzoni, Cheridito, Cox, Davis, Denis, Dolinsky, Dupire, Frittelli, Galichon, Gassiat, Guo, Henry-Labordère, Hobson, Hou, Huesmann, Källblad, Kardaras, Klimmek, Kupper, Maggis, Martini, Mykland, Nadtochiy, Neuberger, Neufeld, Nutz, Obłój, Penker, Perkowski, Possamaï, Prömel, Raval, Riedel, Rogers, Schachermayer, Soner, Spoida, Tan, Tangpi, Temme, Touzi, Wang ...

Robust pricing and hedging

$$\sup_{\mathbb{Q}\in\mathcal{M}_A} \mathbb{E}_{\mathbb{Q}}(g) = \inf\{\mathcal{P}(X) : \exists\, (X,H) \text{ s.t. } X + H\circ S_T \geq g \text{ pathwise on } A\}$$

- traded assets: static trading in options X, dynamic trading in stock S
- $\mathcal{P}(X)$ prices of of statically traded options X at time 0
- trading dates: discrete time vs continuous time
- regularity of the payoff
- quasi sure vs pathwise
- pathspace restriction – beliefs
- information used to choose trading strategies

Superhedging of American options

- An American option may be exercised at any time $t \in \mathbb{T} := \{1, \cdots, T\}$
- It is described by its payoff function $\xi = (\xi_t)_{1 \leq t \leq T}$, where $\xi_t : \Omega \to \mathbb{R}$ belongs to Υ and is the payoff, delivered at time T, if the option is exercised at time t

The superhedging cost of the American option ξ using semi–static strategies is given by

$$\pi^A_\mu(\xi) = \inf \big\{ \mu(\lambda) : \exists (H^1, ..., H^T) \in \mathcal{H}^T \text{ s.t. } H^t_i = H^n_i \ \forall i \leq t \leq n \\ \text{and } \lambda \in \Lambda \ \text{ satisfying } \ (H^t \circ S)_T + \lambda(S) \geq \xi_t \text{ on } \Omega \ \forall t \in \mathbb{T} \big\}$$

- Dynamic trading strategy H^t might be adjusted after disclosure of whether the exercise of American option took place or not
- Consistency: $H^t_i = H^n_i$ whenever $i \leq n \leq t$, H^t is $\mathbb{F}$-predictable
- Asymmetry: there is no way to adjust the static trading strategy due to its nature

Does duality for American option hold ?

Classically, pricing of an American option is recast as an optimal stopping problem and a natural extension of duality for European option would be

$$\pi_\mu^A(\xi) \stackrel{?}{=} \sup_{\tau \in \mathcal{T}(\mathbb{F})} \sup_{\mathbb{Q} \in \mathcal{M}_\mu} \mathbb{E}^{\mathbb{Q}}[\xi_\tau],$$

where $\mathcal{T}(\mathbb{F})$ denotes the set of $\mathbb{F}$-stopping times. The "numerical" reason is that the RHS may be too small since the set $\mathcal{T}(\mathbb{F})$ is too small. Our aim here is to understand fundamental reasons why the duality fails and hence discuss how and why the right hand side should be modified to obtain equality.

Hobson and A. Neuberger (2016a, 2016b), E. Bayraktar, Y. Huang and Z. Zhou (2015), E. Bayraktar and Z. Zhou (2016)

Enlarged space $\overline{\Omega}$

- Let $\overline{\Omega} := \Omega \times \mathbb{T}$ with $\mathbb{T} := \{1, ..., T\}$ with $\overline{\omega} := (\omega, \theta)$
  Extension of S from Ω to $\overline{\Omega}$ as $S(\overline{\omega}) = S(\omega)$.

- The canonical time $\Theta : \overline{\Omega} \to \mathbb{T}$ is given by $\Theta(\overline{\omega}) := \theta$
  The filtration $\overline{\mathbb{F}} := (\overline{\mathcal{F}}_t)_{t=0,1,\ldots,T}$ with $\overline{\mathcal{F}}_t = \mathcal{F}_t \otimes \vartheta_t$ and $\vartheta_t = \sigma(\Theta \wedge (t+1))$, and the σ-field $\overline{\mathcal{F}} = \mathcal{F} \otimes \vartheta_T$
  Θ is an $\overline{\mathbb{F}}$-stopping time

$$\overline{\mathcal{M}}_\mu = \{\overline{\mathbb{Q}} \in \mathfrak{P}(\overline{\Omega}) : \ Law_{\overline{\mathbb{Q}}}(S_t) = \mu_t, \quad \mathbb{E}^{\overline{\mathbb{Q}}}[\Delta S_t | \overline{\mathcal{F}}_{t-1}] = 0 \ \forall t \in \mathbb{T}\}$$

Reformulation of a superhedging of an American option

We identify an American option ξ on Ω with a European option on $\overline{\Omega}$ via

$$\xi(\overline{\omega}) = \xi_\theta(\omega)$$

The superhedging cost of the option ξ on $\overline{\Omega}$

$$\pi^E_\mu(\xi) := \inf\{\mu(\lambda) : \exists\ (\overline{H}, \lambda) \in \overline{\mathcal{H}} \times \Lambda \ \text{ s.t. } \ (\overline{H} \circ S)_T + \lambda(S) \geq \xi \ \}$$

where $\overline{\mathcal{H}}$ is the class of $\overline{\mathbb{F}}$-predictable processes

Theorem

Suppose that $\xi : \overline{\Omega} \to \mathbb{R}$ is bounded from above and upper semicontinuous. Then there exists an optimal measure $\overline{\mathbb{Q}}^* \in \overline{\mathcal{M}}_\mu$ and the pricing duality holds:

$$\pi^A_\mu(\xi) = \pi^E_\mu(\xi) = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_\mu} \mathbb{E}^{\overline{\mathbb{Q}}}[\xi] = \mathbb{E}^{\overline{\mathbb{Q}}^*}[\xi].$$

What models are in $\overline{\mathcal{M}}_\mu$?

- Instead of stopping times relative to $\mathbb{F}$, it allows us to consider any *random* time which can be made into a stopping time under some calibrated martingale measure
- Comparing with formulation on Ω:

$$\sup_{\tau:\text{random time}} \sup_{\mathbb{Q}\in\mathcal{M}_\mu(\mathbb{F}^\tau)} \mathbb{E}^{\mathbb{Q}}[\xi_\tau] = \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}_\mu} \mathbb{E}^{\overline{\mathbb{Q}}}[\xi]$$

- $\overline{\mathcal{M}}_\mu$ is equivalent to *weak formulation*
- Is there a *minimal* way of enlarging a space which is equivalent to $\overline{\mathcal{M}}_\mu$?

Embedding into a larger space $\widehat{\Omega}$

- Presence of statically traded options unables/breaks dynamic programming principle
- Embed the market into a fictitious larger one where both S and all the options $\lambda \in \Lambda$, are traded dynamically
- Let us denote by $\widehat{S} = (S, Y)$ which will now correspond to dynamically traded assets.
- One marginal $\mu = \mu_T$

$\widehat{\Omega}$ is a dynamic extension of Ω

- The canonical space for the measure-valued processes

$$\widehat{\Omega} := \{\mu\} \times (\mathfrak{P}_1(\mathbb{R}_+))^T$$

and $\widehat{X} = (\widehat{X}_t)_{0 \le t \le T}$ the canonical process on $\widehat{\Omega}$

$$\widehat{X}_t(f) = \int_{\mathbb{R}} f(x) \widehat{X}_t(dx)$$

- Define $\mathtt{i} : \widehat{\Omega} \to \Omega$ by $\mathtt{i}(\widehat{\omega}) = \widehat{X}_T(id)(\widehat{\omega})$
$S_t(\widehat{\omega}) = S_t(\mathtt{i}(\widehat{\omega})) = \widehat{X}_t(id)(\widehat{\omega})$
- Define a family of processes $Y = (Y^\lambda)_{\lambda \in \Lambda}$ by $Y_t^\lambda = \widehat{X}_t(\lambda)$ for $t \le T$. Note that $Y_0^\lambda = \mu(\lambda)$ and $Y_T^\lambda = \lambda(\widehat{X}_T(id))$.

MVM measures

Definition

- A probability measure $\widehat{\mathbb{Q}}$ on $(\widehat{\Omega}, \widehat{\mathcal{F}})$ is called a measure-valued martingale measure (MVM measure) if the process $(\widehat{X}_t(f))_{0 \le t \le T}$ is a $(\widehat{\mathbb{Q}}, \widehat{\mathbb{F}})$-martingale for all $f \in \mathcal{C}_1$.
- A MVM measure $\widehat{\mathbb{Q}}$ is terminating if $\widehat{X}_T \in \Delta := \{\eta \in \mathfrak{P}(\mathbb{R}) : \eta = \delta_x,\ x \in \mathbb{R}\}$, $\widehat{\mathbb{Q}}$-a.s.

Let us denote by

$$\widehat{\mathcal{M}}_\mu = \{\widehat{\mathbb{Q}} \in \mathfrak{P}(\widehat{\Omega}) : \widehat{\mathbb{Q}} \text{ is terminating MVM measure}\}.$$

Embedding into a larger space $\widehat{\Omega}$

- Note that for any $\mathbb{Q} \in \mathcal{M}_\mu$ there exists $\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}_\mu$ such that

$$\mathcal{L}_{\mathbb{Q}}(S, Y^{\mathbb{Q}}) = \mathcal{L}_{\widehat{\mathbb{Q}}}(S, Y)$$

  where $Y^{\lambda,\mathbb{Q}} = (\mathbb{E}^{\mathbb{Q}}[\lambda(S_T)|\mathcal{F}_t])_{t \leq T}$

- For any $\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}$ let $\mathtt{I}(\widehat{\mathbb{Q}}) = \widehat{\mathbb{Q}} \circ \mathtt{i}^{-1} \in \mathcal{M}_\mu$. And conversely, from a given $\mathbb{Q} \in \mathcal{M}_\mu$ we may recover its "parent" measure $\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}_\mu$.
- Then the correspondence between $\widehat{\mathcal{M}}_\mu$ and $\mathcal{M}_\mu$ yields to

$$\sup_{\mathbb{Q} \in \widehat{\mathcal{M}}_\mu} \mathbb{E}^{\mathbb{Q}}[g] = \sup_{\mathbb{Q} \in \mathcal{M}_\mu} \mathbb{E}^{\mathbb{Q}}[g] \quad \text{for any } g \in \Upsilon.$$

- The enlargement techniques on space $\widehat{\Omega}$ to obtain $\overline{\widehat{\mathcal{M}}}$ allow to conclude:

$$\sup_{\mathbb{Q} \in \overline{\widehat{\mathcal{M}}}_\mu} \mathbb{E}^{\mathbb{Q}}[\xi] = \sup_{\mathbb{Q} \in \overline{\mathcal{M}}_\mu} \mathbb{E}^{\mathbb{Q}}[\xi] \quad \text{for any } \xi \in \overline{\Upsilon}$$

Duality for an American option on $\widehat{\Omega}$

Theorem

For all upper bounded from above and upper semicontinuous functionals $\xi : \overline{\Omega} \to \mathbb{R}^N$

$$\pi^A_\mu(\xi) = \widehat{\pi}^A(\xi) = \sup_{\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}} \sup_{\tau \in \mathcal{T}(\widehat{\mathbb{F}})} \mathbb{E}^{\widehat{\mathbb{Q}}}[\xi_\tau].$$

The $\widehat{\mathbb{F}}$-stopping time

$$\widehat{\tau}^*(\widehat{\omega}) := \inf \left\{ t \geq 1 : \widehat{\mathcal{E}}_t \left(\xi(\cdot, t) \right) (\widehat{\omega}) = \overline{\widehat{\mathcal{E}}}^t (\xi)(\widehat{\omega}, t) \right\}$$

provides the optimal exercise policy for ξ:

$$\sup_{\overline{\widehat{\mathbb{Q}}} \in \overline{\widehat{\mathcal{M}}}} \mathbb{E}^{\overline{\widehat{\mathbb{Q}}}}[\xi] = \sup_{\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}} \sup_{\tau \in \mathcal{T}(\widehat{\mathbb{F}})} \mathbb{E}^{\widehat{\mathbb{Q}}}[\xi_\tau] = \sup_{\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}} \mathbb{E}^{\widehat{\mathbb{Q}}} [\xi_{\tau^*}] = \overline{\widehat{\mathcal{E}}}^0 (\xi).$$

Follows by

$$\overline{\pi}^E_\mu(\xi) = \pi^A_\mu(\xi) \geq \widehat{\pi}^A(\xi) = \overline{\widehat{\pi}}^E (\xi) \geq \sup_{\mathbb{Q} \in \overline{\widehat{\mathcal{M}}}} \mathbb{E}^{\mathbb{Q}} [\xi] = \sup_{\mathbb{Q} \in \overline{\mathcal{M}}_\phi} \mathbb{E}^{\mathbb{Q}} [\xi] .$$

Conclusions

- Recovering duality for American options
    - Solution 1: American option rendered European option $\bar{\Omega} = \Omega \times \{1, ..., T\}$ and $\bar{\xi}(\omega, t) = \xi_t(\omega)$
    - Solution 2: Presence of statically traded options unables/breaks dynamic programming principle. We allow dynamic trading in these options by enlarging the probability space to $\widehat{\Omega}$.
- Application to Bouchard & Nutz set-up and MOT set-up using measure-valued martingale of Cox & Kälblad

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THANK YOU!

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