On the information generated by a process up to a stopping time[1]

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[1]The support of the Slovene Human Resources Development and Scholarship Fund (contract number 11010-543/2011) is acknowledged.

Outline of talk

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(More on these stochastic control aspects: Part I of the arXiv preprint: “On the informational structure in optimal dynamic stochastic control” (with S. D. Jacka).)

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- The sigma-field generated by the stopped process (possibly completed)?

- Or should these two not be the *same*, anyway!?

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- What is essentially required, however, is a kind-of Galmarino's test, demonstrating that one has, in fact, the equality: $\sigma(X^S) = \mathcal{F}_S^X$.
- In literature this is available for coordinate processes on canonical spaces.
- However, coordinate processes are quite restrictive (e.g. not pertinent to stochastic control).

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- A generalization of (a part of) Galmarino's test to a non-canonical space setting is proved, although full generality could not be achieved.
- Several corollaries and related findings are given, which in particular shed light on the theme of 'informational consistency' (at random /stopping/ times).

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- Let $\mathcal{G} = (\mathcal{G}_t)_{t \in T}$ be a filtration. Then a time $S : \Omega \to T \cup \{\infty\}$ is a stopping time, if $\{S \leq t\} \in \mathcal{G}_t$ for all $t \in T$. In such a case we define $\mathcal{G}_S := \{A \in \mathcal{G}_\infty : A \cap \{S \leq t\} \in \mathcal{G}_t \text{ for all } t \in T\}$, where $\mathcal{G}_\infty := \vee_{t \in T} \mathcal{G}_t$.

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- If P is a complete probability measure on Ω, whose domain contains the σ-field $\mathcal{H}$ thereon, then $\overline{\mathcal{H}}^{\mathsf{P}}$ is the completion of $\mathcal{H}$ under P. If further $\mathcal{G}$ is a filtration, the domain of P including $\mathcal{G}_\infty$, then the completed filtration is denoted $\overline{\mathcal{G}}^{\mathsf{P}}$, and given via $\overline{\mathcal{G}}_t^{\mathsf{P}} := \overline{\mathcal{G}_t}^{\mathsf{P}}$ ($t \in T$).

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What about if one "completes everything"? Then it's trickier ...

Tools

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Blackwell's Theorem. *Let* $(\Omega, \mathcal{F})$ *be a Blackwell space,* $\mathcal{G}$ *a sub-*σ*-field of* $\mathcal{F}$ *and* $\mathcal{S}$ *a separable sub-*σ*-field of* $\mathcal{F}$*. Then* $\mathcal{G} \subset \mathcal{S}$*, if and only if every atom of* $\mathcal{G}$ *is a union of atoms of* $\mathcal{S}$*. In particular, a* $\mathcal{F}$*-measurable real function* g *is* $\mathcal{S}$*-measurable, if and only if* g *is constant on every atom of* $\mathcal{S}$*.*

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Lemma (Key lemma)

Let X be a process (on Ω, with time domain $T \in \{\mathbb{N}_0, [0, \infty)\}$ and values in $(E, \mathcal{E})$), S an $\mathcal{F}^X$-stopping time, $A \in \mathcal{F}^X_S$. If $X_t(\omega) = X_t(\omega')$ for all $t \in T$ with $t \leq S(\omega) \wedge S(\omega')$, then $S(\omega) = S(\omega')$, $X^S(\omega) = X^S(\omega')$ and $\mathbb{1}_A(\omega) = \mathbb{1}_A(\omega')$.

Key results – stopping times

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Theorem (Stopping times)

Let X be a process (on Ω, with time domain $T \in \{\mathbb{N}_0, [0,\infty)\}$ and values in $(E,\mathcal{E})$), $S : \Omega \to T \cup \{\infty\}$ a time. If $T = \mathbb{N}_0$, or else if the conditions:

(1) $\sigma(X|_{[0,t]})$ *and* $\sigma(X^{S\wedge t})$ *are separable,* $(\mathrm{Im}X|_{[0,t]}, \mathcal{E}^{\otimes[0,t]})$ *and* $(\mathrm{Im}X^{S\wedge t}, \mathcal{E}^{\otimes T}|_{\mathrm{Im}X^{S\wedge t}})$ *Hausdorff for each* $t \in [0,\infty)$.

(2) X^S *and* X *are both measurable with respect to a Blackwell* σ*-field* $\mathcal{G}$ *on* Ω.

are met, then the following statements are equivalent:

(i) S *is an* $\mathcal{F}^X$*-stopping time.*

(ii) S *is an* $\mathcal{F}^{X^S}$*-stopping time.*

Key results – Galmarino's test

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Theorem (Generalized Galmarino's test)

Let X be a process (on Ω, with time domain $T \in \{\mathbb{N}_0, [0,\infty)\}$ and values in $(E,\mathcal{E})$), S an $\mathcal{F}^X$-stopping time. If $T = \mathbb{N}_0$, then $\sigma(X^S) = \mathcal{F}^X_S$. Moreover, if X^S is $\mathcal{F}^X_S/\mathcal{E}^{\otimes T}$-measurable (in particular, if it is adapted to the stopped filtration $(\mathcal{F}^X_{t\wedge S})_{t\in T}$) and either one of the conditions:

(1) $\mathrm{Im} X^S \subset \mathrm{Im} X$.

(2) (a) $(\Omega, \mathcal{G})$ *is Blackwell for some* σ*-field* $\mathcal{G} \supset \mathcal{F}^X_\infty$.
(b) $\sigma(X^S)$ *is separable.*
(c) $(\mathrm{Im} X^S, \mathcal{E}^{\otimes T}|_{\mathrm{Im} X^S})$ *is Hausdorff.*

is met, then the following statements are equivalent:

(i) $A \in \mathcal{F}^X_S$.

(ii) $\mathbb{1}_A$ *is constant on every set on which* X^S *is constant and* $A \in \mathcal{F}^X_\infty$.

(iii) $A \in \sigma(X^S)$.

Key results – informational consistency

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Corollary (Observational consistency)

Let X and Y be two processes (on Ω, with time domain $T \in \{\mathbb{N}_0, [0,\infty)\}$ and values in $(E,\mathcal{E})$), S an $\mathcal{F}^X$ and an $\mathcal{F}^Y$-stopping time. Suppose furthermore $X^S = Y^S$. If any one of the conditions

(1) $T = \mathbb{N}_0$.

(2) $\mathrm{Im}X = \mathrm{Im}Y$.

(3) (a) $(\Omega,\mathcal{G})$ *(resp.* $(\Omega,\mathcal{H})$*) is Blackwell for some* σ*-field* $\mathcal{G} \supset \mathcal{F}^X_\infty$ *(resp.* $\mathcal{H} \supset \mathcal{F}^Y_\infty$*).*
(b) $\sigma(X^S)$ *(resp.* $\sigma(Y^S)$*) is separable and contained in* $\mathcal{F}^X_S$ *(resp.* $\mathcal{F}^Y_S$*).*
(c) $(\mathrm{Im}X^S, \mathcal{E}^{\otimes T}|_{\mathrm{Im}X^S})$ *(resp.* $(\mathrm{Im}Y^S, \mathcal{E}^{\otimes T}|_{\mathrm{Im}Y^S})$*) is Hausdorff.*

is met, then $\mathcal{F}^X_S = \mathcal{F}^Y_S$.

Key results – monotonicity of information

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Proposition (Monotonicity of information)

Let Z be a process (on Ω, with time domain $T \in \{\mathbb{N}_0, [0,\infty)\}$ and values in $(E, \mathcal{E})$), $U \leq V$ two stopping times of $\mathcal{F}^Z$. If either $T = \mathbb{N}_0$ or else the conditions:

1. *$(\Omega, \mathcal{G})$ is Blackwell for some σ-field $\mathcal{G} \supset \sigma(Z^V) \vee \sigma(Z^U)$.*
2. *$(\mathrm{Im} Z^V, \mathcal{E}^{\otimes T}|_{\mathrm{Im} Z^V})$ is Hausdorff.*
3. *$\sigma(Z^V)$ is separable.*

are met, then $\sigma(Z^U) \subset \sigma(Z^V)$.

Similarities and differences. . .

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4. Are true, if the stopping times are predictable ...

A counter-example

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Example

Let $\Omega = (0,\infty) \times \{-2,-1,0\}$ be endowed with the law $\mathsf{P} = \mathrm{Exp}(1) \times \mathrm{Unif}(\{-2,-1,0\})$, defined on the tensor product of the Lebesgue σ-field on $(0,\infty)$ and the power set of $\{-2,-1,0\}$. Denote by e, respectively I, the projection onto the first, respectively second, coordinate. Define the process $X_t := I(t-e)\mathbb{1}_{[0,t]}(e)$, $t \in [0,\infty)$, and the process $Y_t := (-1)(t-e)\mathbb{1}_{[0,t]}(e)\mathbb{1}_{\{-1,-2\}} \circ I$, $t \in [0,\infty)$. The completed natural filtrations of X and Y are already right-continuous. The first entrance time S of X into $(-\infty,0)$ is equal to the first entrance time of Y into $(-\infty,0)$, and this is a stopping time of $\overline{\mathcal{F}^X}^{\mathsf{P}}$ as it is of $\overline{\mathcal{F}^Y}^{\mathsf{P}}$ (but not of $\mathcal{F}^X$ and not of $\mathcal{F}^Y$). Moreover, $X^S = 0 = Y^S$. Finally, consider the event $A := \{I = -1\}$. Then $A \in \overline{\mathcal{F}^X}^{\mathsf{P}}_S$, however, $A \notin \overline{\mathcal{F}^Y}^{\mathsf{P}}_S$. $\diamond$

In discrete time. . .

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Lemma

Let $T = \mathbb{N}_0$, $\mathcal{G}$ a filtration on Ω. Let furthermore P *be a complete probability measure on Ω, whose domain includes $\mathcal{G}_\infty$; S a $\overline{\mathcal{G}}^{\mathsf{P}}$-stopping time. Then S is* P*-a.s. equal to a stopping time S' of $\mathcal{G}$; and for any $\mathcal{G}$-stopping time U,* P*-a.s. equal to S, $\overline{\mathcal{G}_U}^{\mathsf{P}} = \overline{\mathcal{G}}^{\mathsf{P}}_S$. Moreover, if U is another random time,* P*-a.s equal to S, then it is a $\overline{\mathcal{G}}^{\mathsf{P}}$-stopping time, and $\overline{\mathcal{G}}^{\mathsf{P}}_S = \overline{\mathcal{G}}^{\mathsf{P}}_U$.*

Handling the predictable case (in continuous time)

Handling the predictable case (in continuous time)

Similarly:

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Proposition

Let $T=[0,\infty)$, $\mathcal{G}$ be a filtration on Ω. Let furthermore P *be a complete probability measure on Ω, whose domain includes $\mathcal{G}_\infty$; S a predictable stopping time relative to $\overline{\mathcal{G}}^{\mathsf{P}}$. Then S is* P*-a.s. equal to a predictable stopping time P of $\mathcal{G}$. Moreover, if U is any $\mathcal{G}$-stopping time,* P*-a.s. equal to S, then $\overline{\mathcal{G}}^{\mathsf{P}}_S=\overline{\mathcal{G}_U}^{\mathsf{P}}$. Finally, if S' is another random time,* P*-a.s equal to S, then it is a $\overline{\mathcal{G}}^{\mathsf{P}}$-stopping time, and $\overline{\mathcal{G}}^{\mathsf{P}}_S=\overline{\mathcal{G}}^{\mathsf{P}}_{S'}$.*

Further work?

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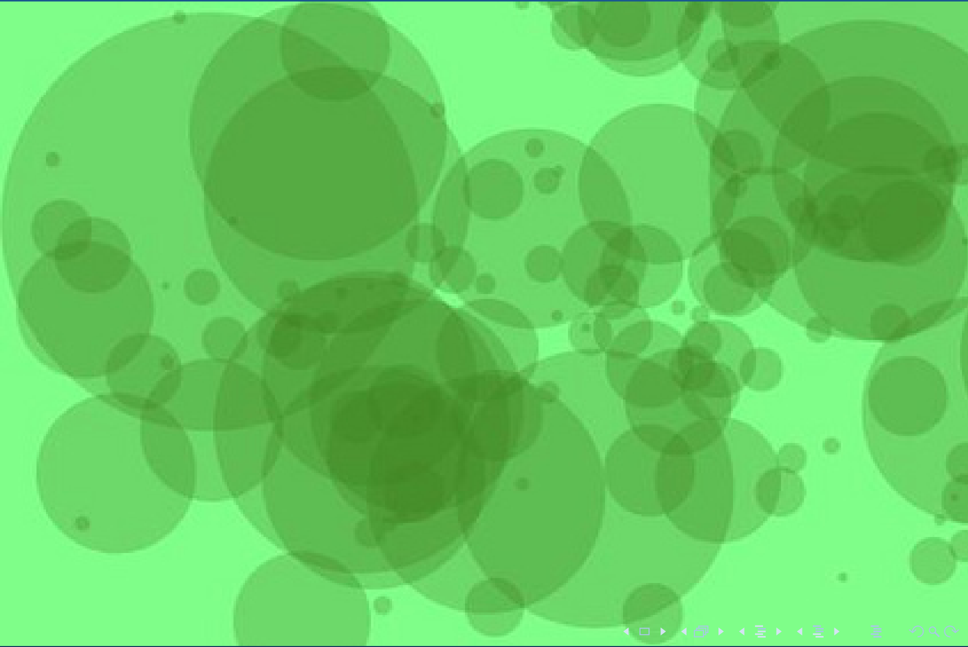
- Try and relax/drop the Blackwell-ian assumption.
- Alternatively (or in addition) find relevant counter-examples!



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(Interested in the details/proofs? Part II of the arXiv preprint.)