Weak and strong moments of l_r-norms of log-concave vectors

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(based on joint work with Rafał Latała)

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Log-concave vectors

Definition

We say that a random vector X in $\mathbb{R}^n$ is log-concave if for any compact subsets A, B of $\mathbb{R}^n$ and any $\lambda \in (0,1)$ we have

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If X has a log-concave density (i.e. of a form $e^{-\psi}$, where the function $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is convex), then X is log-concave.
If X is log-concave and its support is not contained in a nontrivial linear subspace of $\mathbb{R}^n$, then X has a log-concave density.

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Proposition

If X is a log-concave vector, $\|\cdot\|$ seminorm on $\mathbb{R}^n$ and $1 \leqslant p \leqslant q$, then

$$(\mathbb{E}\|X\|^p)^{1/p} \geqslant C\frac{p}{q}(\mathbb{E}\|X\|^q)^{1/q}.$$

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Log-concavity = vectors uniformly distributed on convex bodies
+ affine transformations + weak limits.

The Paouris inequality

Theorem [Paouris, 2006]

For a log-concave vector X in $\mathbb{R}^n$ and any $p \geqslant 1$ we have

$$\left(\mathbb{E}\|X\|_2^p\right)^{1/p} \leqslant C\left(\mathbb{E}\|X\|_2 + \sigma_X(p)\right),$$

where $\sigma_X(p)$ is the p-th weak moment of X defined by

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By Chebyshev's inequality this implies the large deviation inequality

$$\mathbb{P}\left(\|X\|_2 \geqslant 2Ct\mathbb{E}\|X\|_2\right) \leqslant \exp\left(-\sigma_X^{-1}\left(t\mathbb{E}\|X\|_2\right)\right) \quad \text{for t} \geqslant 1.$$

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For a simpler proof see R. Adamczak, R. Latała, A. E. Litvak, K. Oleszkiewicz, A. Pajor, and N. Tomczak-Jaegermann, 2014.

Comparison of weak and strong moments conjecture

Conjecture

For any log-concave vector X in a separable Banach space $(F, \|\cdot\|)$ and $p \geqslant 1$ we have

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- i.i.d case.

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- It suffices to prove the theorem in the case of l_r.
- It suffices to prove the theorem in the case of $(\mathbb{R}^n, \|\cdot\|_r)$.

Modified Paouris inequality

Theorem [Latała 2014]

For a log-concave isotropic vector X in $\mathbb{R}^n$ and $p \geqslant 1$ we have

$$\mathbb{E}\left(\sum_{i=1}^{n} X_i^2 \mathbf{1}_{\{|X_i| \geqslant t\}}\right)^p \leqslant (C\sigma_X(p))^{2p} \qquad \text{for } t \geqslant C\log\left(\frac{n}{\sigma_X(p)^2}\right). \quad (1)$$

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The proof uses the Paouris inequality, but (1) is formally stronger than the Paouris inequality.

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In i.i.d. case the best threshold is $C\sigma_X^{-1}\left(\log\left(\frac{nt^2}{\sigma_X(p)^2}\right)\right)$.

Generalization of the modified Paouris inequality

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For a log-concave vector X in $\mathbb{R}^n$, $p \geqslant 1$ and $r \geqslant 2$ we have

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$$\mathbb{E}\left(\sum_{i=1}^{n}|X_i|^r\mathbf{1}_{\{|X_i|\geqslant td_i\}}\right)^{p/r}\leqslant (Cr\sigma_{X,r}(p))^p \qquad \text{for } t\geqslant C\log\left(\frac{d}{\sigma_{X,r}(p)}\right),$$

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$$\begin{aligned}(\mathbb{E}\|X\|_r^p)^{1/p} \leqslant (\mathbb{E}\|X\|_r^{\tilde{p}})^{1/\tilde{p}} \leqslant Cr\sigma_{r,X}(\widetilde{p}) = Cr\max\{d, \sigma_{r,X}(p)\} \\ \leqslant Cr(\mathbb{E}\|X\|_r + \sigma_{r,X}(p)).\end{aligned}$$

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This gives the main theorem in the case of $(\mathbb{R}^n,\|\cdot\|_r)$.

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We divide the sum into several parts. Define sets

$$I_0 := \left\{(i_1, k_1, \ldots, i_l, k_l)\colon \ \mathbb{P}(B_{i_1,k_1,\ldots,i_l,k_l}) > e^{-rl}\right\},$$

$$I_j := \left\{(i_1, k_1, \ldots, i_l, k_l)\colon \ \mathbb{P}(B_{i_1,k_1,\ldots,i_l,k_l}) \in (e^{-rl2^j}, e^{-rl2^{j-1}}]\right\}.$$

Crucial proposition

Proposition

Let X, r, d_i and d be as before and $A := \{X \in K\}$, where K is a convex set in $\mathbb{R}^n$ satisfying $0 < \mathbb{P}(A) \leqslant 1/e$. Then
(i) for every $t \geqslant r$,

$$\sum_{i=1}^{n} \mathbb{E}|X_i|^r \mathbf{1}_{A \cap \{X_i \geqslant td_i\}} \leqslant C^r \mathbb{P}(A) \left(r^r \sigma_{r,X}^r(-\log(\mathbb{P}(A))) + (dt)^r e^{-t/C} \right),$$

(ii) for every $t > 0$, $u \geqslant 1$,

$$\sum_{k=0}^{\infty} 2^{kr} \sum_{i=1}^{n} d_i^r \mathbf{1}_{\{\mathbb{P}(A \cap \{X_i \geqslant 2^k td_i\}) \geqslant e^{-u}\mathbb{P}(A)\}} \leqslant \frac{(Cu)^r}{t^r} \left(\sigma_{r,X}^r(-\log(\mathbb{P}(A))) + d^r \mathbf{1}_{\{t \leqslant uC\}} \right).$$

Thank you
for your attention!