Local Laws in RMT

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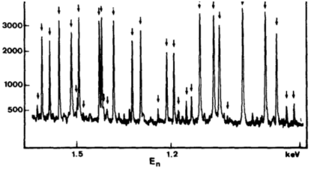
Wigner's Nuclear Physics

Experimental data for scattering energies of neutrons bouncing off a heavy nucleus:



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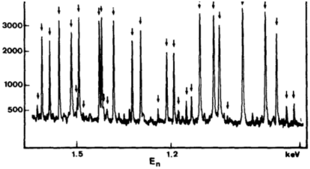

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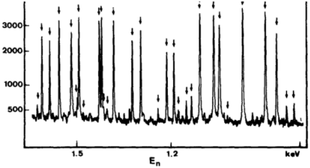

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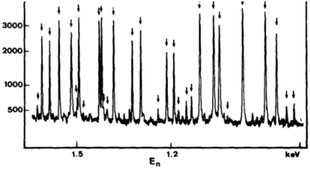
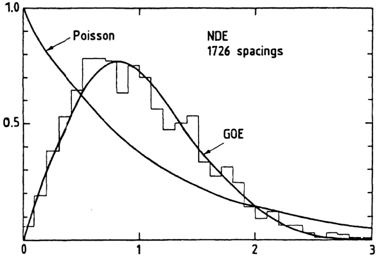


Question: How to understand the spacing between peaks in the above data?

- Structure of large nuclei is very complicated
- Not feasible to compute these energies either theoretically or numerically
- Wigner's idea: the Hamiltonian can be modeled by a very **large random matrix**

Data for spacing between energies with eigenvalue spacing overlaid:



Definition

Let h be a probability distribution on $\mathbb{C}$. A **Wigner matrix** is a $N \times N$ matrix that is:

- Hermitian,
- entries are iid complex *random variables* with distribution h,
- mean 0 and variance $1/\mathrm{N}$.

The variance is chosen so the spectrum has compact support.

Universality

Consider a Wigner matrix (Hermitian $N \times N$ matrix with iid entries drawn from *any* distribution h). Let

- $\mathcal{N}[a;b]$ = number of eigenvalues in the interval $[a;b]$
- ρ_{sc} is a probability density on $\mathbb{R}$

$$\rho_{sc}(E) = \begin{cases} \frac{1}{2\pi}\sqrt{1-\frac{E^2}{4}}, & \text{if } |E| \leq 2 \\ 0 & \text{if } |E| > 2 \end{cases} .$$

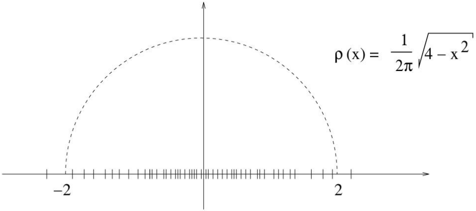
Wigner's key result – universality:
For $\delta > 0$ and all a, b, Wigner proved that

$$\lim_{N\to\infty} \mathbb{P}\left(\left|\frac{\mathcal{N}[a;b]}{N(b-a)} - \frac{1}{(b-a)}\int_a^b \rho_{sc}(s)d\,s\right| \geq \delta\right) = 0$$

Universality - the big picture

- Want to derive *macroscopic* properties of large systems with *unknown* or random interactions of their constituents.
  Examples:
  - laws of thermodynamics
  - law of large numbers
- Random matrices model various physical systems
  - quantum and wave chaos
  - many particle systems
  - scattering energies of large atoms
- For random matrices, universality can be defined and proved
  - A lot done for Wigner matrices
  - Still many open problems for other ensembles, such as *non-Hermitian* matrices
  - Universality of fluctuations?
- **Universality in random matrices suggests a mechanism of universality in physical systems**

Zooming in



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Counting number of eigenvalues of Wigner matrices in short intervals:

- In Wigner' work, interval $[a, b]$ independent of N
- What if we take $[a(N), b(N)]$ with $b(N) - a(N) \to 0$ as $N \to \infty$?

Local Laws

- *Erdös-Schlein-Yau 2008, 2009, 2010*. Let $\delta > 0$ and E away from spectral edge. There exists a constant K_δ such that with $\eta = K_\delta / N$,

$$\mathbb{P}\left(\left|\frac{\mathcal{N}(E-\eta, E+\eta)}{2N\eta} - \rho_{sc}(E)\right| > \delta\right) < Ce^{-c\delta^2\sqrt{N\eta}}$$

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Theorem 1 (M.-Schlein 2011)

For **any** *sequence* $\eta(N) \to 0$

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Need: Gaussian decay, two derivatives, and $\int \left(\frac{f'(x)}{f(x)}\right)^2 f(x)dx < \infty$.

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[Cacciapuoti -M. - Schlein 2014] For $\alpha = 1, \ldots, N$, *let* $\hat{\alpha} = \min\{\alpha, N+1-\alpha\}$. *Then there exist constants* $C, c, N_0, \varepsilon > 0$ *such that*

$$\mathbb{P}\left(|\lambda_\alpha - \gamma_\alpha| \geq \frac{K \log N}{N} \left(\frac{N}{\hat{\alpha}} \right)^{\frac{1}{3}} \right) \leq \frac{(Cq)^{cq^2}}{K^q} \tag{2.1}$$

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- Eigenvalues are expected to fluctuate on the scale $\sqrt{\log N}/N$ (Gustavson for GUE)
- Improved from $\frac{(\log N)^{\log N}}{N}$ (up to constants) in prior work of Erdös-Yau-Yin

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2-pt correlation function:

$$p^{(2)}(\lambda_1, \lambda_2) := \int_{\mathbb{R}^{N-2}} p_N(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_N) d\lambda_3 \ldots d\lambda_N,$$

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- Tao - Vu used a moment method to prove it for Wigner

Other problems solved and unsolved

Local laws in other ensembles

- Covariance matrices - mostly solved!
- Non-Hermitian iid matrices - somewhat solved
- Sparse - somewhat solved
- Heavy-tailed and heavier-tailed Wigner - a little bit of work done but mostly open

Technicalities

$$G_{jj} = \frac{1}{h_{jj} - z - \mathbf{a}_j^* G^{(j)} \mathbf{a}_j} \tag{5.1}$$

Let $\Lambda = m - m_{sc}$, then

$$G_{jj} = -\frac{1}{m_{sc} + z + \Lambda + \Upsilon_j} \tag{5.2}$$

where

$$\Upsilon_j = -h_{jj} - \frac{1}{N} G_{jj} - \frac{1}{N} \sum_{k \neq j} (G_{kk} - G_{kk}^{(j)}) + (I - \mathbb{E}_j) \mathbf{a}_j^* G^{(j)} \mathbf{a}_j$$

where $\mathbb{E}_j$ denotes the expectation with respect to $\mathbf{a}_j$.

Solve for the fluctuation

We set $R = \frac{1}{N}\sum_j \Upsilon_j\, G_{jj}$, then so that Eq. (5.2) reads

$$m_{sc}\Lambda^2 + (m_{sc}^2 - 1)\Lambda + m_{sc}R = 0\,. \tag{5.3}$$

We solve the equation for Λ and choose the correct solution

$$\Lambda = \frac{-(m_{sc}^2 - 1) + \sqrt{(m_{sc}^2 - 1)^2 - 4m_{sc}^2 R}}{2m_{sc}}\,.$$

$$|\Lambda| \leq C\frac{|m_{sc}||R|}{\sqrt{|m_{sc}^2 - 1|^2 + 4|m_{sc}|^2|R|}} \leq C\min\left\{\frac{|R|}{|m_{sc}^2 - 1|}, \sqrt{|R|}\right\}\,. \tag{5.4}$$

Hanson & Wright: fluctuations of quadratic forms

For $j = 1, \ldots, N$ let $x_j = \text{Re } x_j + i \,\text{Im } x_j$, where $\{\text{Re } x_j, \text{Im } x_j\}_{j=1}^{N}$ is a sequence of $2N$ real iid random variables, whose common distribution ν has subgaussian decay. Let $A = (a_{ij})$ be a $N \times N$ complex matrix. Then there exist constants $c, C > 0$ such that, for any $\delta > 0$

$$\mathbb{P}\left(\left|\sum_{i,j=1}^{N} a_{ij}\left(x_i \bar{x}_j - \mathbb{E} x_i \bar{x}_j\right)\right| \geq \delta\right) \leq Ce^{-c\delta/\sqrt{\text{Tr } A^* A}}. \tag{5.5}$$

Let $z = E + i\eta$. For any $j = 1, ..., N$, set

$$Z_j = (I - \mathbb{E}_j)\mathbf{a}_j^* G^{(j)} \mathbf{a}_j \,. \tag{5.6}$$

- Setting $A = G$ and integrating over δ

$$\mathbb{E}|Z_j|^{2q} \leq (Cq)^{2q}\left(\frac{(\text{Im } m_{sc})^q + \mathbb{E}|\Lambda|^q}{(N\eta)^q} + \frac{1}{(N\eta)^{2q}}\right), \tag{5.7}$$

- $\mathbb{E}|G_{12}|^{2q} < \frac{C_q}{(N\eta)^q}$ – tricky!