Free fermions confined in a Gaussian potential

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Zero temperature wave function

Consider a spinless particle confined in an external field $V(x)$, the wave functions φ_n for its position solve the equation

$$-\nabla^2\varphi_n + V(x)\varphi_n = \varepsilon_n\varphi_n \qquad \text{and} \qquad \varepsilon_0 < \varepsilon_1 < \cdots \ .$$

When $V(x) = x^2$, this equation has explicit solutions:

$$\varphi_n(x) = \frac{(-1)^n}{\sqrt{n!2^n\sqrt{\pi}}} e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2} \qquad \text{and} \qquad \varepsilon_n = 2n+1 \ .$$

If we consider N non-interacting fermions confined by $V(x) = x^2$ at temperature $T = 0$, their positions are described by the ground state wave function

$$\Phi_N(x_1,\ldots,x_N) = \frac{1}{\sqrt{N!}}\det\left[\varphi_n(x_j)\right]_{\substack{n=0,\ldots,N-1\\ j=1,\ldots,N}} \ .$$

Determinantal structure

It means that the joint density of the N particles is given by

$$\rho_N^N(\mathbf{x}) = |\Phi_N(x_1, \ldots, x_N)|^2 = \frac{1}{N!}\left(\det\left[\varphi_n(x_j)\right]\right)^2 \ . \tag{1}$$

If we let

$$K_0^N(x, y) = \sum_{n=0}^{N-1} \varphi_n(x)\varphi_n(y) \ ,$$

using the identity $\det(A^2) = (\det A)^2$, we can rewrite formula (1) as

$$\rho_N^N(\mathbf{x}) = \frac{1}{N!}\det\left[K_0^N(x_i, x_j)\right]_{i,j=1,\ldots,N} \ . \tag{2}$$

$\Rightarrow$ the distribution of the free fermions at $T = 0$ is a **determinantal point process** on $\mathbb{R}$ with **correlation kernel** K_0^N.

The Gaussian Unitary (Invariant) Ensemble

A GUE matrix is an $N \times N$ Hermitian matrices H whose entries are independent and satisfies for all $i < j$,

$$H_{ii} \sim \mathcal{N}_{\mathbb{R}}(0, 1/2) \ , \qquad H_{ij} \sim \mathcal{N}_{\mathbb{C}}(0, 1) \ .$$

We are interested in the distribution of the eigenvalues $(\lambda_1, \ldots, \lambda_N)$ of H.

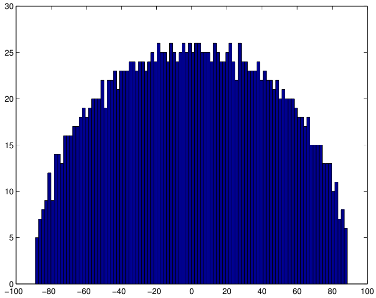


Figure: Histogram of the eigenvalues of a 2000×2000 GUE matrix

Equilibrium distribution (Wigner)

If we consider the empirical measure

$$\mu_N = \frac{1}{N}\sum_{n=1}^{N} \delta_{\lambda_n/\sqrt{N}} \ .$$

we have seen that, almost surely, as $N \to \infty$,

$$\mu_N \Rightarrow \frac{1}{2\pi}\sqrt{4-x^2}\mathbb{1}_{|x|<2}dx \ .$$

Limiting fluctuations

Observe that the typical distance between eigenvalues is $\sim N^{-1/2}$, so we can consider the measures

$$\nu_N^{0,\alpha}(f) = \sum_{n=1}^{N} \delta_{\lambda_n N^{1/2-\alpha}} \ ,$$

for any $0 \leq \alpha \leq 1$.

Theorem (Fyodorov-K-S, Bourgade-E-Y-Y, L)

$\forall 0 < \alpha < 1$, $\forall f \in H^{1/2}$ with compact support, as $N \to \infty$,

$$\nu_N^{0,\alpha}(f) - \mathbb{E}\left[\nu_N^{0,\alpha}(f)\right] \Rightarrow \mathcal{N}\left(0, \|f\|^2_{H^{1/2}}\right) \ .$$

Free fermions at positive temperature

The joint density of N fermions at temperature $T > 0$ is given by

$$p_{N,T}(x_1, \dots, x_N) = \frac{1}{Z_N(T)N!} \sum_{n_1 < \cdots < n_N} \left| \det \left[\varphi_{n_i}(x_j) \right]_{i,j=1,\dots,N} \right|^2 \exp\left(-\frac{1}{T} \sum_{i=1}^{N} \varepsilon_{n_i} \right) .$$

This p.d.f. does not define a determinantal point process. However, it is known that the corresponding **grand-canonical ensemble** is determinantal with correlation kernel

$$K_T^N(x,y) = \sum_{n=0}^{\infty} \frac{1}{e^{(\varepsilon_n - \mu)/T} + 1} \varphi_n(x) \varphi_n(y) . \qquad (3)$$

The chemical potential μ is chosen so that the expected number of fermions is $\mathbb{E}\left[\#\right] = N$.

From GUE to Poisson statistics

It turns out that the right scaling to study the transition from GUE statistics to Poisson is

$$\begin{aligned} T &= 2\tau N^{\nu} \qquad \text{where } 0 < \nu < 1 \text{ and } \tau > 0 \\ \mu &= 2N + 1 \ . \end{aligned}$$

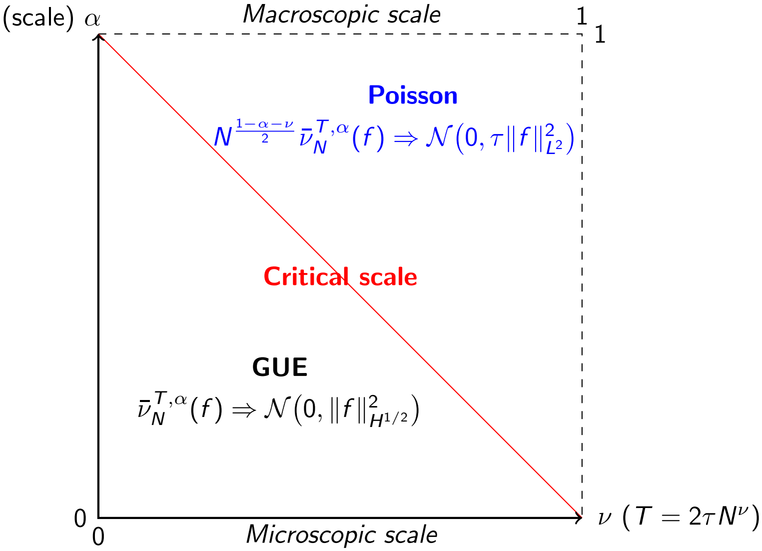
The transition also depends on the scaling of the process. If f is a compactly supported function and $0 \le \alpha \le 1$, we consider the random variable

$$\bar{\nu}_N^{T,\alpha}(f) = \sum_{k=1}^{\#} f(\lambda_k N^{1/2-\alpha}) - \mathbb{E}\left[\sum_{k=1}^{\#} f(\lambda_k N^{1/2-\alpha})\right] \ ,$$

where $(\lambda_1, \ldots, \lambda_{\#})$ are distributed according to the determinantal point process with correlation kernel

$$K_T^N(x,y) = \sum_{n=0}^{\infty} \frac{1}{e^{(n-N)/\tau N^{\nu}} + 1} \varphi_n(x)\varphi_n(y) \ .$$

The transition



The cumulants method to prove a CLT

If Z is a real-valued random variable, we define its cumulants $\mathrm{C}^n[Z]$ by

$$\log \mathbb{E}\left[e^{tZ}\right] = \sum_{n=1}^{\infty} \mathrm{C}^n[Z] \frac{t^n}{n!}$$

Note that $\mathrm{C}^2[Z] = \mathbb{V}\mathrm{ar}\,[Z]$ and, if Z is a Gaussian, then $\mathrm{C}^n[Z] = 0$ for all $n > 2$.

Theorem (Marcinkiewicz)

If there exists $k \geq 3$, such that $\mathrm{C}^n[Z] = 0$ for all $n \geq k$, then the random variable Z is Gaussian.

The Critical regime

Theorem (Johansson-L)

For any $f \in H^{1/2}(\mathbb{R})$ with compact support and any $0 < \alpha < 1$, the linear statistic $\bar{\nu}_N^{T,\alpha}(f)$ converges in distribution as $N \to \infty$ to a random variable $X_\tau(f)$ whose cumulants are given by

$$\mathrm{C}^n\left[X_\tau(f)\right] = 2 \sum_{|\mathbf{m}|=n} \mathrm{M}(\mathbf{m}) \int_{u_1+\cdots+u_n=0} d^{n-1}u \int_{x_1<\cdots<x_n} d^n x \;\Re\left\{\prod_{i=1}^{n} \frac{\hat{f}(u_i)e^{x_i}}{(1+e^{x_i})^2}\right\} \mathrm{G}_\tau^{\mathbf{m}}(u,x)\ .$$

The sum is over all compositions $\mathbf{m} = (m_1, m_2, \ldots, m_\ell)$ of the integer n,

$$\mathrm{M}(\mathbf{m}) = \frac{(-1)^{\ell+1}}{\ell} \frac{n!}{\prod m_j!}\ ,$$

and $G_\tau^{\mathbf{m}}(u,x)$ is a complicated function...

Thank you!

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