Exponential moments of fixed points of the nonhomegenous smoothing transform

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joint work with Gerold Alsmeyer *(University of Münster)*

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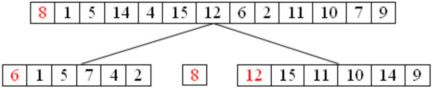
Warsaw Summer School in Probability

Example (Quicksort)



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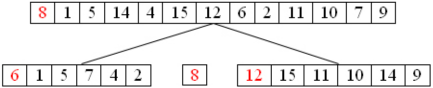

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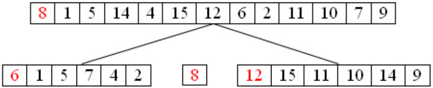

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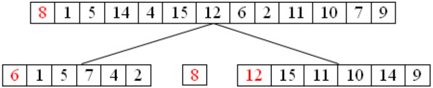

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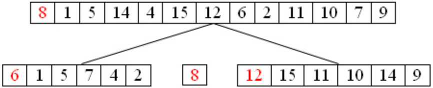

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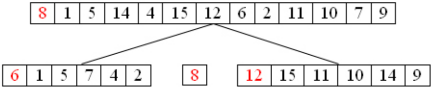

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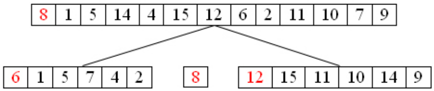

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as $n \to \infty$ we get

$$Y \stackrel{d}{=} UY^{(1)} + (1 - U)Y^{(2)} + g(U).$$

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and define $\mathcal{S}\colon \mathcal{M} \to \mathcal{M}$ by: for $\eta \in \mathcal{M}$ take $(Y_k)_k$ iid(η) and put

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Then

$$X \stackrel{d}{=} \sum_{k=1}^{N} T_k X_k + C \Leftrightarrow \mathcal{S}(\mu) = \mu.$$

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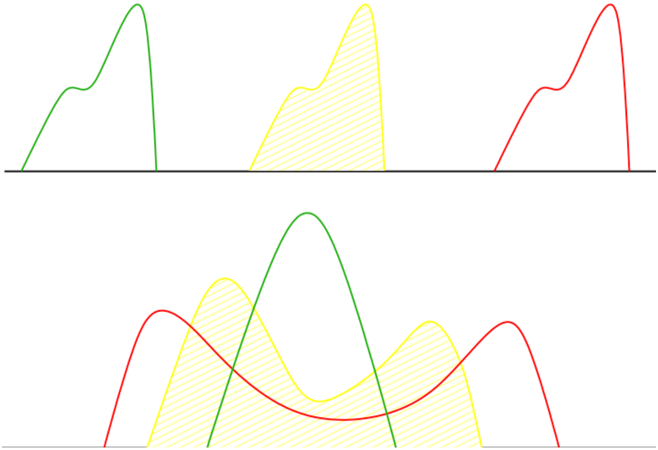
This distance can be interpreted as an optimal transportation problem.



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Assume that

$$\mathbb{E}\left[\sum_{k=1}^{N} T_k\right] < 1, \quad 0 < \mathbb{E}\left[|C|\right] < \infty. \tag{$\spadesuit$}$$

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Corollary

Assume (♠), then there exists a unique solution (in $\mathcal{M}$*) of*

$$X \stackrel{d}{=} \sum_{k=1}^{N} T_k X_k + C.$$

Furthermore, for any $\eta \in \mathcal{M}$ *we have* $\mathcal{S}^n(\eta) \xrightarrow{\mathrm{d}} \mu$ *as* $n \to \infty$*.*

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Let $\Theta > 0$ then $\Theta \in \mathbb{D}_\Psi$ if, and only if $\exists\, \Phi\colon [0,\Theta] \to (0,+\infty)$, $\Phi(0) = 1$, $\Phi(\theta) > \delta > 0$, differentiable at 0 such that

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Proof.

There exists c such that

$$\mathbb{E}[\exp\{\theta Z_0\}] = \exp\{c\theta\} \le \Phi(\theta) \qquad \text{for } \theta \in [0,\Theta].$$

Consider $Z_n \stackrel{d}{=} \mathcal{S}^n(\delta_c)$.

Proof continued.

$Z_n \stackrel{d}{=} \mathcal{S}(Z_{n-1})$ with $Z_0 = c$.

$\square$

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$Z_n \stackrel{d}{=} \mathcal{S}(Z_{n-1})$ with $Z_0 = c$. Let $\Psi_n(\theta) = \mathbb{E}[\exp\{\theta Z_n\}]$ and thus, by induction we obtain for any $n \in \mathbb{N}$

$$\Psi_n(\theta) = \mathbb{E}\left[\exp\{\theta C\} \prod_{k=1}^{N} \Psi_{n-1}(T_k\theta)\right]$$
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On the other hand if $\Theta \in \mathbb{D}_\Psi$ then $\Phi(\theta) = \Psi(\theta)$ satisfies

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Rough idea.

For Φ increasing sufficiently fast

$$\mathbb{E}\left[\exp\{\theta C\}\Phi(T_1\theta)\prod_{k=2}^{N}\Phi(T_k\theta)\right] \approx \mathbb{E}\left[e^{\theta C}\mathbb{1}_{\{\max_k T_k = 1\}}\right]\Phi(\theta) \le \Phi(\theta).$$

Example

Suppose $A \stackrel{d}{=} B(\alpha, 1)$, and let $N = n \geq 1$ such that $\alpha < \frac{2}{n-1}$, $T_1 = T_2 = \ldots = T_n = A$ and take C independent of A with $\mathbb{E}[C] = 0$ and $\varphi(\theta) = \mathbb{E}[\exp\{\theta C\}]$.

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and so

$$\Psi^{n-1}(\theta) = \frac{\varphi(\theta)^{n-1}}{1 - \int_0^\theta (\varphi(s)^n - 1) \left(\frac{\theta}{s}\right)^{\alpha(n-1)+1} \alpha(n-1) s^{-1} \, \mathrm{d}s}$$