ℓ_1 minimisation with noisy data

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Abstract

Compressed sensing aims at recovering a sparse signal $x \in \mathbb{R}^N$ from few nonadaptive, linear measurements $\Phi(x)$ given by a measurement matrix Φ. One of the fundamental recovery algorithms is an ℓ_1 minimisation. In this paper we investigate the situation when our measurement $\Phi(x)$ is contaminated by arbitrary noise under the assumption that the matrix Φ satisfies the restricted isometry property. This complements results from [4] and [8].

1 Introduction

Compressed sensing is a new scheme which shows that some signals can be reconstructed from fewer measurements than previously considered. The mathematical formulation is the following. Our signal is a vector $x \in \mathbb{R}^N$. We have a matrix Φ with N columns and d rows called *measurement matrix*

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and our measurements are represented by $y = \Phi(x) \in \mathbb{R}^d$. We also need a decoder Δ (which generally is non-linear) which produces $\Delta(y) \in \mathbb{R}^N$ which should be an approximation to x. The main point in compressed sensing as expressed in recent papers (see e.g. [3, 5, 9, 7]) is that it is actually possible to recover the essential information about x from relatively few non-adaptive measurements $d << N$. Substantial progress have been made in recent years in understanding the performance of various measurement matrices Φ and decoders Δ. Generally we have also an integer $k \leq d$ which measures the amount of information we wish to recover. The standard initial requirement is that for every k–sparse vector (i.e. $x \in \Sigma_k$) we have $\Delta(\Phi(x)) = x$. This clearly forces $\Phi|\Sigma_k$ to be one to one. But for Δ to be numerically friendly we must have the corresponding systems of equations well conditioned. This leads to the restricted isometry property RIP (also called in the literature uniform uncertainty property – UUP).

By Σ_μ we will mean the set of all vectors from $\mathbb{R}^s$ (where s should be clear from the context) which have at most μ non-zero coefficients. For a vector $x \in \mathbb{R}^N$ and $\mu = 1, 2, \ldots, N$ we define the error of the best μ-term approximation in ℓ_1 norm as

$$\sigma^1_\mu(x) = \inf\{\|x - z\|_1 \;\;:\;\; z \in \Sigma_\mu\}.$$

We say that matrix Φ satisfies RIP(k, δ) where $0 < \delta < 1$ and $k \in \mathbb{N}$ if

$$(1 - \delta)\|c\|_2 \leq \|\Phi(c)\|_2 \leq (1 + \delta)\|c\|_2 \tag{1}$$

for all vectors $c \in \Sigma_k$. One of the most popular decoders (see e.g. [2, 4, 8, 9]) is ℓ_1 minimization given by

$$\Delta_1(y) = \text{Argmin}\{\|z\|_1 \;:\; \Phi z = y\}. \tag{2}$$

For $\epsilon \geq 0$ we will also consider the decoder cf. [4]

$$\Delta^\epsilon(y) = \text{Argmin}\{\|z\|_1 \;:\; \|\Phi(z) - y\|_2 \leq \epsilon\}. \tag{3}$$

Note that $\Delta^0 = \Delta_1$. It seems to be a general belief in the compressed sensing community that ℓ_1 minimization for RIP measurement matrices is robust with respect to the measurement errors. This is based on many results proven under various assumptions about noise see e.g. [7, Sect. 1.5.2] for an overview. The aim of this paper is to examine this belief in some detail in the worst case situation i.e. we put no restriction on the structure of the error and we want to have estimates valid for *all* errors. Basically there are two results dealing with this question. First we have the classical theorem of Candes-Romberg-Tao [4] (with later improvements) which says

Theorem 1.1. *Suppose that Φ satisfies $RIP(2k,\delta)$ with $\delta < \sqrt{2}-1$ and let $\epsilon \geq 0$ be fixed. Then for any $x \in \mathbb{R}^N$ and any $e \in \mathbb{R}^d$ with $\|e\|_2 \leq \epsilon$*

$$\|\Delta^\epsilon(\Phi(x)+e) - x\|_2 \leq C_0 k^{-1/2}\sigma_k^1(x) + C_1\epsilon \tag{4}$$

where constants C_0 and C_1 depend only on δ.

Secondly we have theorems from [12] and [8] (see also [13]) which can be summarized as follows

Theorem 1.2. *Suppose that Φ is a random matrix with all entries being independent copies of a symmetric, subgaussian random variable η such that $\mathbb{E}(\eta^2) = d^{-1}$. For $x \in \mathbb{R}^d$ we define*

$$\|x\|_J = \max(\sqrt{\log N}\|x\|_\infty, \|x\|_2). \tag{5}$$

Then with overwhelming probability matrix Φ has the following property: for every $x \in \mathbb{R}^N$ and $e \in \mathbb{R}^d$ we have

$$\|\Delta_1(\Phi(x)+e) - x\|_2 \leq C(\|e\|_J + k^{-1/2}\sigma_k^1(x)) \tag{6}$$

for $k = cd/\log N$ where constants C and c depend on η. When η is Gaussian variable we can replace $\|x\|_J$ by $\|x\|_2$.

Let us note that when we use a constant we understand that this constant does not depend on N, d, k but may depend on other parameters of the problem e.g. δ.

When we know that there is no error both Theorems reduce to Theorem 2.1. There are however several important differences between those theorems.

- Theorem 1.1 applies to a wide, abstract class of matrices while Theorem 1.2 deals with special matrices. Actually the proof of Theorem 1.2 requires RIP (which in this case is known to hold with overwhelming probability) plus some other properties of the matrix.
- Theorem 1.1 requires an a priori upper bound on the size of the error and uses a decoder which depends on this upper bound. It says nothing

about errors with $\|e\|_2 > \epsilon$. But if we pessimistically estimate $\|e\|_2 < \epsilon$ while in reality e is much smaller or $e = 0$ the estimate (4) still contains the term $C_1\epsilon$. Contrary to this Theorem 1.2 uses a fixed decoder which gives the error of the decoding dependent on the real (unknown) error.

This analysis rises several questions

1. What is the actual performance of Δ^ϵ for noisy data with $\|e\|_2 \ll \epsilon$ and $\|e\|_2 > \epsilon$?

2. Can we produce an estimate like (5) for arbitrary RIP measurement matrix with $\|.\|_J$ replaced by a suitable norm?

We will discuss those question in the worst case situation i.e. we put no restriction on the structure of the error and we want to have estimates valid for *all* errors. The results our discussion can be summarized as follows

1. For arbitrary matrix Φ we construct a norm $\|.\|_\Phi$ such that analog of Theorem 1.2 holds when this norm replaces $\|.\|_J$. We show that in general this norm gives an optimal estimate.

2. We show examples of matrices with very good RIP properties such that for some measurements an unlucky but very small measurement error can result in huge reconstruction error of Δ_1 decoder. The same may happen for Δ^ϵ when the error is bigger then ϵ.

3. We show ways to modify measurement matrices so that the modified matrix will be robust with respect to measurement error.

2 General estimates

We fix a matrix $\Phi : \mathbb{R}^N \to \mathbb{R}^d$ and we assume that it satisfies $RIP(k, \delta)$.We will always denote by $(\phi_j)_{j=1}^N$ columns of the matrix Φ. Symbols Δ_1 and Δ^ϵ will denote decoders defined in (2) and (3).

Let us now make some remarks about those decoders:

1. In order for Δ_1 and Δ^ϵ to be well defined we must have $\Phi(\mathbb{R}^N) = \mathbb{R}^d$ and we will always assume this.

2. By compactness for every $y \in \mathbb{R}^d$ there exists $\tilde{y} \in \mathbb{R}^d$ such that $\|y - \tilde{y}\|_2 \leq \epsilon$ and $\Delta^\epsilon(y) = \Delta_1(\tilde{y})$. Actually by elementary convexity considerations we see that we must have $\|y - \tilde{y}\|_2 = \epsilon$.

3. Since unit ball in a euclidean space is strictly convex this $\tilde{y}$ is unique.

A fundamental role in our considerations will be played by the following Theorem of E.Candes, J.Romberg and T.Tao [4]:

Theorem 2.1. *Suppose the matrix* Φ *satisfies RIP*$(2k, \delta)$ *with* $\delta < \sqrt{2} - 1$. *Then there exists a constant* K *such that*

$$\|x - \Delta_1(\Phi x)\|_2 \leq \frac{K\sigma_k^1(x)}{\sqrt{k}} \tag{7}$$

for all $x \in \mathbb{R}^N$.

Note that in particular this theorem gives $\Delta_1(\Phi(x)) = x$ for all $x \in \Sigma_k \subset \mathbb{R}^N$. This Theorem was formulated without proof in [1]. It appeared for the first time in [4] but with the assumption that Φ satisfies RIP$(3k, \delta_1)$ and RIP$(4k, \delta_2)$ and $\delta_1 + 3\delta_2 < 2$. The present version was proved in [2].

For a fixed k and a matrix Φ we define a norm $\|y\|_\Phi$ on $\mathbb{R}^d$ by the formula

$$\|y\|_\Phi = \max(\|y\|_2, \frac{1}{\sqrt{k}} \inf\{\sum_{j=1}^N |y_j| \ : \ \sum_{j=1}^N y_j\phi_j = y\}). \tag{8}$$

Since we assume $\Phi(\mathbb{R}^N) = \mathbb{R}^d$ this is a well defined norm. Its unit ball equals $U = \sqrt{k}\Phi(B_1^N) \cap B_2^d$. Note that in particular we have $\|x\|_\Phi \geq \|x\|_2$. The proof of the following Proposition and Theorem repeat almost verbatim arguments from [12] which follow arguments from [4]. We present them for completeness of the paper.

Proposition 2.2. *For every* $x \in \mathbb{R}^N$ *there exists* $\tilde{x} \in \mathbb{R}^N$ *such that*

1. $\Phi(x) = \Phi(\tilde{x})$

2. $\|\tilde{x}\|_1 \leq \sqrt{k}\|\Phi(x)\|_\Phi$

3. $\|\tilde{x}\|_2 \leq C\|\Phi(x)\|_\Phi$

Proof. From the definition of the norm $\|\Phi(x)\|_\Phi$ we can find a vector $\tilde{x} = (\tilde{x}_j)_{j=1}^N$ such that $\Phi(x) = \sum_{j=1}^N \tilde{x}_j \phi_j = \Phi(\tilde{x})$ with $\frac{1}{\sqrt{k}} \sum_{j=1}^N |\tilde{x}_j| \leq \|\Phi(x)\|_\Phi$ which gives 1. and 2.. To estimate $\|\tilde{x}\|_2$ we split the set $\{1, 2, \ldots, N\}$ into disjoint k–element sets $S_0, S_1, \ldots$ such that $|\tilde{x}_j| \geq |\tilde{x}_l|$ whenever $j \in S_\nu$ and $l \in S_{\nu+1}$. Clearly we have

$$\|\tilde{x}|S_{\nu+1}\|_2 \leq \frac{1}{\sqrt{k}} \|\tilde{x}|S_\nu\|_1. \tag{9}$$

From 2. and (9) we get

$$\|\tilde{x}|S_0^c\|_2 \leq \sum_{\nu \geq 1} \|\tilde{x}|S_\nu\|_2 \leq \frac{1}{\sqrt{k}} \|\tilde{x}\|_1 \leq \|\Phi(x)\|_\Phi. \tag{10}$$

Also using (10) and RIP condition we get

$$\|\Phi(\tilde{x}|S_0^c)\|_2 \leq \sum_{\nu \geq 1} \|\Phi(\tilde{x}|S_\nu)\|_2 \leq (1+\delta) \sum_{\nu \geq 1} \|\tilde{x}|S_\nu\|_2 \leq (1+\delta)\|\Phi(x)\|_2. \tag{11}$$

Now we use RIP condition, (11) and the definition of $\|.\|_\Phi$ to get

$$\begin{aligned}
\|\tilde{x}|S_0\|_2 &\leq \frac{1}{1-\delta} \|\Phi(\tilde{x}|S_0)\|_2 = \frac{1}{1-\delta} \|\Phi(\tilde{x}) - \Phi(\tilde{x}|S_0^c)\|_2 \\
&\leq \frac{1}{1-\delta} \left(\|\Phi(x)\|_2 + \|\Phi(\tilde{x}|S_0^c)\|_2 \right) \\
&\leq \frac{1}{1-\delta} \left(\|\Phi(x)\|_2 + (1+\delta)\|\Phi(x)\|_2 \right) \\
&\leq \frac{2+\delta}{1-\delta} \|\Phi(x)\|_\Phi.
\end{aligned}$$

which together with (10) gives 3 with $C = \sqrt{1 + \left(\frac{2+\delta}{1-\delta}\right)^2}$

□

Theorem 2.3. *Suppose that the matrix Φ satisfies RIP$(2k, \delta)$ with the constant $\delta < \sqrt{2} - 1$ and is surjective. Then for any $x \in \mathbb{R}^N$ and any $e \in \mathbb{R}^d$*

$$\|\Delta_1(\Phi(x) + e) - x\|_2 \leq C\Big(\|e\|_\Phi + \frac{\sigma_k^1(x)}{\sqrt{k}} \Big). \tag{12}$$

If we also have $\|e\|_2 \geq \epsilon$ then we have

$$\|\Delta^\epsilon(\Phi(x) + e) - x\|_2 \leq C \left(\epsilon + (1 - \epsilon\|e\|_2^{-1})\|e\|_\Phi + k^{-1/2}\sigma_k^1(x) \right). \tag{13}$$

Proof. Since Φ is surjective we infer that there exists $z \in \mathbb{R}^N$ such that $\Phi(z) = e$. From Proposition 2.2 we infer that we can choose z such that $\|z\|_1 \leq \sqrt{k}\|e\|_\Phi$ and $\|z\|_2 \leq C\|e\|_\Phi$. Since $\Phi(x+z) = \Phi(x)+e$ from Theorem 2.1 we get

$$\|\Delta(\Phi(x)+e) - (x+z)\|_2 \leq K\frac{\sigma_k^1(x+z)}{\sqrt{k}}$$

so using Proposition 2.2 we get

$$\begin{aligned} \|\Delta(\Phi(x)+e) - x\|_2 &\leq \|z\|_2 + K\frac{\sigma_k^1(x+z)}{\sqrt{k}} \\ &\leq C\|e\|_\Phi + K\frac{\sigma_k^1(x) + \|z\|_1}{\sqrt{k}} \\ &\leq (C+K)\|e\|_\Phi + K\frac{\sigma_k^1(x)}{\sqrt{k}}. \end{aligned}$$

Now we prove (13). Let us put $\Phi(x) + e = y$ and $\beta = 1 - \epsilon\|e\|_2^{-1}$. Using Proposition 2.2 let us fix $z \in \mathbb{R}^N$ such that $\Phi(z) = e$ and $\|z\|_1 \leq \sqrt{k}\|e\|_\Phi$ and $\|z\|_2 \leq C\|e\|_\Phi$ and let us put $v = x + \beta z$ so $\|y - \Phi(v)\|_2 = \epsilon$. From Theorem 2.1 we get

$$\|\Delta^\epsilon(y) - v\|_2 \leq C_1\epsilon + C_0 k^{-1/2}\sigma_k^1(v)$$

so

$$\begin{aligned} \|\Delta^\epsilon(y) - x\|_2 &\leq \|\Delta^\epsilon(y) - v\|_2 + \beta\|z\|_2 \leq C_1\epsilon + C_0 k^{-1/2}\sigma_k^1(v) + C\beta\|e\|_\Phi \\ &\leq C_1\epsilon + C_0 k^{-1/2}\sigma_k^1(x) + C_0 k^{-1/2}\beta\|z\|_1 + C\beta\|e\|_\Phi \\ &\leq C(\epsilon + \beta\|e\|_\Phi + k^{-1/2}\sigma_k^1(x)). \end{aligned}$$

□

Now we want to show that the decoder Δ^ϵ has an error ϵ build in; we analyze the situation when in reality there is no measurement error.

Proposition 2.4. *Suppose that Φ satisfies assumptions of Theorem 2.1 and let $\epsilon > 0$ be fixed. Then for any vector $x \in \mathbb{R}^N$ we have*

1. *If $\|\Phi(x)\|_2 \leq \epsilon$ then $\Delta^\epsilon(\Phi(x)) = 0$.*

2. If x is k–sparse and $\|\Phi(x)\|_2 > \epsilon$ then

$$\|x - \Delta^\epsilon(\Phi(x))\|_1 \geq \frac{\epsilon}{1+\delta}\frac{\|x\|_1}{\|x\|_2} \tag{14}$$

and

$$\|x - \Delta^\epsilon(\Phi(x))\|_2 \geq (1-\delta)\epsilon. \tag{15}$$

Proof. The first claim follows directly from the definition. Let us denote $x^\# = \Delta^\epsilon(\Phi(x))$. To prove (14) let us put $x_0 = \xi x$ for $\xi = 1 - \frac{\epsilon}{\|\Phi(x)\|_2}$. Since $\|\Phi(x) - \Phi(x_0)\|_2 = |1-\xi|\|\Phi(x)\|_2 = \epsilon$ we get $\|x^\#\|_1 \leq \|\Delta_1(\Phi(x_0))\|_1$. From Theorem 2.1 we infer that $\Delta_1(\Phi(x_0)) = x_0$ so we get $\|x^\#\|_1 \leq \xi\|x\|_1$ and

$$\|x - x^\#\|_1 \geq \|x\|_1 - \xi\|x\|_1 = \epsilon\frac{\|x\|_1}{\|\Phi(x)\|_2} \geq \frac{\epsilon}{1+\delta}\frac{\|x\|_1}{\|x\|_2}$$

so we have (14).

Now let us denote $A = \operatorname{supp} x$. If $\|x - x^\#|A\|_2 > \frac{\epsilon}{1+\delta}$ then $\|x - x^\#\|_2 \geq \|x - x^\#|A\|_2 > \frac{\epsilon}{1+\delta} > (1-\delta)\epsilon$, so we have (15). If $\|x - x^\#|A\|_2 \leq \frac{\epsilon}{1+\delta}$ using RIP we get $\|\Phi(x) - \Phi(x^\#|A)\|_2 \leq \epsilon$. From the definition of Δ^ϵ we infer that $\|x^\#\|_1 \leq \|\Delta_1(\Phi(x^\#|A)\|_1$. But $\sigma_k^1(x^\#|A) = 0$ so Theorem 2.1 gives $\Delta_1(\Phi(x^\#|A) = x^\#|A$. Thus we get $\|x^\#\|_1 \leq \|x^\#|A\|_1$. This implies that $\operatorname{supp} x^\# \subset A$. But from the definition of Δ^ϵ we see that we must have $\|\Phi(x) - \Phi(x^\#)\|_2 = \epsilon$ so using RIP we get $\|x - x^\#\|_2 \geq (1-\delta)\epsilon$. Thus (15) holds. □

3 Examples and comments

Examples Before we proceed let us build some geometric intuitions. Clearly the best norm we can hope for in place of $\|.\|_\Phi$ in Theorem 2.3 is the euclidean norm $\|.\|_2$. Nothing really smaller will work as we see if take any k-sparse signal x and $e = \Phi(z)$ with $\operatorname{supp}(z) \subset \operatorname{supp}(x)$. It was shown in [12] that for a random Gaussian measurement matrix Φ we have $\|y\|_\Phi \leq c\|y\|_2$ for $y \in \mathbb{R}^d$. In [13] we gave some other matrices with this property.

Thus if we are looking for examples where Theorem 2.3 is precise we need vectors e such that $\|e\|_\Phi$ is big while $\|e\|_2$ is very small. From (8) we see that it happens when any representation of e as a linear combination of Φ_j's must have big ℓ_1 norm of coefficients. Another way to express it is to say that every euclidean ball $B(0,r) \subset \operatorname{conv}\{\pm\phi_j\}_{j=1}^N$ has very small radius r. We see from (8) that if $\operatorname{conv}\{\pm\phi_j\}_{j=1}^N$ contains euclidean ball $B(0,\frac{c}{\sqrt{k}})$ then $\|y\|_\Phi \leq c^{-1}\|y\|_2$ for all $y \in \mathbb{R}^d$. With this in mind we can produce the following

Example: Let us start with a matrix $\Phi : \mathbb{R}^N \to \mathbb{R}^d$ with columns $(\phi_j)_{j=1}^N$ such that

1. Φ has $RIP(4k,\delta)$ for some k and $\delta > 0$
2. $\operatorname{conv}\{\pm\phi_j\}_{j=k+1}^N \supset \frac{\mu}{\sqrt{k}}B_2^d$, in particular Φ is surjective.

It follows from arguments in [12] that random Gaussian matrices with $d \sim k\ln N$ satisfy those conditions with overwhelming probability.

Let us fix $\gamma > 0$ and assume that $\gamma + \delta < \sqrt{2} - 1$. We define our measurement matrix $\tilde{\Phi} : \mathbb{R}^N \to \mathbb{R}^{d+k}$ by defining columns $\tilde{\phi}_j$ as follows: $\tilde{\phi}_j = \phi_j$ for $j = k+1, k+2, \ldots, N$ and $\tilde{\phi}_j = \phi_j + \gamma e_{d+j}$ for $j = 1,\ldots,k$. We identify vectors $(a_1,\ldots,a_d) \in \mathbb{R}^d$ with vectors $(a_1,\ldots,a_d,0,\ldots,0) \in \mathbb{R}^{d+k}$. One checks that $\tilde{\Phi}$ maps $\mathbb{R}^N$ onto $\mathbb{R}^{d+k}$ and that for any set A with $\#A \leq 4k$ we have

$$(1-\delta)\sqrt{\sum_{j\in A}|a_j|^2} \leq \|\sum_{j\in A} a_j\tilde{\phi}_j\|_2 \leq \sqrt{(1+\delta)^2+\gamma^2}\sqrt{\sum_{j\in A}|a_j|^2}. \tag{16}$$

so $\tilde{\Phi}$ satisfies $RIP(4k,\delta+\gamma)$.

Now let us put $e = \frac{\eta}{\sqrt{k}}\sum_{j=d+1}^{d+k} e_j$ so $\|e\|_2 = \eta$. If $\tilde{\Phi}(z) = e$ then $z_j = \frac{\eta}{\gamma\sqrt{k}}$ for $j = 1,\ldots,k$. This implies that

$$\begin{aligned} \|z\|_1 &\geq \|z|\{1,\ldots,k\}\|_1 = \frac{\eta\sqrt{k}}{\gamma} \\ \|z\|_2 &\geq \|z|\{1,\ldots,k\}\|_2 = \frac{\eta}{\gamma} \end{aligned}$$

so $\|e\|_\Phi \geq \frac{\eta}{\gamma}$. Also $\sum_{j=k+1}^N z_j\phi_j = -\sum_{j=1}^k z_j\phi_j$. Since $\|\sum_{j=1}^k z_j\phi_j\|_2 \leq (1+\delta)\eta/\gamma$, from our assumptions we infer that we can find such $(z_j)_{j=k+1}^N$ satisfying $\sum_{j=k+1}^N |z_j| \leq \frac{\eta(1+\delta)\sqrt{k}}{\gamma\mu}$ which yields $\|e\|_\Phi \leq \frac{\eta(1+\delta)}{\gamma\mu}$, so

$$\frac{\eta}{\gamma} \leq \|e\|_\Phi \leq \frac{1+\delta}{\mu}\frac{\eta}{\gamma}. \tag{17}$$

Now we want to analyze the performance of decoders Δ_1 and Δ^ϵ for measurement matrix $\tilde{\Phi}$ and measurement error e. Let us take any k-sparse signal x with $\operatorname{supp} x \subset \{k+1, \dots, N\}$ and put $y = \Phi(x) = \tilde{\Phi}(x)$. For $z \in \mathbb{R}^N$ such that $\tilde{\Phi}(z) = y + e$ we see that $z_j = \frac{\eta}{\gamma\sqrt{k}}$ for $j = 1, \dots, k$. This implies that $\|z\|_2 \geq \|z|\{1, \dots, k\}\|_2 = \frac{\eta}{\gamma}$, so $\|\Delta_1(y + e) - x\|_2 \geq \frac{\eta}{\gamma}$. Since $\sigma_k^1(x) = 0$ from Theorem 2.3 and (17) we get

$$\frac{\mu}{1+\delta}\|e\|_\Phi \leq \|\Delta_1(\tilde{\Phi}(x) + e) - x\|_2 \leq C\|e\|_\Phi. \tag{18}$$

Now let us take $\epsilon < \eta$ and let us analyze $\|\Delta^\epsilon(\tilde{\Phi}(x) + e) - x\|_2$. From the definition of Δ^ϵ we see that there exists $\tilde{e} \in \mathbb{R}^d$ such that $\Delta^\epsilon(\tilde{\Phi}(x) + e) = \Delta_1(\tilde{\Phi}(x)+e+\tilde{e})$ and $\|\tilde{e}\|_2 \leq \epsilon$. If for some $z \in \mathbb{R}^N$ we have $\tilde{\Phi}(z) = \tilde{\Phi}(x)+e+\tilde{e}$ then

$$(z|\{1, \dots, k\}) = \gamma^{-1}((\tilde{e}+\tilde{\Phi}(x)+e)|\{d+1, \dots, d+k\}) = \gamma^{-1}((\tilde{e}+e)|\{d+1, \dots, d+k).$$

This implies that

$$\|z|\{1, \dots, k\}\|_2 \geq \gamma^{-1}\|e + \tilde{e}\|_2 \geq \frac{\eta - \epsilon}{\gamma}$$

so $\|z-x\|_2 \geq \|(z-x)|\{1, \dots, k\}\|_2 = \|z|\{1, \dots, k\|_2 \geq \gamma^{-1}(\eta-\epsilon)$ in particular

$$\|\Delta^\epsilon(\tilde{\Phi}(x) + e) - x\|_2 \geq \gamma^{-1}(\eta - \epsilon).$$

This example shows that

1. For Δ_1 decoder and a measurement matrix satisfying RIP even with very small constant it may happen that very small measurement error can produce very big decoding error. Thus RIP is not a sufficient assumption to guarantee robust recovery.

2. In some situations the norm $\|e\|_\Phi$ used in Theorem 2.3 is an optimal control of decoding error.

3. The decoder Δ^ϵ for errors with $\|e\|_2 > \epsilon$ can produce very big decoding errors. In particular if $\|e\|_2 \geq 2\epsilon$ the decoding error can be comparable with $\|e\|_\Phi$.

Improving the measurement matrix Now we want to show that despite previous results, when the matrix Φ satisfies RIP the corrupted measurement $\Phi(x)+e$ contains enough information to recover x with accuracy $C(\|e\|_2 + k^{-1/2}\sigma_k^1(x))$ using ℓ_1 minimization as the decoder. This is not very supprising as some of the greedy decoders have similar performance guarantee (see e.g. [11, 6]). We will achieve this goal by enlarging the matrix Φ, which may seem a bit counterintuitive.

The basic idea for enlargement was already sketched in [13]. Suppose we have a matrix $\Phi : \mathbb{R}^N \to \mathbb{R}^d$ which satisfies $RIP(2k,\delta)$ with $\delta < \sqrt{2}-1$. Suppose also that we can find vectors $\psi_1,\ldots,\psi_s \in \mathbb{R}^d$ such that the matrix A with columns $[\phi_1,\ldots,\phi_N,\psi_1,\ldots,\psi_s]$ maps $\mathbb{R}^N \oplus \mathbb{R}^s = \mathbb{R}^{N+s}$ into $\mathbb{R}^d$, satisfies $RIP(2k,\delta')$ with $\delta' < \sqrt{2}-1$ and $\|y\|_A < C\|y\|_2$ for all $y \in \mathbb{R}^d$. In this situation we can use the following recovery procedure (decoder) based on ℓ_1 minimization.

1. We identify a signal $x \in \mathbb{R}^N$ with a vector in $\mathbb{R}^{N+s}$.
2. For the noisy measurement $y = \Phi(x) + e \in \mathbb{R}^d$ we use the Δ_1 decoder defined using matrix A i.e. we find
$$x^{\#} = \operatorname{Argmin}\{\|z\|_1 \ : \ Az = y, \quad z \in \mathbb{R}^{N+s}\}.$$
3. We define $x^{\#}|\{1,\ldots,N\} \in \mathbb{R}^N$ as the output of our algorithm.

Note that $\Phi(x) = A(x)$, so from Theorem 2.3 (in this case it is [12, Theorem 3.4]) and our assumptions about A we get

$$\|x^{\#}|\{1,\ldots,N\} - x\|_2 \le \|x^{\#} - x\|_2 \le C(\|e\|_2 + \frac{\sigma_k^1(x)}{\sqrt{k}}).$$

It should be pointed out that measurements used by this algorithm are done using matrix Φ, matrix A is only a part of our recovery procedure.

In [13] it is shown that if $k \sim \frac{cn}{\ln N}$ then $s \sim N$ random Gaussian vectors $\psi_1,\ldots,\psi_s$ does the job with overwhelming probability.

In general the problem of finding vectors $\psi_1,\ldots,\psi_s$, i.e. extending the RIP matrix so that the new matrix will meet our requirements, clearly requires farther thought. Geometrically speaking we have matrix Φ such that the set $\operatorname{conv}\{\pm\phi_j\}_{j=1}^N$ does contains only a small euclidean ball $B(0,r)$. Our aim is to add vectors so that this convex will be enlarged to contain euclidean ball of radius $\sim \frac{1}{\sqrt{k}}$ but that the bigger matrix will still have good RIP constant. The method described in [13] is basically to add vectors so that the convex of new vectors alone will be big enough. This is clearly wasteful.

Geometric remarks To make more precise some of the geometric ideas discussed above for a matrix Φ we consider the Kolmogorov widths of $\Phi(B_N^1)$. Those are classical tools in approximation theory [10, Chap. 13]. They are defined as follows

$$\mathbf{d}_s = \inf_{\dim V \leq s} \sup_{x \in \Phi(B_N^1)} \operatorname{dist}(x, V) \tag{19}$$

where V denotes a linear subspace of $\mathbb{R}^d$ and $s = 0, 1, \ldots, d$. It is easy to check that we have

$$\mathbf{d}_s = \inf_{\dim V \leq s} \sup_j \operatorname{dist}(\phi_j, V). \tag{20}$$

The distances in those definitions are usual euclidean distances. Clearly $(\mathbf{d}_s)$ is a decreasing sequence with $\mathbf{d}_d = 0$ and it is easy to see that if Φ has $RIP(2, \delta)$ then $d_1 \geq 1 - \delta$. The following Proposition relates Kolmogorov widths $(\mathbf{d}_s)$ with our discussion.

Proposition 3.1. *Suppose $\Phi : \mathbb{R}^N \to \mathbb{R}^d$ satisfies $RIP(k, \delta)$.*

1. *Let $\mathbf{d}_s < \frac{\gamma}{\sqrt{k}}$ for some $s < d$. Let V be a subspace of dimension s such that $\sup_j \operatorname{dist}(\phi_j, V) < \frac{\gamma}{\sqrt{k}}$ and let P be an orthogonal projection onto V. Then the matrix $P\Phi : \mathbb{R}^d \to V$ satisfies $RIP(k, \delta + \gamma)$.*

2. *Suppose that we have $\mathbf{d}_{d-1} \geq \frac{\gamma}{\sqrt{k}}$. Then $\|y\|_\Phi \leq \gamma^{-1} \|y\|_2$ for all $y \in \mathbb{R}^d$.*

Proof. First note that subspace V exists by our assumption about $\mathbf{d}_s$. We write $\phi_j = P\phi_j + (I - P)\phi_j := P\phi_j + v_j$ and we know that $\|v_j\|_2 \leq \frac{\gamma}{\sqrt{k}}$. For any $A \subset \{1, \ldots, N\}$ with $\#A \leq k$ we have

$$\begin{aligned}
\|\sum_{j \in A} a_j P\phi_j\|_2 &\leq \|\sum a_{j \in A} a_j \phi_j\|_2 + \|\sum_{j \in A} a_j v_j\|_2 \\
&\leq (1 + \delta)\sqrt{\sum_{j \in A} |a_j|^2} + \sqrt{\sum_{j \in A} |a_j|^2}\sqrt{\sum_{j \in A} \|v_j\|_2^2} \\
&\leq (1 + \delta + \gamma)\sqrt{\sum_{j \in A} |a_j|^2}.
\end{aligned}$$

The estimate from below follows from the same calculation with obvious modifications, so the first claim follows. If $\mathbf{d}_{d-1} \geq \frac{\gamma}{\sqrt{k}}$ then for every vector $\xi \in \mathbb{R}^d$ with $\|\xi\|_2 = 1$ there exists j such that $|\langle \phi_j, \xi \rangle| \geq \frac{\gamma}{\sqrt{k}}$ because $\mathrm{dist}(\phi_j, ker\, \xi) = |\langle \phi_j, \xi \rangle|$ and $ker\, \xi$ is a subspace of dimension $d-1$. This implies that $\Phi(B_1^N) \supset \frac{\gamma}{\sqrt{k}} B_2^d$. Indeed if not then some $y \in \mathbb{R}^d$ with $\|y\|_2 < \frac{\gamma}{\sqrt{k}}$ is not in $\Phi(B_1^N)$ so using the fact that $\Phi(B_1^N)$ is a convex set we apply Hahn-Banach theorem to get $\xi \in \mathbb{R}^d$ with $\|\xi\|_2 = 1$ such that

$$\sup_j |\langle \xi, \phi_j \rangle| < \langle \xi, y \rangle \leq \|\xi\|_2 \|y\|_2 < \frac{\gamma}{\sqrt{k}}.$$

This contradicts our assumption that $\mathbf{d}_{d-1} \geq \frac{\gamma}{\sqrt{k}}$. One easily checks from (8) that the inclusion $\Phi(B_1^N) \supset \frac{\gamma}{\sqrt{k}} B_2^d$ implies the second claim. □

References

[1] E.Candes, *People Hearing Without Listening: An introduction to compressive sampling*, IEEE Signal Processing Magazine March (2008) 21-30.

[2] E.Candes, *The restricted isometry property and its implications for compressed sensing* C.R. Acad. Sci. Paris Ser. I 346(2008) 589-592

[3] E.Candes, T.Tao, *Decoding by linear programming*, IEEE Trans. Inform. Theory 51(2005) 4203-4215

[4] E.Candes, J.Romberg, T.Tao, *Stable signal recovery from incomplete and inacurate measurements*, Comm. Pure Appl. Math., 59(8),(2006) 1207-1223

[5] A.Cohen, W.Dahmen, R. DeVore, *Compressed sensing and best k-term approximation*, J. Amer. Math. Soc. 22 (2009) 211-245

[6] A.Cohen, W.Dahmen, R. DeVore,*Instance optimal decoding by tresholding in compressed sensing* in Harmonic analysis and partial differential equations, 128, Contemp. Math., 505, Amer. Math. Soc., Providence, RI, 2010.

[7] M.A.Davenport, M.F.Duarte,Y.C.Eldar,G.Kutyniok, *Introduction to Compressed Sensing* in **Compressed Sensing: Theory and Applications**, Cambridge University Press, (to appear)

[8] R.DeVore, G.Petrova, P.Wojtaszczyk, *Instance optimality in probability with ℓ_1-minimization decoder*, Appl. Comput. Harmon. Anal. 27(2009) 275-288

[9] D.Donoho, *Compressed sensing*, IEEE Trans. Inf. Theory 52(2006) 1289-1306

[10] G.G.Lorentz, M.Golitschek, Y.Makovoz,**Constructive Approximation, Advanced Problems**, Springer Verlag 1996

[11] D.Needell, J. A.Tropp *CoSaMP: Iterative signal recovery from incomplete and inaccurate samples*, Appl. Comput. Harmon. Anal. 26(2008) 301-321

[12] P. Wojtaszczyk *Stability and instance optimality for Gaussian measurements in compressed sensing* Found. Comput. Math. 10(2010) 1-13

[13] P.Wojtaszczyk, *Stability of ℓ_1-minimization in compressed sensing*, Proc. SPARS09, St. Malo `http:/hal.inria.fr/view_by _stamp.php?label=SPARS09&action_todo=home&langue=en`