

Exam: Stochastic Simulation

MIM UW, June 2019.

1. 2-dimensional random variable (X, Y) has the density given by

$$f(x, y) = \frac{1}{z} \exp \left[-\frac{y(x^2 + 1)}{2} \right],$$

for $y > 0$, $-\infty < x < \infty$, where z is a normalizing constant.

- Identify full conditional distributions $f(y|x)$, $f(x|y)$. (*Warning:* there is no need to compute anything! Just look.)
- Describe how to construct a Gibbs Sampler targeting the joint distribution $f(x, y)$. Assume that you can use built-in functions in R to generate random variables from standard distributions.
- Compute the marginal distributions $f_X(x)$, $f_Y(y)$.
- Describe how to use the method of conditional distributions to generate (X, Y) *exactly* from the joint distribution $f(x, y)$. Assume that you can use built-in functions in R to generate from standard distributions.

2. Random variables X_1, \dots, X_n are independent and have the identical distributions with density

$$p(x) = \begin{cases} x^{-2} & \text{for } x \geq 1; \\ 0 & \text{for } x < 1. \end{cases}$$

Consider the Monte Carlo estimator

$$\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^n X_i^{-1}.$$

- Give $\theta = \lim_{n \rightarrow \infty} \hat{\theta}_n$.
- Compute $\text{Var} \hat{\theta}_n$.
- Give n such that $\mathbb{P}(|\hat{\theta}_n - \theta| \leq 0.01) \approx 0.95$. Use the normal approximation based on the Central Limit Theorem. You can assume that 0.975-quantile of standard normal distribution is ≈ 2 .

3. Markov chain on the state space $\mathcal{X} = (0, \infty)$ is defined as follows:

- Initial state is $X_0 = 1$;
- If $X_n = X$ then X_{n+1} is sampled as follows:
Gen $X^* \sim \text{Ex}(1)$;
Gen $U \sim \text{U}(0, 1)$;
if $U < \frac{X^*}{X}$ then $X_{n+1} := X^*$ else $X_{n+1} := X$.

(Ex(1) denotes the exponential distribution with density $q(x) = e^{-x}$ for $x > 0$).

- Find the stationary distribution π of the chain $X_0, X_1, \dots, X_n, \dots$
- Compute $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} X_i$.

Hint: The chain X_n is a special case of Independent Metropolis-Hastings algorithm, in which the probability distribution of „proposal“ X^* does not depend on X , i.e. $X^* \sim q(\cdot)$ for some q . The acceptance probability is then $a(X, X^*) = \frac{\pi(X^*)q(X)}{\pi(X)q(X^*)} \wedge 1$.

4. Assume that p is a probability density on a space \mathcal{X} (equipped with a fixed reference measure denoted dx) and $f : \mathcal{X} \rightarrow \mathbb{R}$ is a bounded function. Let q be another (instrumental) probability density on \mathcal{X} . Assume that $p(x) > 0$ and $q(x) > 0$ for all $x \in \mathcal{X}$. Consider the following algorithm (written in a “pseudo-code”):

```

Gen  $X \sim q$ ;
if  $p(X) > q(X)$  then
     $W := p(X)/q(X)$ ;
    else
    Gen  $U \sim U(0, 1)$ ;
    if  $U < p(X)/q(X)$  then  $W := 1$  else  $W := 0$ ;
return( $X, W$ ).

```

- Derive a formula for $\mathbb{E}(f(X)W)$.
- Derive a formula for $\mathbb{P}(W = 0)$.

Hint: Consider separately the integrals over $A = \{x : p(x) > q(x)\}$ and over $\mathcal{X} \setminus A = \{x : p(x) \leq q(x)\}$. (On A we have *importance sampling*; on $\mathcal{X} \setminus A$ we have something like *rejection sampling*.)

5. Consider the following R code.

```

n <- 10^6
U <- runif(n)
Mu <- ifelse(U<0.5,-1,1)
X <- rnorm(n,mean=Mu,sd=1)
m <- mean(X); s2 <- var(X)

```

- What is the probability distribution of (components of vector) X ? (You may e.g. write the density.)
- Guess the value of m .
- Guess the value of $s2$.

(You should guess with error < 0.001 .)

6. Random variable X is generated by the following algorithm:

```

Gen  $W_1 \sim \text{Ex}(1)$ ;
repeat Gen  $W_2 \sim \text{Ex}(1)$  until  $W_2 > W_1$ ;
 $X := W_2$ ;
return( $X$ ).

```

(Symbol $\text{Ex}(1)$ denotes the exponential distribution with parameter $\lambda = 1$.)

- Compute $f(x|w_1)$.
- Compute the (unconditional) probability distribution of X .
- Compute $\mathbb{E}X$.
- Compute the expected number of times W_2 is generated until acceptance. (*Warning:* this expectation turns out to be rather *unexpected!*)