Star exercises - series II
Languages, automata and computations II

Deadline: February 4th, 2020
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1. Universality for deterministic VASS
Consider a labelled VASS, where every transition is labelled by some letter from a finite alphabet Σ, no ε-labelled transitions are allowed. Such a VASS together with initial configuration $(p, v) \in Q \times \mathbb{N}^d$ and a final set of states $F \subseteq Q$ defines a *VASS language* in a natural way. A VASS is *deterministic* if for any state $q \in Q$ and letter $a \in \Sigma$ there exists at most one transition outgoing from q and a-labelled. Provide a deterministic polynomial time algorithm, which for a language L of a deterministic VASS decides whether L is universal, i.e. whether $L = \Sigma^*$.

2. Short run in $\mathbb{N} \times \mathbb{Z}$-VAS
Consider a system called $\mathbb{N} \times \mathbb{Z}$-VAS defined by a set of transitions $T \subseteq \mathbb{Z}^2$. Configuration in this system is a vector $v \in \mathbb{N} \times \mathbb{Z}$, i.e. first coordinate has to be nonnegative, but second not necessarily. Transitions between configurations are defined as usual in VASes. Let M be the maximal absolute value of a number occurring in a transition. Show that if there is a path from $(0, 0)$ to $(0, 1)$ then there exists such a path of length at most polynomial in M.
Hint: Use the following Steinitz lemma. Let vectors $v_1, \ldots, v_n \in \mathbb{R}^d$ sum up to zero vector and all the coordinates of all v_i have absolute value at most 1 (i.e. their infinity norm is at most 1). Then there exists a permutation $v'_1, \ldots, v'_n$ of the above vectors such that every point reachable by a path starting in zero vector and firing consecutively vectors $v'_1, \ldots, v'_n$ has the infinite norm bounded by d.

3. Marble automata
A deterministic two-way automaton with k-marbles is a model that can mark some positions with marbles as it moves around the input word. All marbles are different, so it knows if it sees the 1st or the 5th marble. The marbles in the word have to satisfy the *interval condition* – the set of marbles that is currently "in use" has to be an interval ($[1..i]$ for some i), or be empty.
Formally a two-way automaton with k-marbles consists of:

- a finite input alphabet Σ
- a finite set of states Q,
- a transition function

$$f : (\Sigma + \{\vdash, \dashv\}) \times Q \times \overbrace{P(\{1, 2, \ldots k\})}^{\text{currently seen marbles}} \to \text{Actions} \times Q$$

  where Actions is the following set:

  – accept/reject

- go left/right
- put the *smallest* unused marble in the current position
- collect marbles $[i, i+1, \dots k]$ from the word (for any i).

The automaton runs on the word $\vdash w \dashv$, for $w \in \Sigma^*$. If the automaton loops, or exits the word the automaton rejects it. Note that the automaton doesn't have to see the marble in order to collect it.
Prove that for every k deterministic two-way automata with k marbles recognize exactly regular languages.