# Star exercises - series I
# Languages, automata and computations II

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**1. Boundedly repeating words**

We say that an $\omega$-word $w \in \Sigma^\omega$ is *boundedly repeating* if for every finite infix $v \in \Sigma^*$ that occurs infinitely many times in $w$ there exists a bound $B$ such that infix $v$ occurs in every infix of $w$ of length $B$. In other words for every $i$ word $v$ is an infix of $w[i]w[i+1]\cdots w[i+B-1]$. Every ultimately periodic word is boundedly repeating, but there exists boundedly repeating words, which are not ultimately periodic. A notable example is the Thue-Morse word. A *cross product* of two $\omega$-words $u \in \Sigma^\omega$, $v \in \Gamma^\omega$ is a word $w \times v \in (\Sigma \times \Gamma)^\omega$ defined as $(u \times v)[i] = (u[i], v[i])$. Decide whether cross product of two boundedly repeating words is also a boundedly repeating word.

**2. Characterization of $\omega$-regular languages**

For any $\omega$-language $L \subseteq \Sigma^\omega$ we say that relation $\sim \subseteq \Sigma^* \times \Sigma^*$ is *L-compatible* if both following conditions hold:

1. For all infinite sequences $u_i, v_i$ of finite words such that $u_i \sim v_i$ for all $i \in \mathbb{N}$ it holds $u_1 u_2 \cdots \in L \iff v_1 v_2 \cdots \in L$.
2. For all $u_1, u_2, v_1, v_2 \in \Sigma^*$ it holds: if $u_1 \sim v_1$ and $u_2 \sim v_2$ then $u_1 u_2 \sim v_1 v_2$.

Show that language $L$ is $\omega$-regular if and only if there exists an $L$-compatible relation with finite index.

**Hint:** It can be useful to use Infinite Ramsey Theorem: in every infinite clique with edges colored on finite number of colors there exists a monochromatic infinite clique.

**3. Fixed ambiguous automata**

A finite automaton $\mathcal{A}$ is *k-ambiguous* if for every word accepted by $\mathcal{A}$ there is exactly $k$ accepting runs of $\mathcal{A}$ on that word. Decide whether there is a polynomial algorithm which decides universality, i.e. answers whether a given $k$-ambiguous automaton $\mathcal{A}$ fulfills $L(\mathcal{A}) = \Sigma^*$.

**Remark:** An NP-hardness or coNP-hardness is treated as a solution, as this means that there is no polynomial time algorithm unless P = NP.

**4. Co-finiteness of UFA**

Decide whether there exists a polynomial time algorithm deciding whether language of a given unambiguous finite automaton $\mathcal{A}$ is co-finite, i.e. whether $\Sigma^* \setminus L(\mathcal{A})$ is finite.

**Remark:** An NP-hardness or coNP-hardness is treated as a solution, as this means that there is no polynomial time algorithm unless P = NP.

**5. Distance automata with more counters**

Consider the following extension of a distance automaton. Instead of having a set of costly transitions, we have two counters $\{1, 2\}$ and each transition is labelled by an instruction from the following toolkit:

- do nothing;
- increment counter 1;
- reset counter 1;
- reset counter 1 and increment counter 2;
- reset both counters.

The value of a run is the biggest value attained by any counter. Prove that limitedness is decidable for these automata, using the limitedness game.

**6. Separation**

Prove that the following problem is decidable:

- **Input:** Regular word languages $L, K \subseteq \Sigma^*$, given say by deterministic automata.
- **Question:** Is there a language of star height 1 which contains $L$ but is disjoint with $K$? A language of star height 1 is a language which can be defined by a regular expression, without complement, where the Kleene star is allowed, but it cannot be nested.

As a hint, use the previous exercise.