The Reachability Problem for Petri Nets is Not Elementary

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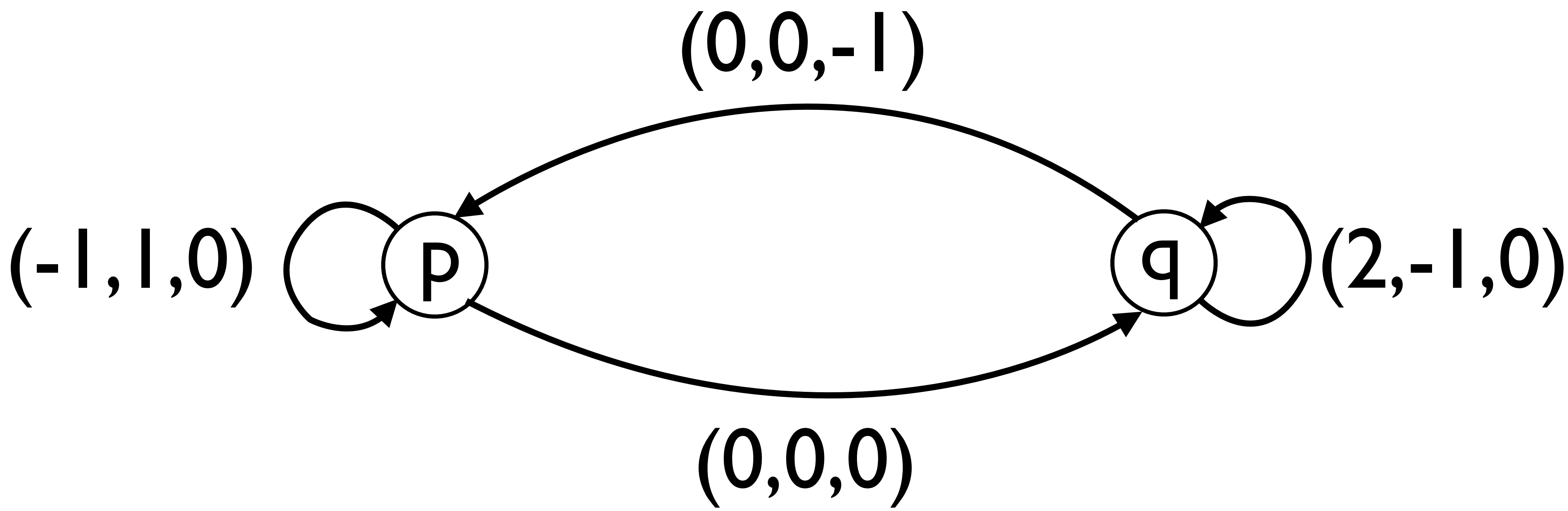
Vector Addition Systems with States

Nonnegative counters

No zero-tests



$p(1,0,4) \longrightarrow p(0,1,4) \longrightarrow q(0,1,4) \longrightarrow q(2,0,4) \longrightarrow p(2,0,3)$

Decision Problems

Given: a VASS, two configurations s and t

Reachability problem: is there a run from s to t?

Coverability problem: is there a run from s above t?

Short history - reachability

Lipton `76: ExpSpace-hardness

Mayr `81: decidability

Kosaraju `82, Lambert `92: simplifications

Leroux, Schmitz `15: cubic-Ackermann

Big complexity gap

Common feeling: possibly in ExpSpace

Main result

Theorem

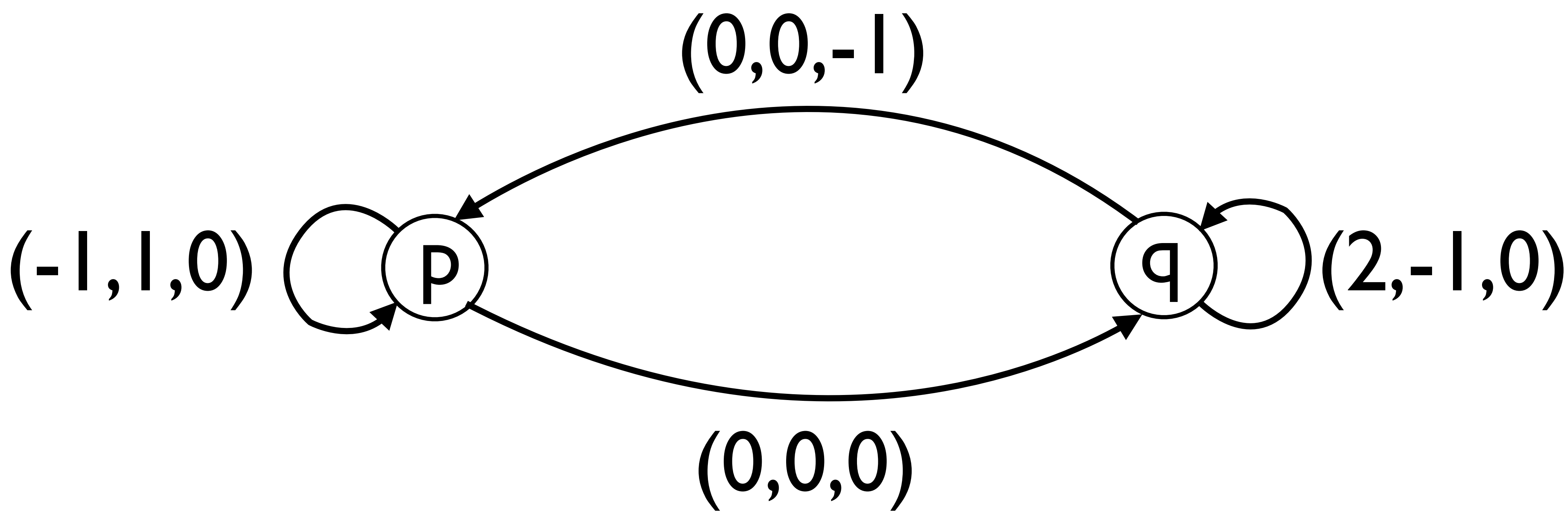
The Reachability Problem for Vector Addition Systems with States is not elementary.

Not in time: $2^{2^{2^{\cdot^{\cdot^{\cdot^{2^{n}}}}}}}$ ← fixed height

Plan

- classic VASS example
- nested VASS example
- big counters matter
- first trick: from k to k! once
- second trick: from k to k! many times

Hopcroft-Pansiot example



$$p(k,0,n) \longrightarrow p(0,k,n) \longrightarrow q(0,k,n) \longrightarrow q(2k,0,n) \longrightarrow p(2k,0,n-1)$$

$$p(1,0,n) \longrightarrow p(2,0,n-1) \longrightarrow \dots \longrightarrow p(2^n,0,0)$$

Nesting the example

$$p(1, 0, n, 1) \longrightarrow p(2^n, 0, 0, 1)$$

$$\downarrow$$

$$r(0,0,0, 2^{2^n}) \longleftarrow r(2^n, 0, 0, 1)$$

Size of finite reachability set can 2-exp, 3-exp, ...

Even up to Ackermann size is possible

But this VASS can reach also r(0,0,0,5)

Without zero test it is hard to enforce big values

Big counters

If one can manipulate precisely counters up to value K then reachability problem is K-space-complete

Lemma

For a counter automata with zero tests the problem if there exists a run reaching final configuration with all counters along the run bounded by 3!!…!! (n times) is tower-complete

Zero tests

How to implement zero tests on VASSes?

Idea:

two counters: x, y

initialize: $x = 0, y = K$

keep invariant: $x + y = K$

zero test x counter: y -= K, y += K

We need to be able to repeat things K times

Reaching K!

Assume we may manipulate counters up to K

$$2 / 1 \cdot 3 / 2 \cdot 4 / 3 \cdot \ldots \cdot k / k\text{-}1 = k$$

set x, y and z to M

for i in {1,..., k-1}

multiply weakly x by i+1 / i

multiply weakly z by i+1

check if $x = k \cdot y$ [$z = k! \cdot M$]

enforces strong computation!

Zero tests many times

How to assure strongness many times?

Having $(x, y, z) = (C \cdot K, C, K)$ we can do it C times

```
loop
    y--
    loop
        operation
        z--    ž++
        x--
    swap(z, ž)
test if x = 0
```

Lifting the construction

Lemma 1

There exists a VASS, which starting from $(C \cdot K, C, K)$ computes $(D \cdot K!, D, K)$ for some guessed D.

Lemma 2

There exists a VASS, which starting from $(C \cdot K, C, K)$ computes $(D \cdot K!!\ldots!!, D, K)$ for some guessed D.

we can zero test K!!…!!-bounded counters many times!

Thank you!