Challenges of the Reachability Problem in Infinite-State Systems

Wojciech Czerwiński

MFCS 2024

Plan

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- basic notions

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- when reachability is undecidable

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- when there is a hope

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- when there is a hope
- what is known

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- several examples and challenges

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- when reachability is undecidable
- when there is a hope
- what is known
- several examples and challenges
- goal: roadmap and inspiration

Computation model

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Turing machine = automaton with unbounded tape

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finite automaton

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pushdown automaton

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automaton with counters

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automaton with some structure

Reachability problem

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Given: a model, two its configurations s and t

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Question: is there a path from s to t?

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Reachability problem

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Why this problem?

Central one for a computation model

Halting problem for TM

Halting problem for TM

undecidable

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undecidable

the same as reachability problem

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what for other models?

Undecidable models

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Automaton with:

Undecidable models

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- two stacks (simulate tape)

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Undecidable models

Automaton with:

- two stacks (simulate tape)
- three zero-tested counters (simulate two stacks)
- two zero-tested counters (simulate three counters)
- Hilbert's Tenth Problem easily reduces to reachability in automata with zero-tested counters

Simulation

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stack can be simulated by two counters

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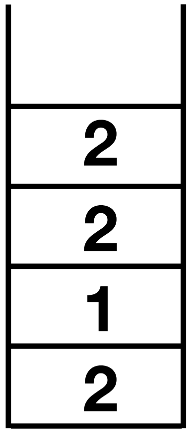

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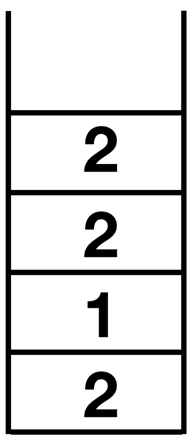

<2122> in ternary = 71

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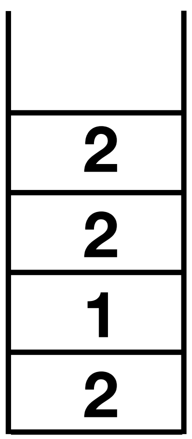

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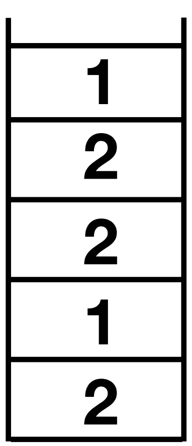

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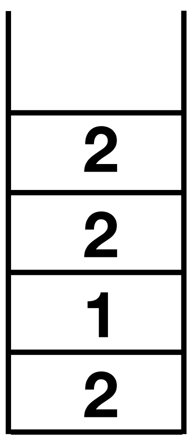

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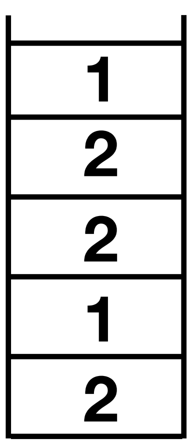

<21221> in ternary = $3 \cdot 71+1$

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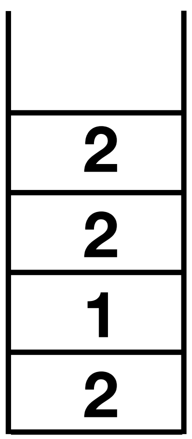

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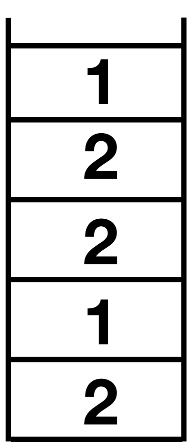
(71,0)



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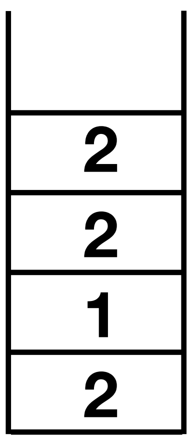

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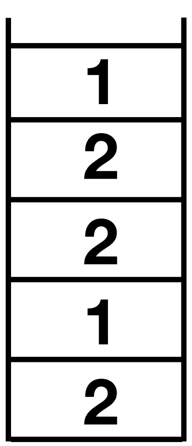

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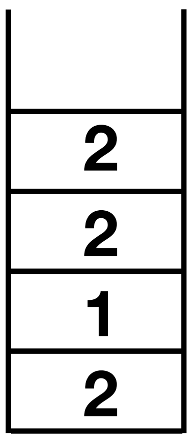

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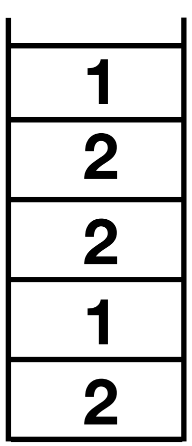

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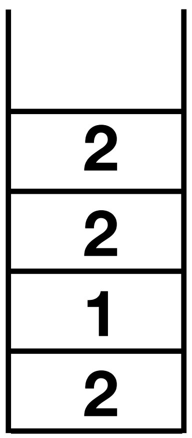

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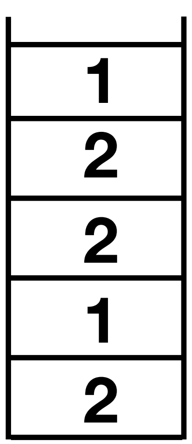

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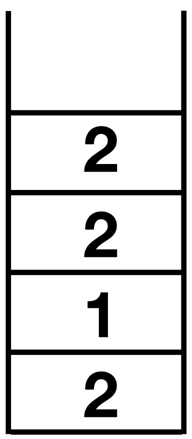

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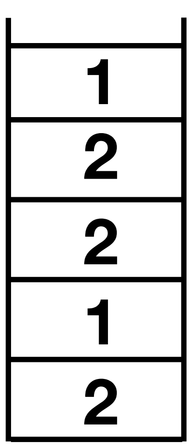

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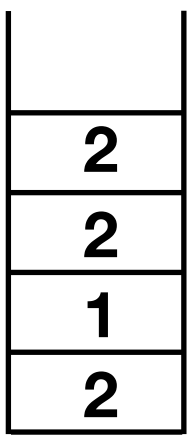

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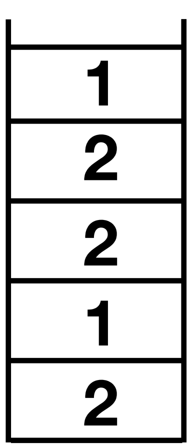

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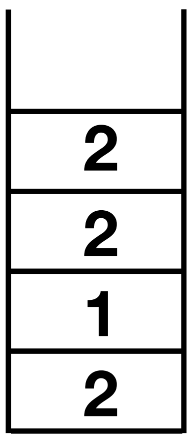

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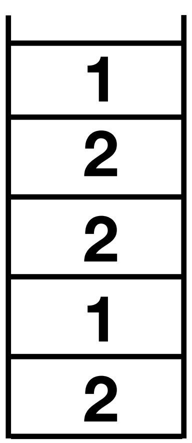

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Some hopeful restrictions

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- other structures

Vector Addition Systems with States (VASS)

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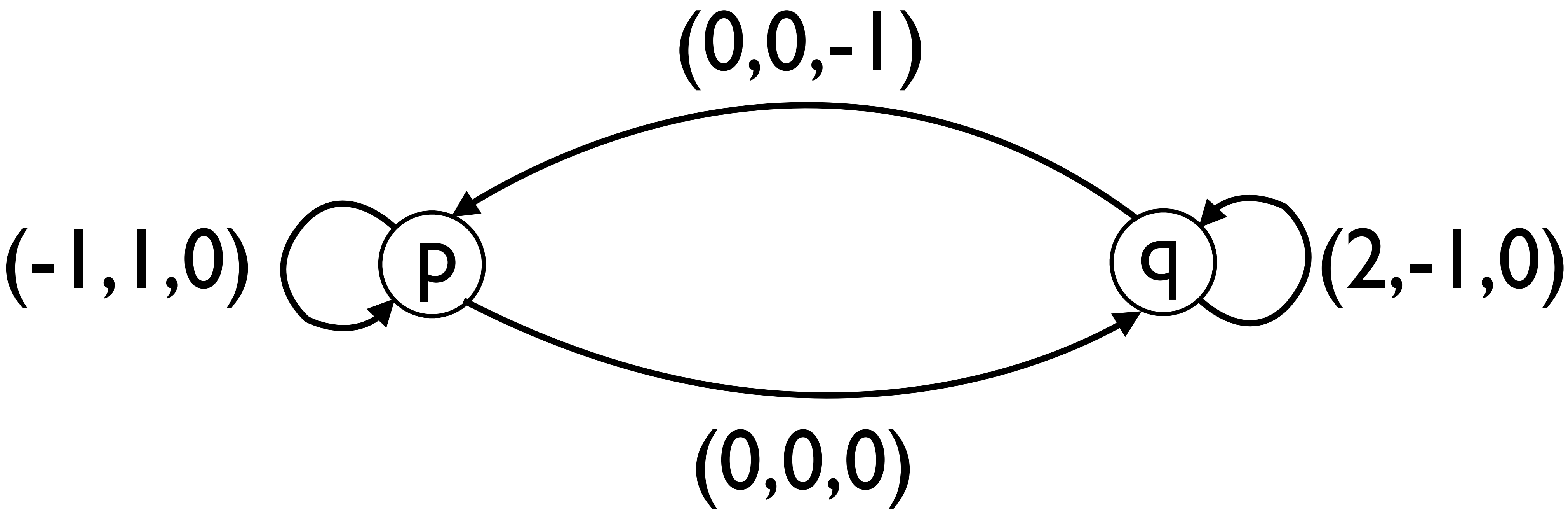

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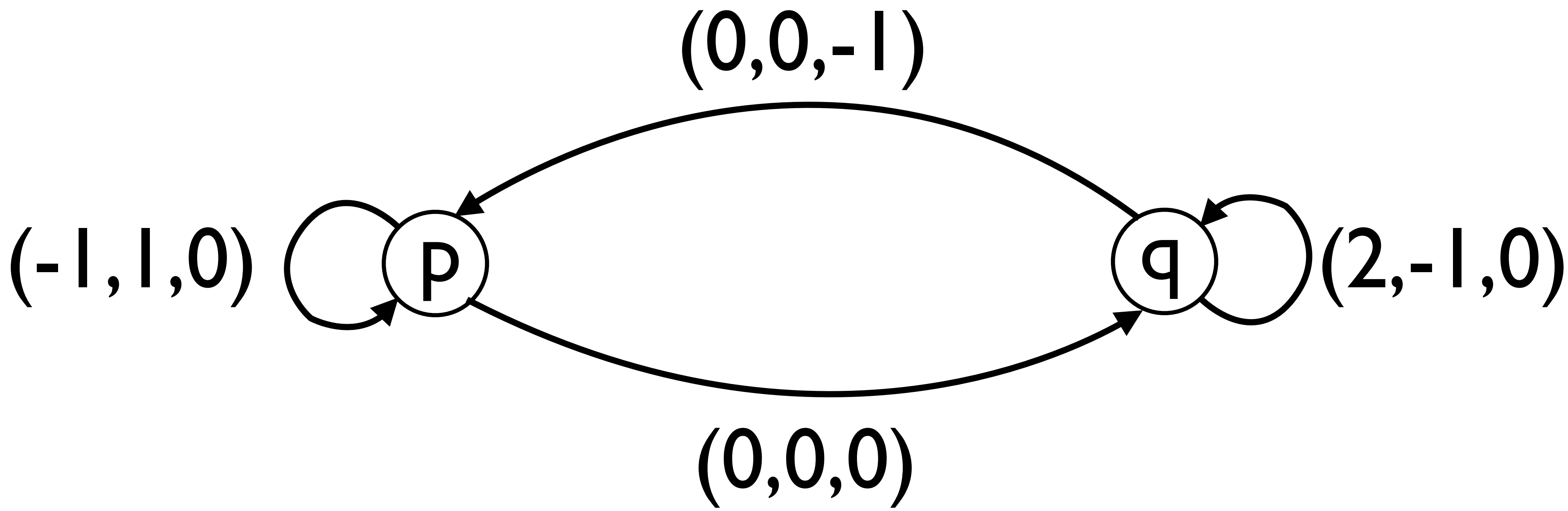
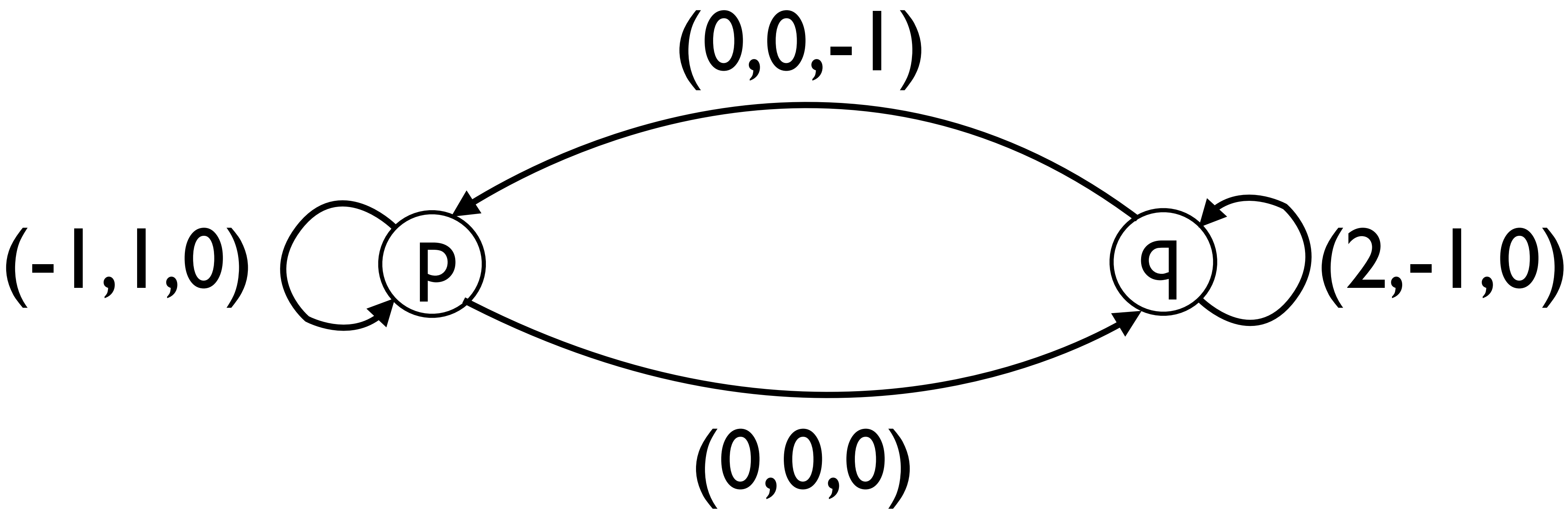


p(2,0,7)

Vector Addition Systems with States (VASS)



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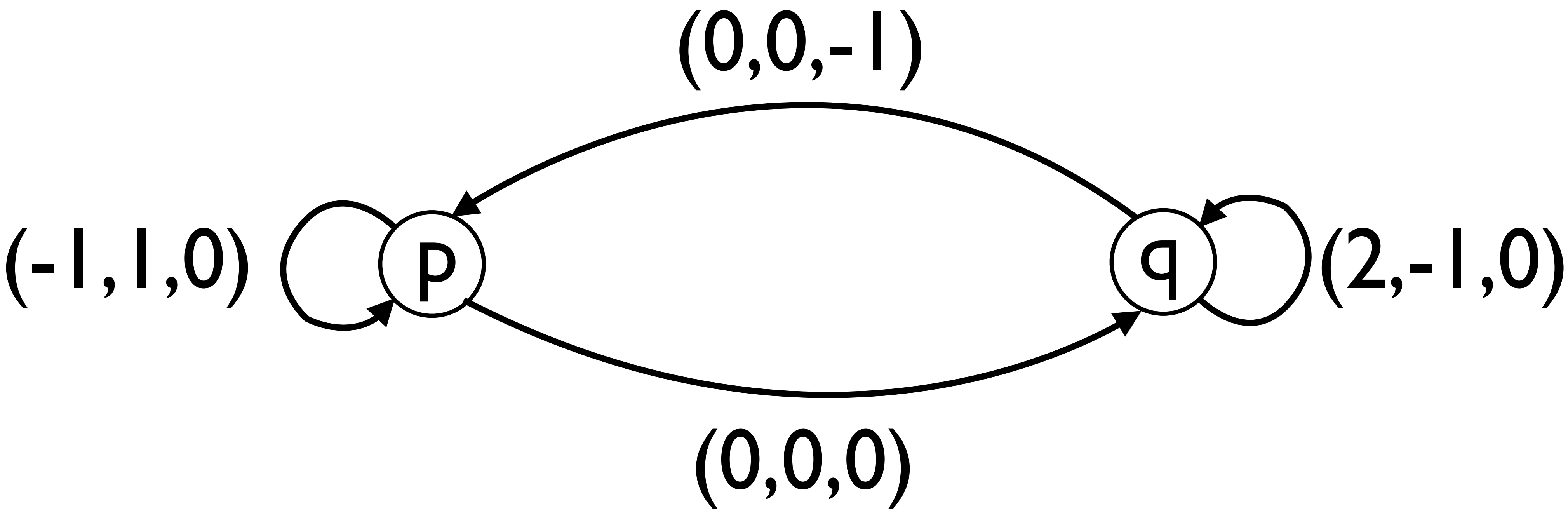
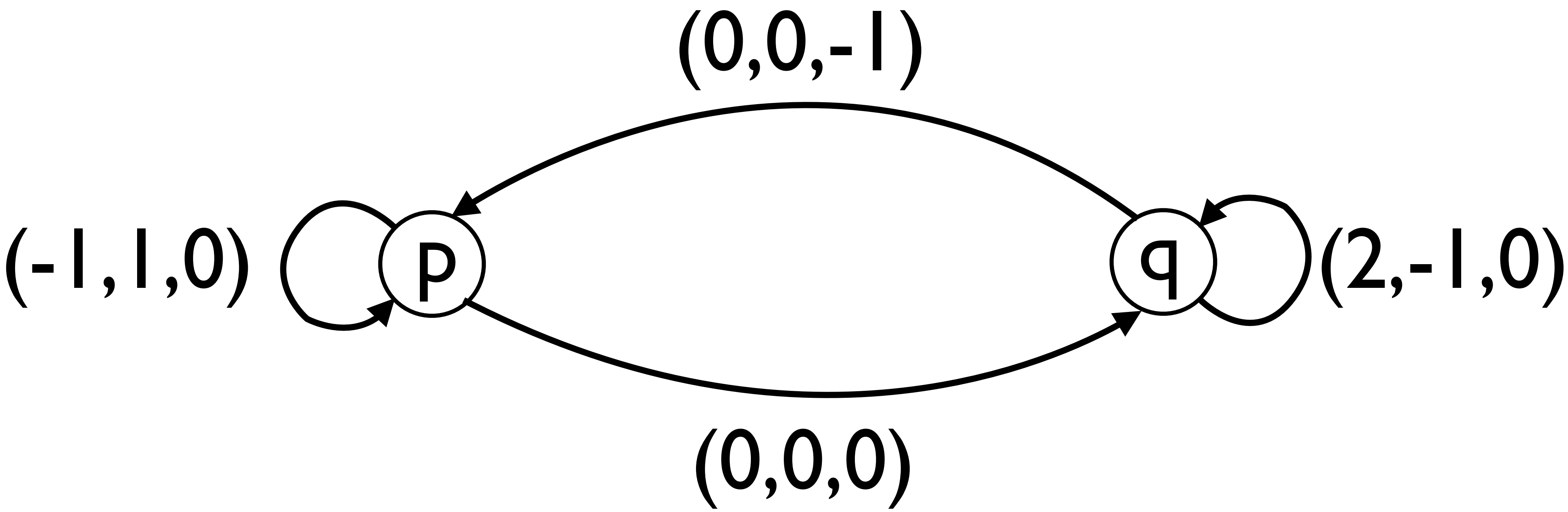


p(2,0,7) ⟶ p(1,1,7) ⟶ p(0,2,7)

Vector Addition Systems with States (VASS)



$$p(2,0,7) \longrightarrow p(1,1,7) \longrightarrow p(0,2,7) \longrightarrow q(0,2,7)$$

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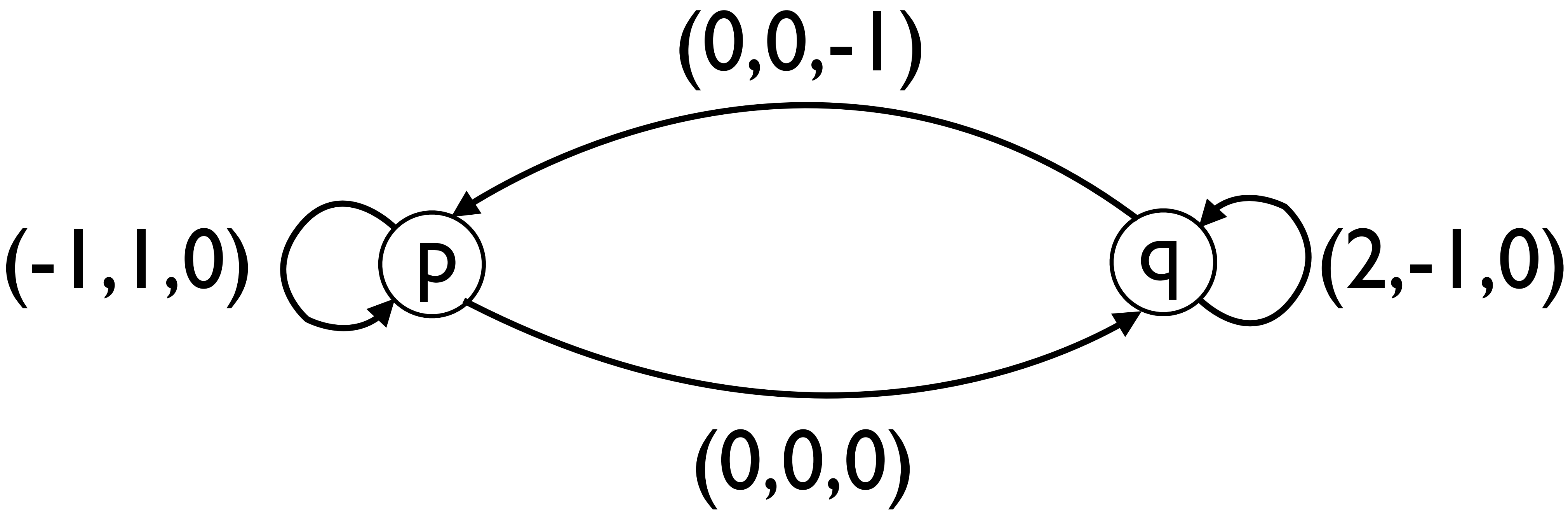

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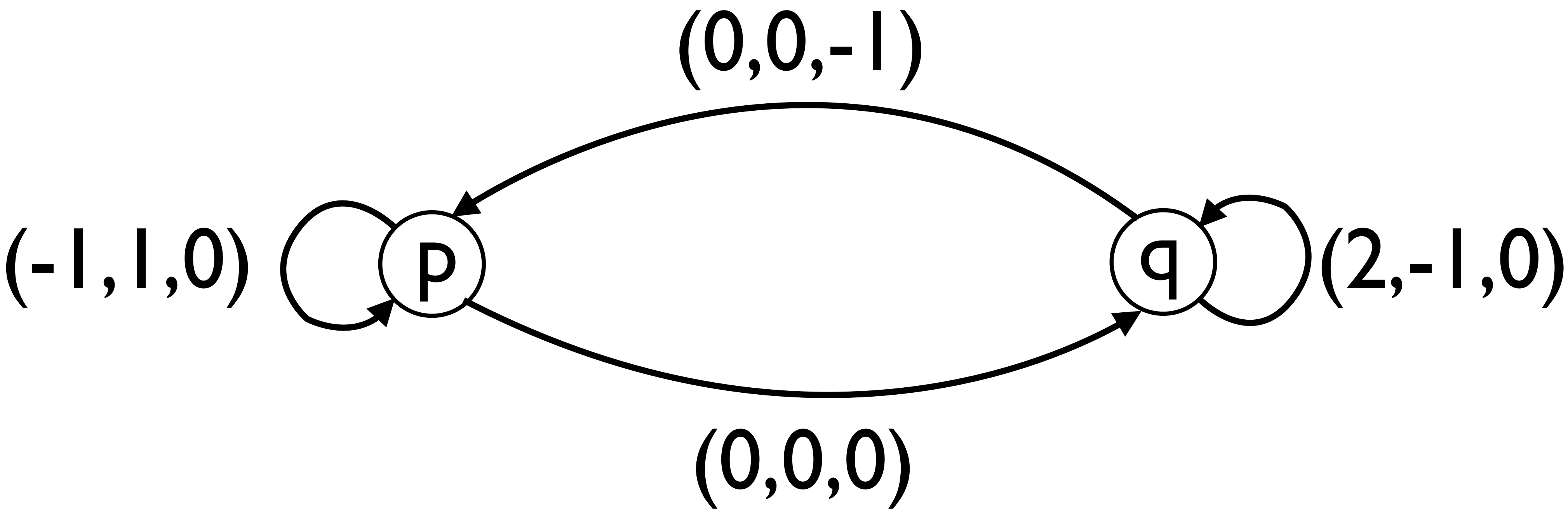

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$$\longrightarrow q(4,0,7)$$

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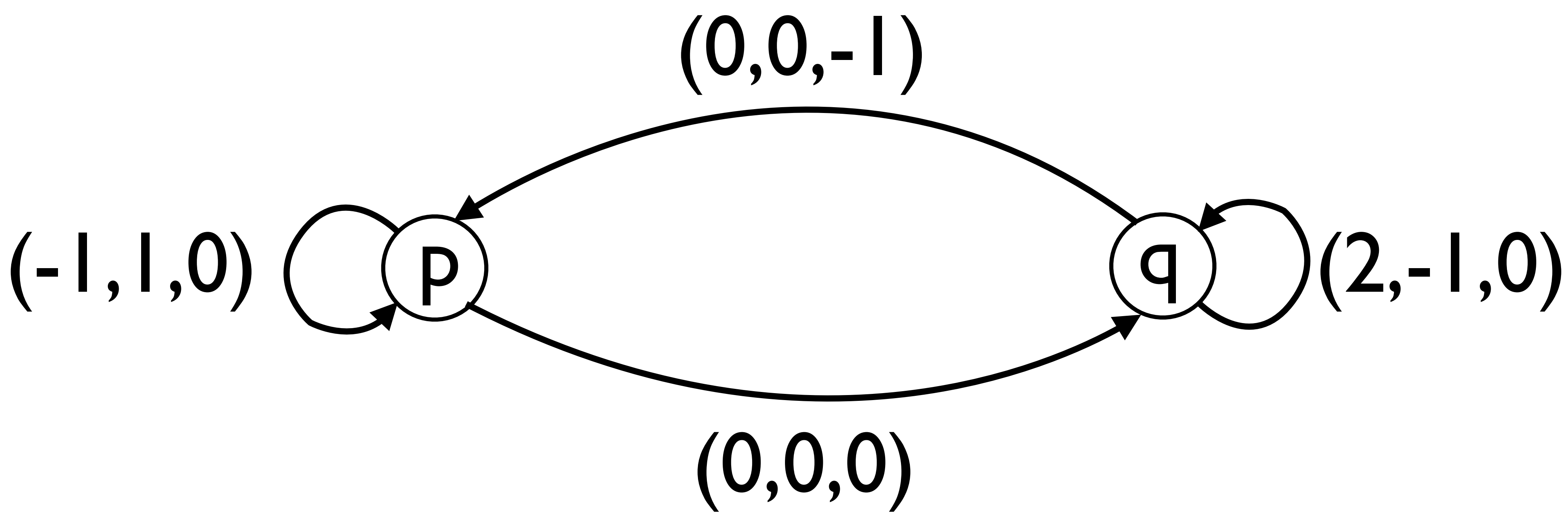
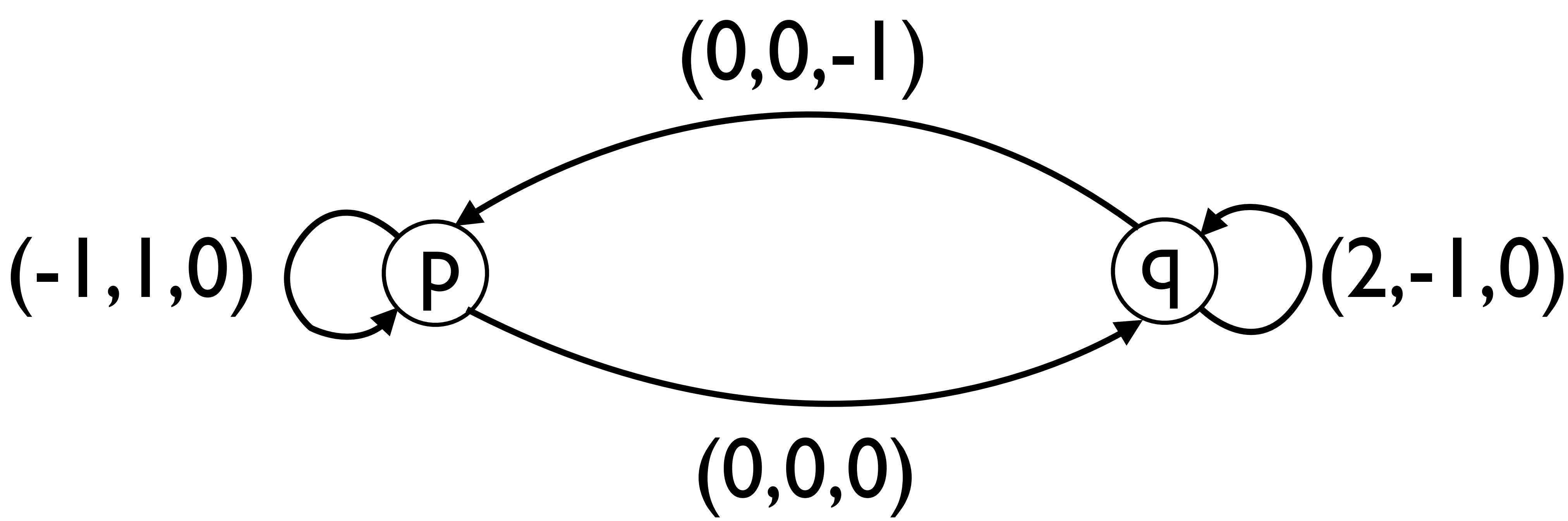


$$p(2,0,7) \longrightarrow p(1,1,7) \longrightarrow p(0,2,7) \longrightarrow q(0,2,7) \longrightarrow q(2,1,7)$$

$$\longrightarrow q(4,0,7) \longrightarrow p(4,0,6)$$

Vector Addition Systems with States (VASS)



$p(2,0,7) \longrightarrow p(1,1,7) \longrightarrow p(0,2,7) \longrightarrow q(0,2,7) \longrightarrow q(2,1,7)$

$\longrightarrow q(4,0,7) \longrightarrow p(4,0,6)$

Petri nets

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- Challenge 1: doubly-exponential path example in 3-VASS

Doubly-exp 4-VASS

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exponential example for unary 3-VASS

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$$(a_1 / b_1)^{2^1} \cdot (a_2 / b_2)^{2^2} \cdot \ldots \cdot (a_k / b_k)^{2^k} = a / b$$

Higher dimensions

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- 8-VASS: Tower-hard

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- message: path length important

Reachability for Pushdown VASS

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- Ackermann-size reachability set in 1-GVAS

Challenge

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Challenge 3: doubly-exponential example for 1-GVAS

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1-GVAS G and $s, t \in \mathbb{N}$ such that
minimal derivation from s to t in G is
doubly-exponential

Big reachability set

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$S \longrightarrow n\,X$

Big reachability set

$$S \longrightarrow n\ X$$

$$X \longrightarrow -1\ X\ 2 \mid 0$$

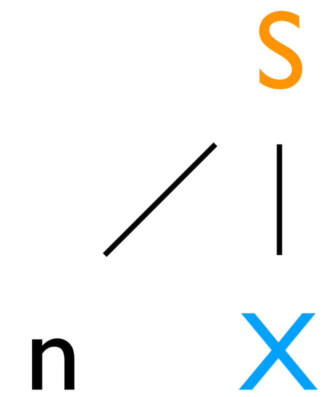
S

Big reachability set

S ⟶ n X

X ⟶ -1 X 2 | 0

Big reachability set



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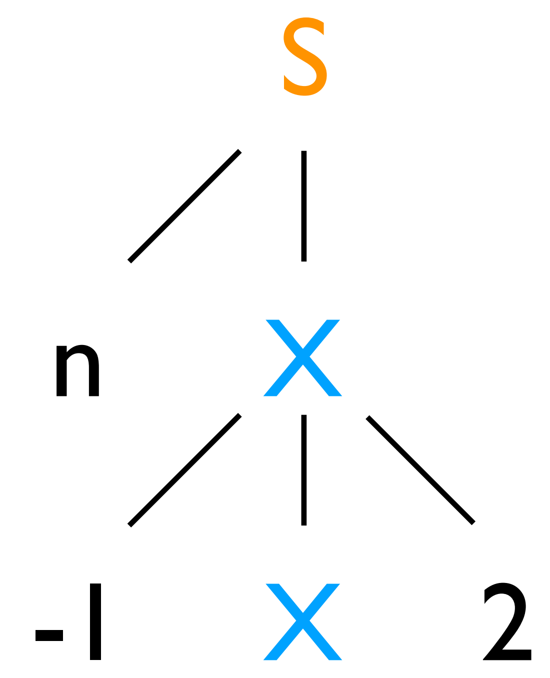

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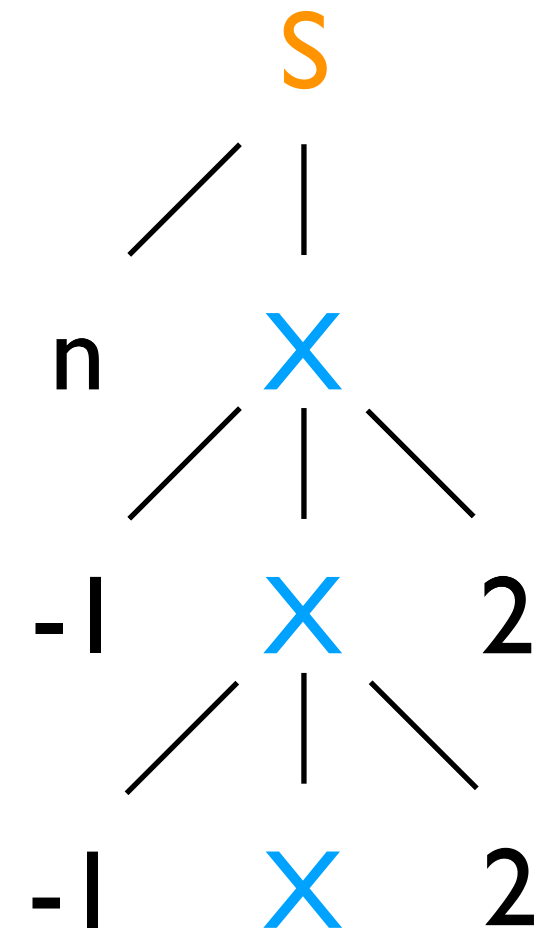
Big reachability set

S → n X

X → -1 X 2 | 0



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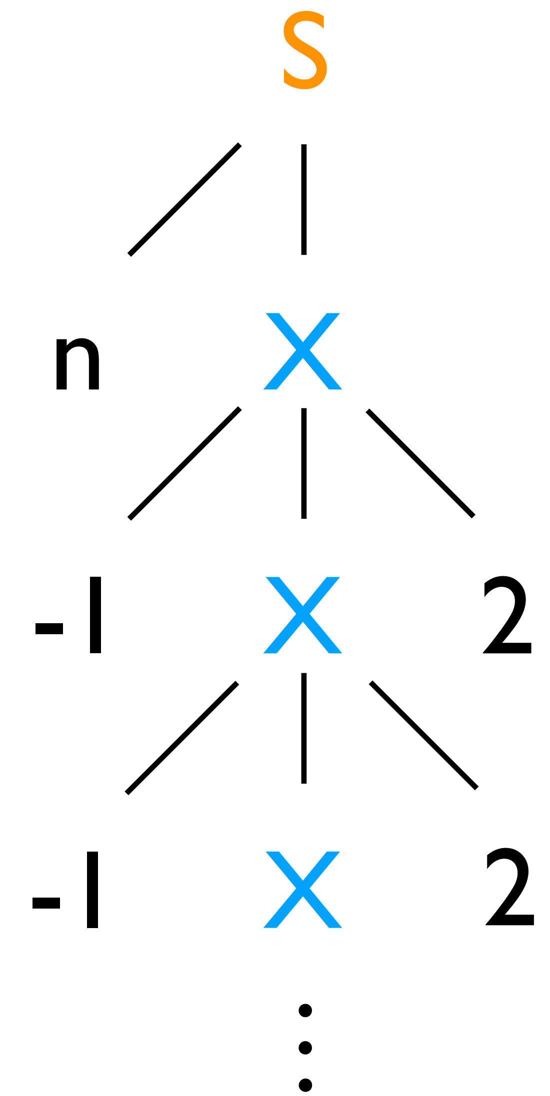

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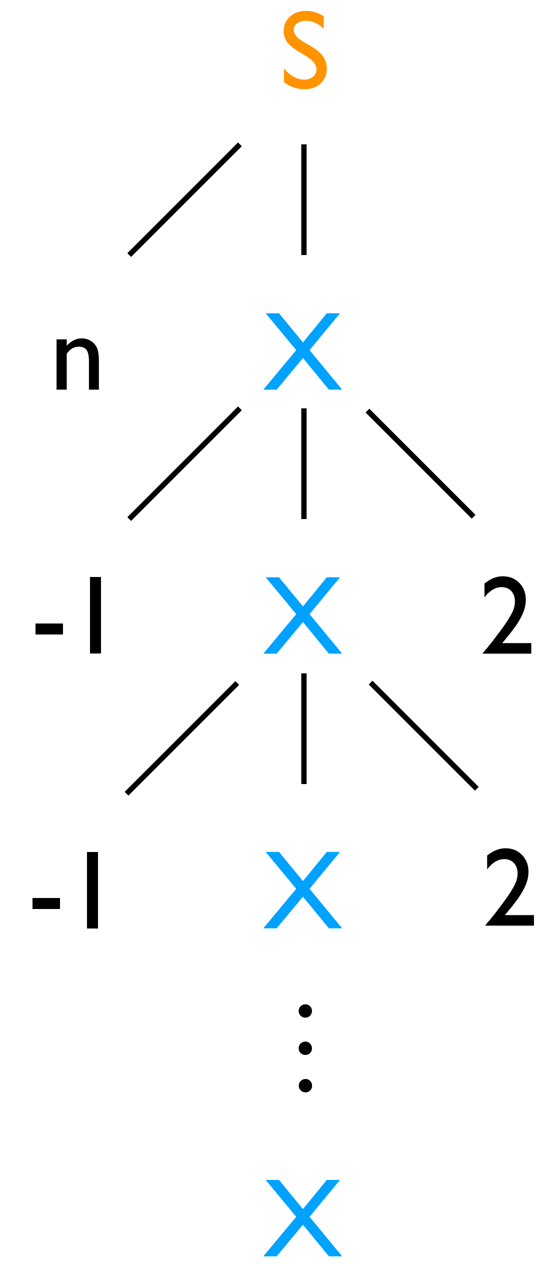
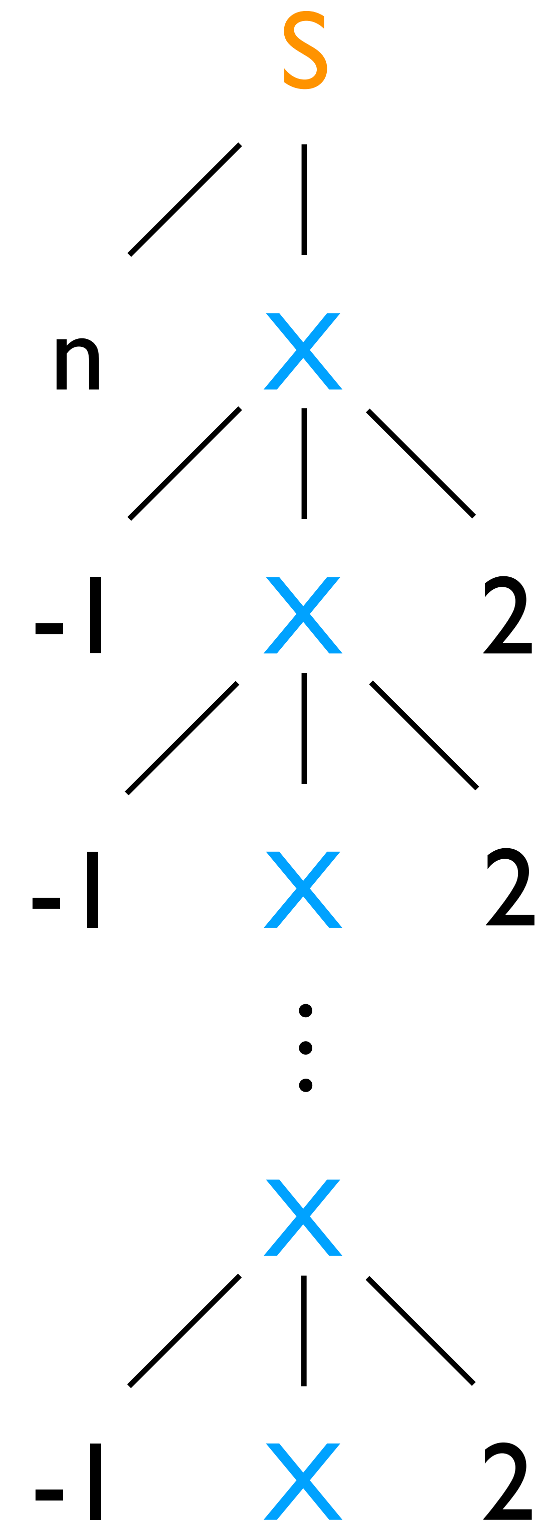


Big reachability set

S ⟶ n X

X ⟶ -1 X 2 | 0



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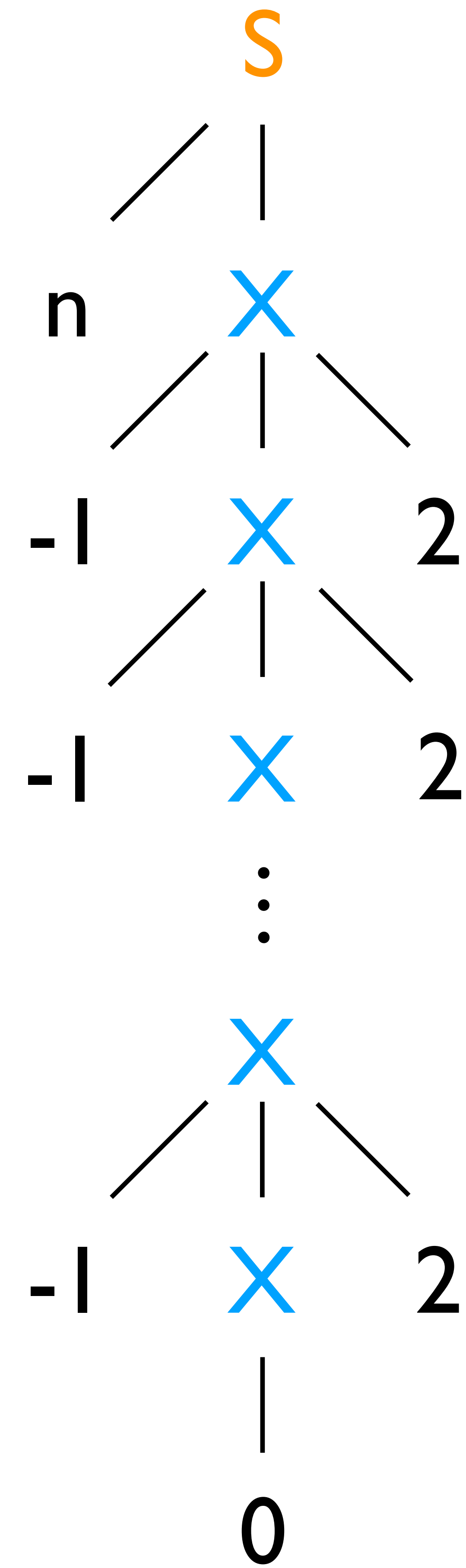

Big reachability set

$S \longrightarrow n\ X$

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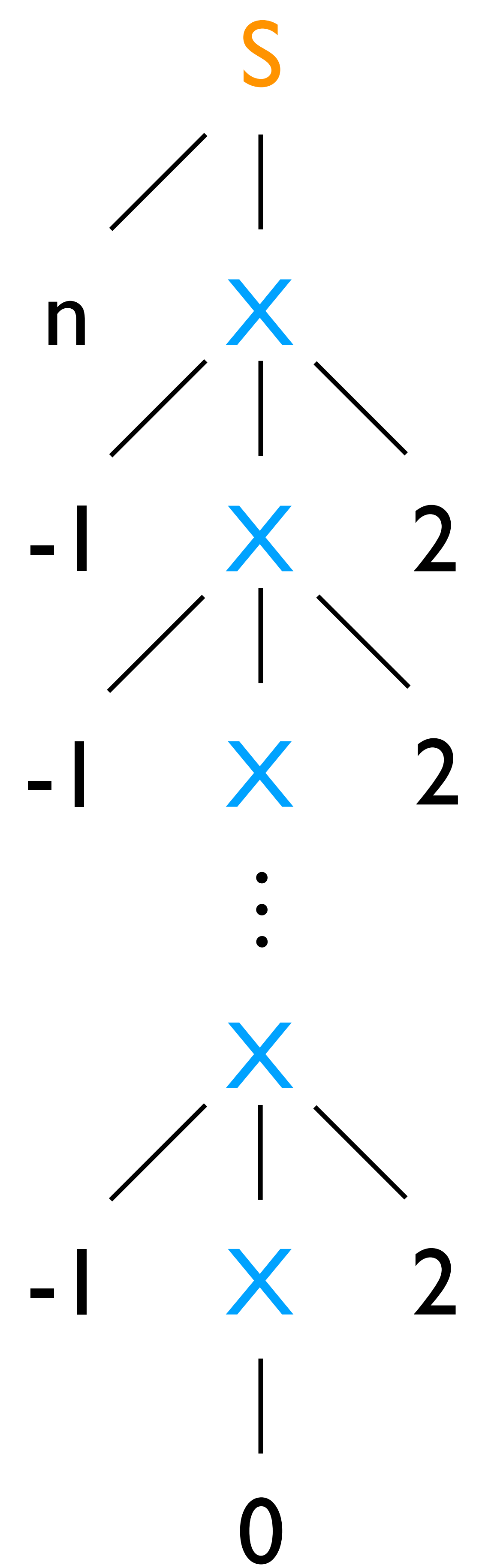
$k \xrightarrow{X} 2k$



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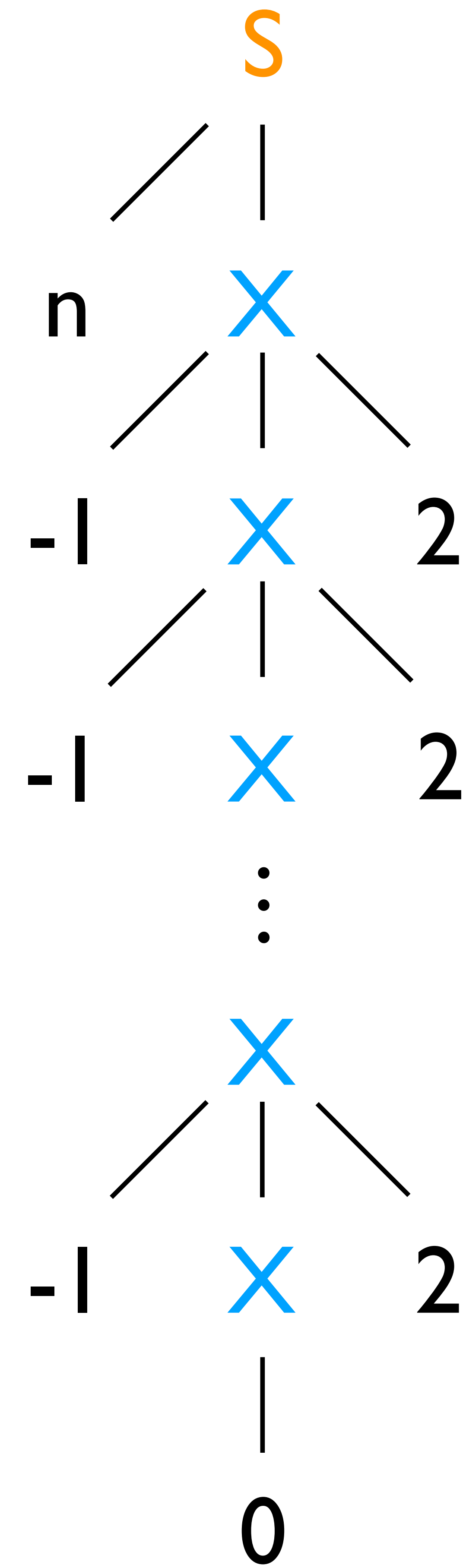
maximally



Big reachability set

Big reachability set

$S \longrightarrow nY$

Big reachability set

$$S \longrightarrow n\,Y$$

$$Y \longrightarrow -1\;Y\;X \mid 1$$

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Big reachability set

S ⟶ n Y

Y ⟶ -1 Y X | 1

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Y

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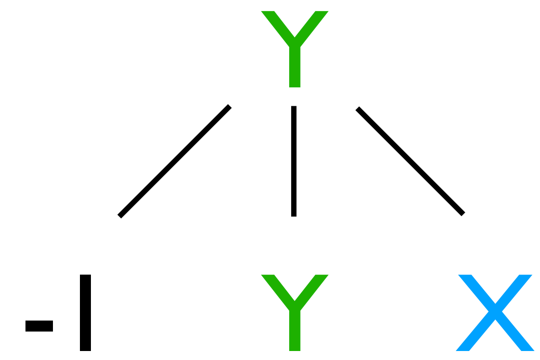

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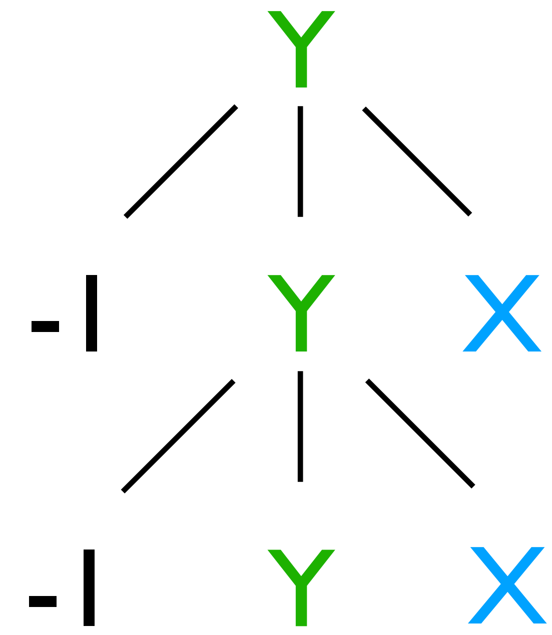

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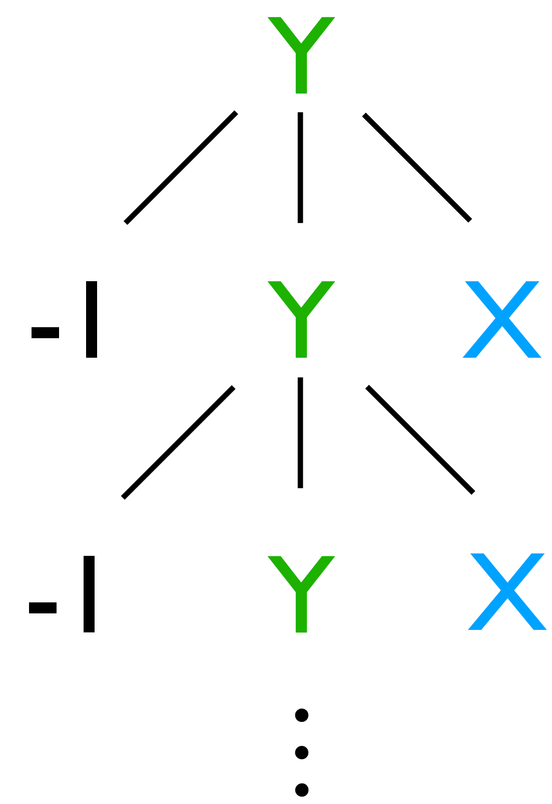

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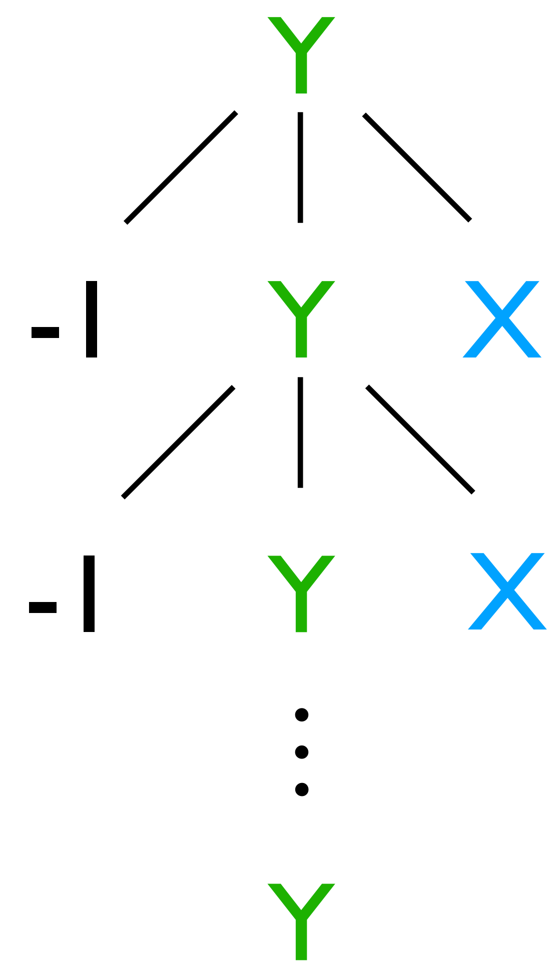

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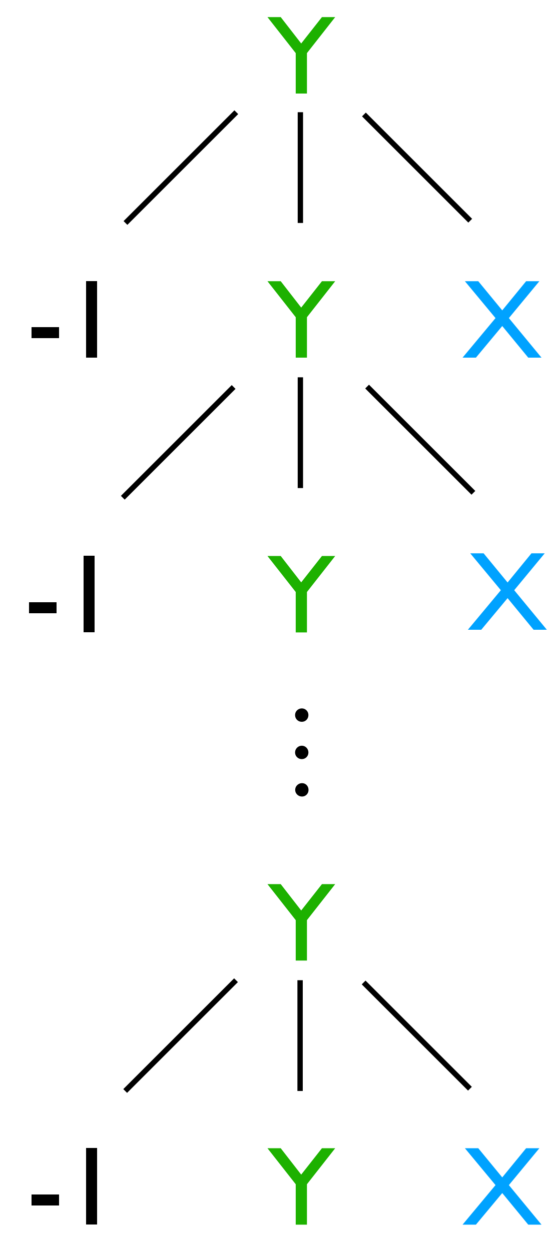

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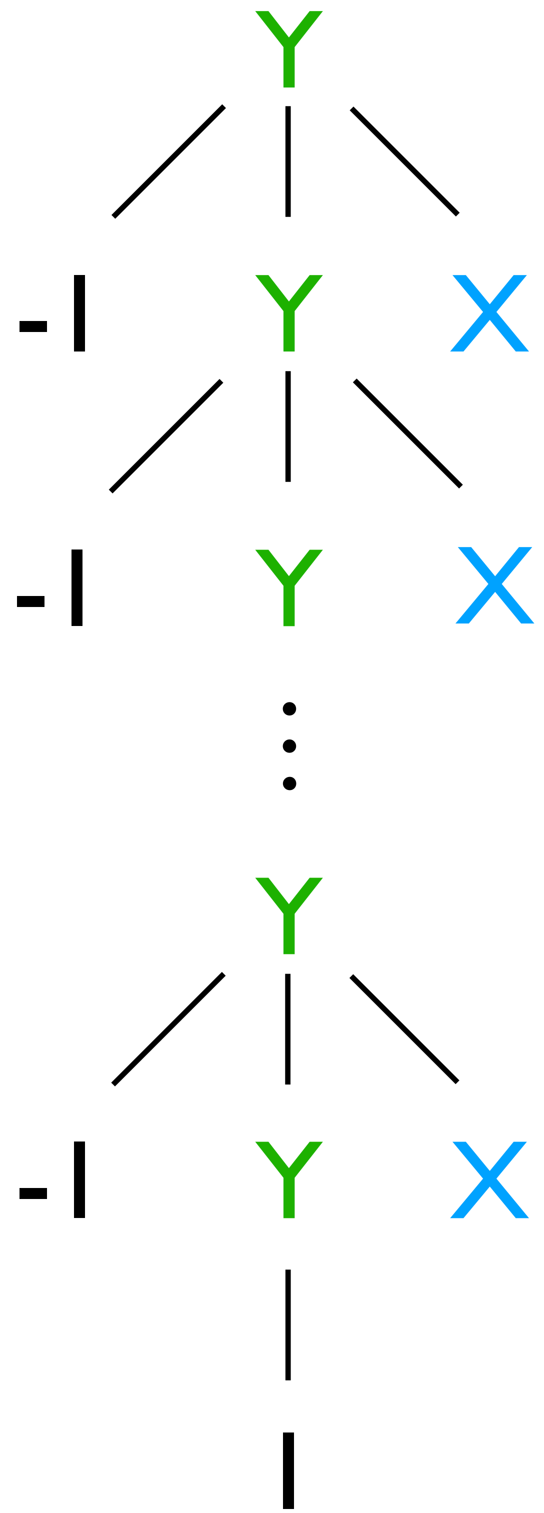

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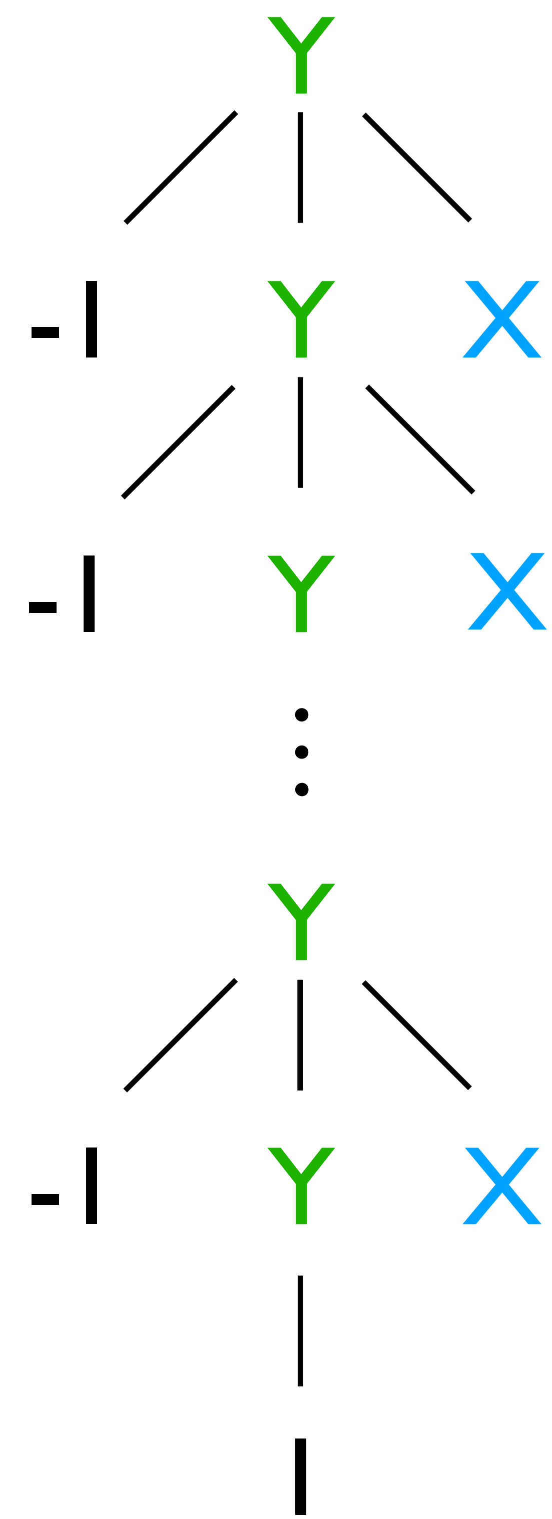

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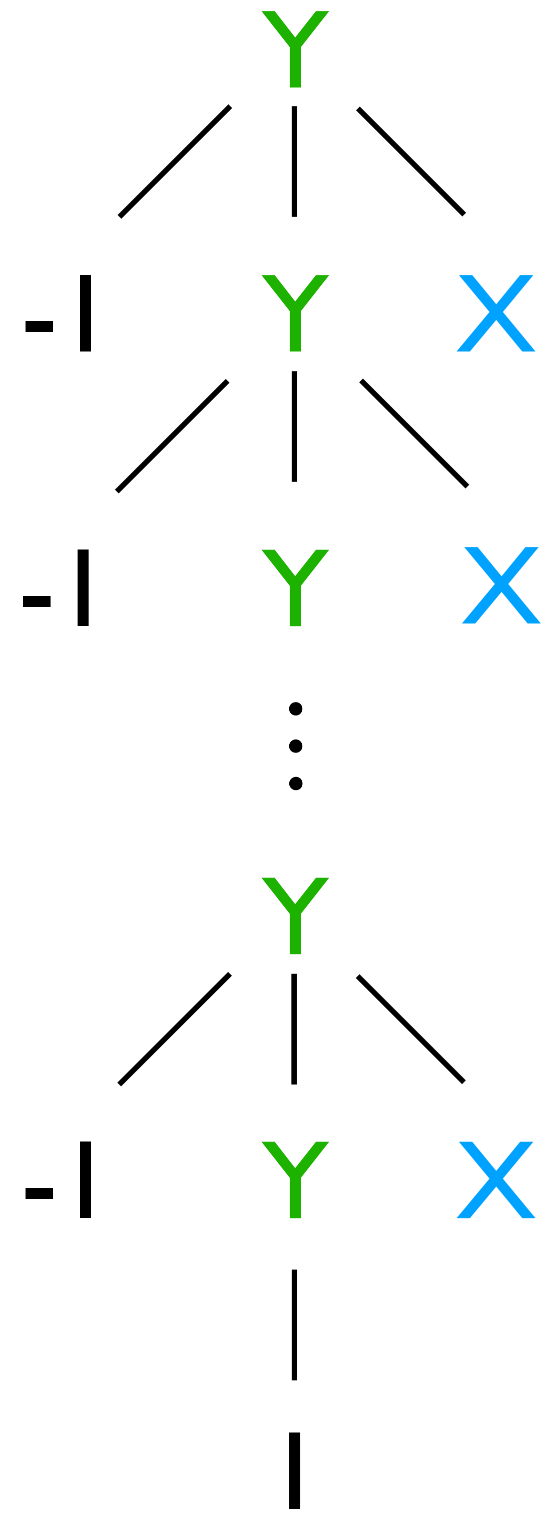

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$d+1$ nonterminals: reachability set of size $F_d(n)$

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- Challenge 4: find an example with 2-exp long path for 2-VASS + $\mathbb{Z}$-counters

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- possibly with zero-tests and parity checks
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- exponential path: multiply n times, then decrease
- no example with over-exponential path known
- Challenge 5: find an example with 2-exp long path
- important: find other interesting, decidable models

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- thank you!