Lower Bounds for the Reachability Problem in Fixed Dimensional VASSes

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Plan

Plan

- definitions

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- definitions
- short history

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- motivation

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- main result

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- techniques

Reachability problem

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Given: a Vector Addition System with States (VASS) V,
two configurations s and t

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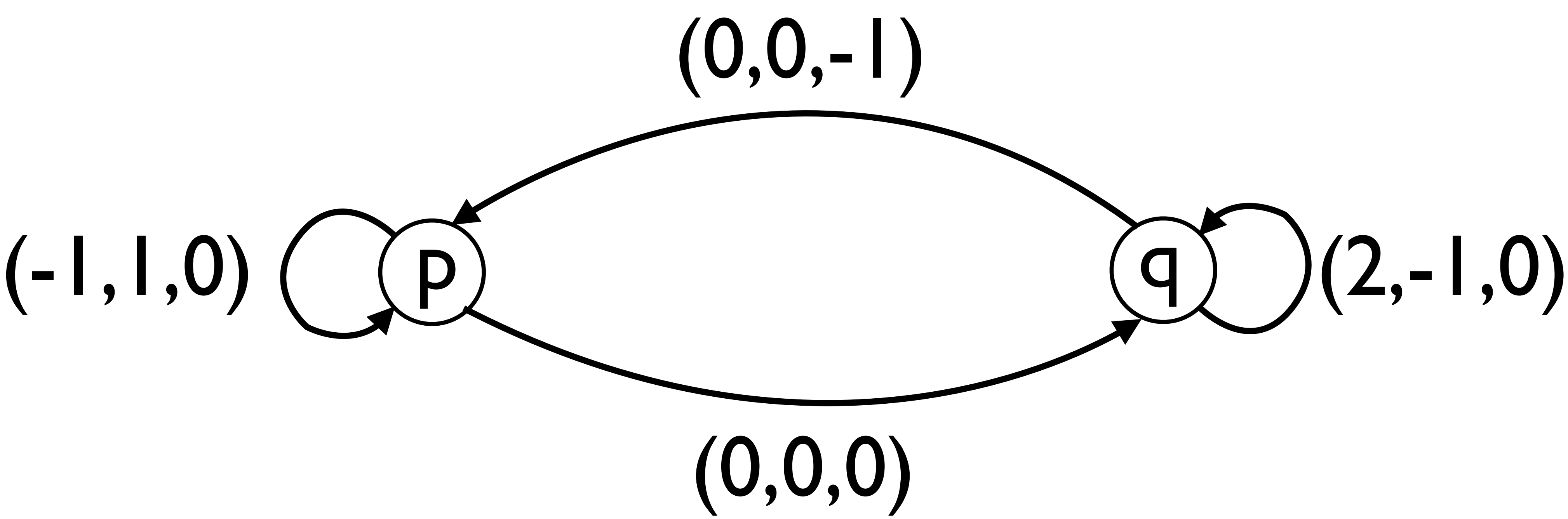
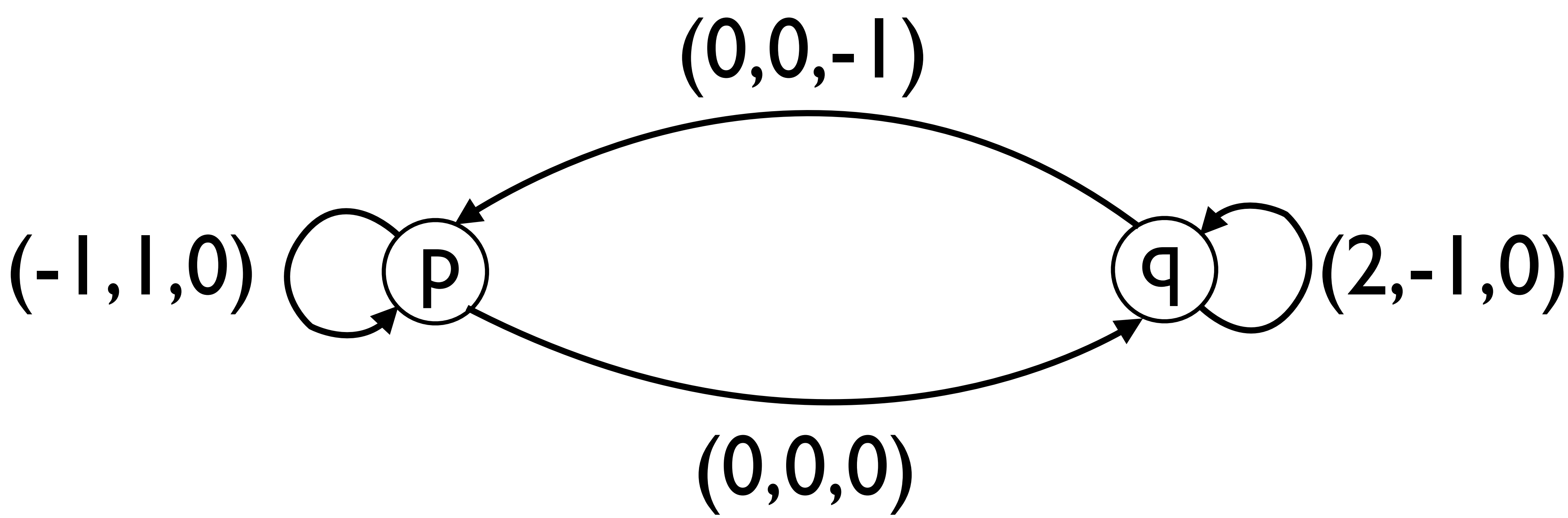


Reachability problem

Given: a Vector Addition System with States (VASS) V, two configurations s and t

Question: is there a run from s to t in V?



Recent history

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Leroux, Schmitz `19: Ackermann

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Cz., Lasota, Lazic, Leroux, Mazowiecki `19:
Tower-hard

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Leroux & Cz., Orlikowski `21:
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Recent history

Leroux, Schmitz `19: Ackermann

Cz., Lasota, Lazic, Leroux, Mazowiecki `19:
Tower-hard

Leroux & Cz., Orlikowski `21:
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everything solved?

Small dimensions

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2-VASSes:

Small dimensions

2-VASSes: PSpace-complete

Small dimensions

2-VASSes: PSpace-complete (binary)

Small dimensions

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3-VASSes:

Small dimensions

2-VASSes: PSpace-complete (binary)

3-VASSes: PSpace-hardness

Small dimensions

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in F_7 (more than Tower)

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in F_7 (more than Tower)

d-VASSes:

Small dimensions

2-VASSes: PSpace-complete (binary)

3-VASSes: PSpace-hardness

in F_7 (more than Tower)

d-VASSes: for which d in Elementary?

Small dimensions

2-VASSes: PSpace-complete (binary)

3-VASSes: PSpace-hardness

in F_7 (more than Tower)

d-VASSes: for which d in Elementary?

experience: small dimensions
lead to progress!

The result

The result

Theorem

The reachability problem is:

The result

Theorem

The reachability problem is:

1) NP-hard for unary, flat 3-VASSes

The result

Theorem

The reachability problem is:

1) NP-hard for unary, flat 3-VASSes (4-VASSes)

The result

Theorem

The reachability problem is:

1) NP-hard for unary, flat 3-VASSes (4-VASSes)

2) PSpace-hard for unary 5-VASSes

The result

Theorem

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1) NP-hard for unary, flat 3-VASSes (4-VASSes)

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3) ExpSpace-hard for binary 6-VASSes

The result

Theorem

The reachability problem is:

1) NP-hard for unary, flat 3-VASSes (4-VASSes)

2) PSpace-hard for unary 5-VASSes

3) ExpSpace-hard for binary 6-VASSes

4) Tower-hard for unary 8-VASSes

Techniques

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controlling counter

Techniques

controlling counter used in 1, 2, 4

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controlling counter used in 1, 2, 4

multiplication triples

Techniques

controlling counter used in 1, 2, 4

multiplication triples used in 3, 4

Techniques

controlling counter	used in 1, 2, 4
multiplication triples	used in 3, 4
quadratic pairs	

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controlling counter	used in 1, 2, 4
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Techniques

controlling counter	used in 1, 2, 4
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new

Quadratic pairs

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assure exact computations

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$(k, 0, k^2)$

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$(-1, 1, -1)$ in a loop

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Quadratic pairs

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$(1, -1, -1)$ in a loop

$(a, b, \geq k^2-2k)$

Quadratic pairs

assure exact computations

(k, 0, k^2) (-1, 1, -1) in a loop

(1, -1, -1) in a loop

(a, b, $\geq k^2-2k$) (-1, 0, 1) once

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Invariant: $x_3 \geq (x_1+x_2)^2$

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Invariant: $x_3 \geq (x_1+x_2)^2$

equality ⇔ all exact

NP-hardness in 3-VASSes

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reduce from 3-SAT

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encode valuation as a counter

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k variables

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reduce from 3-SAT

k variables

valuation 1 0 0 1 1

encode valuation as a counter

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$p_1 \; n_2 \; n_3 \; p_4 \; p_5$

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counter one keeps the encoding

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counter two helps multiplying

counter three assures exactness

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controlling counter

Message

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new techniques

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Thank you!