Reachability in 3-VASS is Elementary

Wojciech Czerwiński

Ismaël Jecker

Sławomir Lasota

Łukasz Orlikowski

Plan

Plan

- basic notions

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- short history

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- main contribution

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- basic notions
- short history
- main contribution
- proof ideas

Vector Addition Systems with States (VASS)

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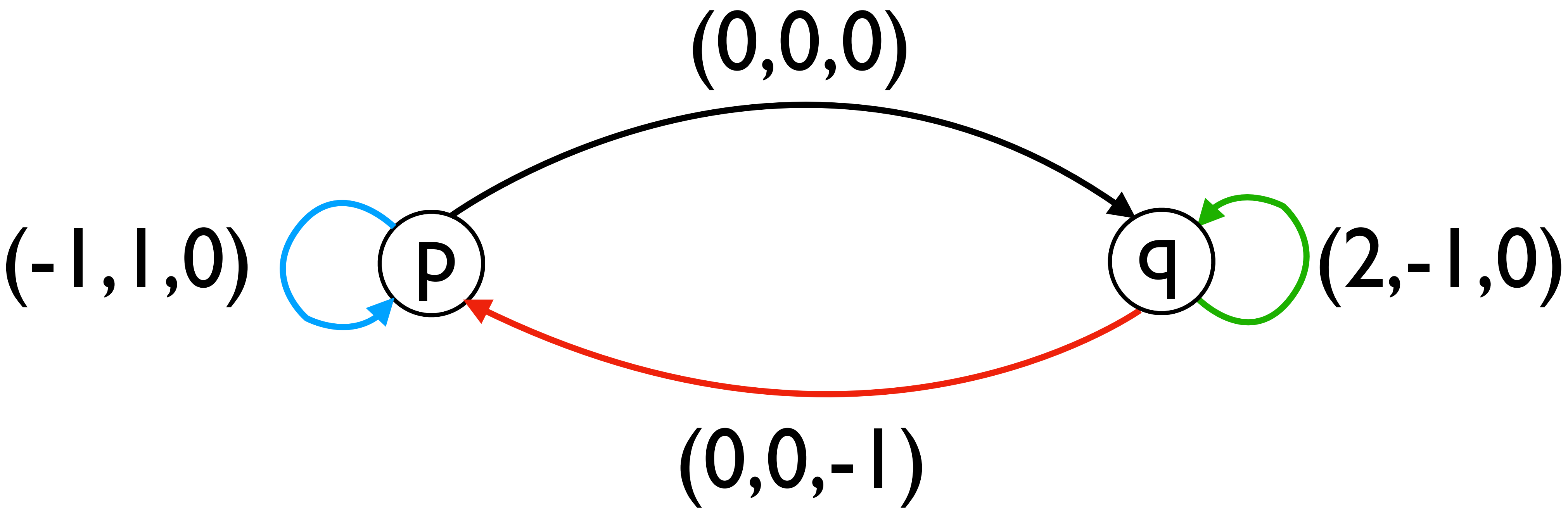

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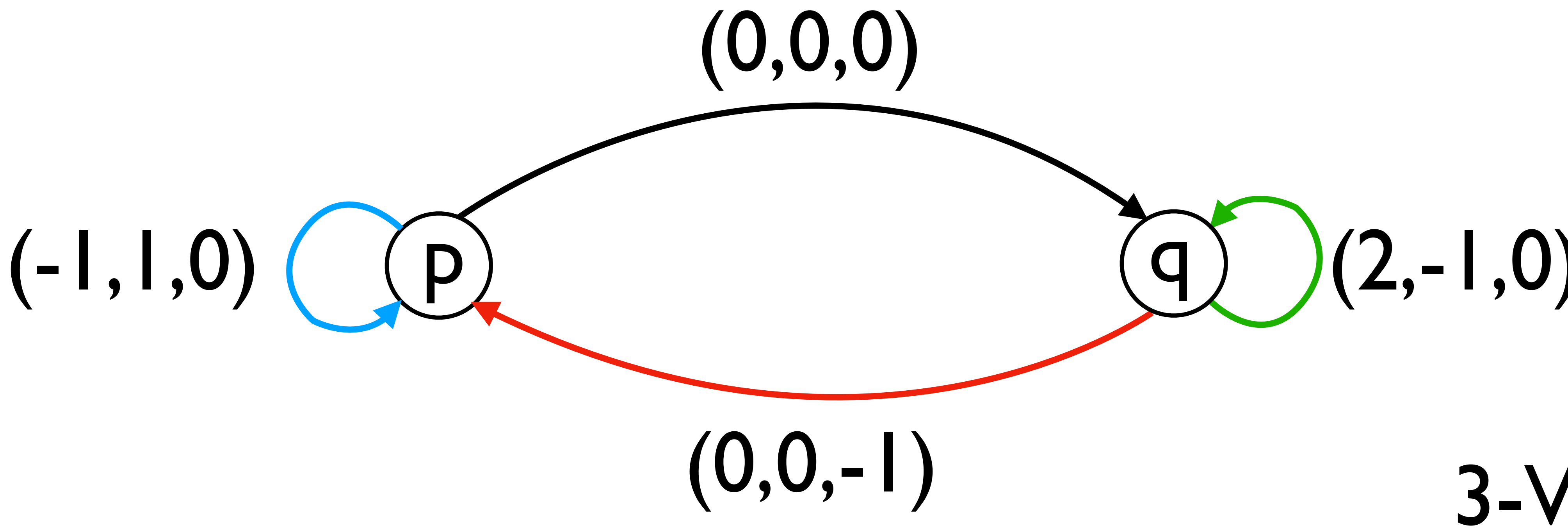
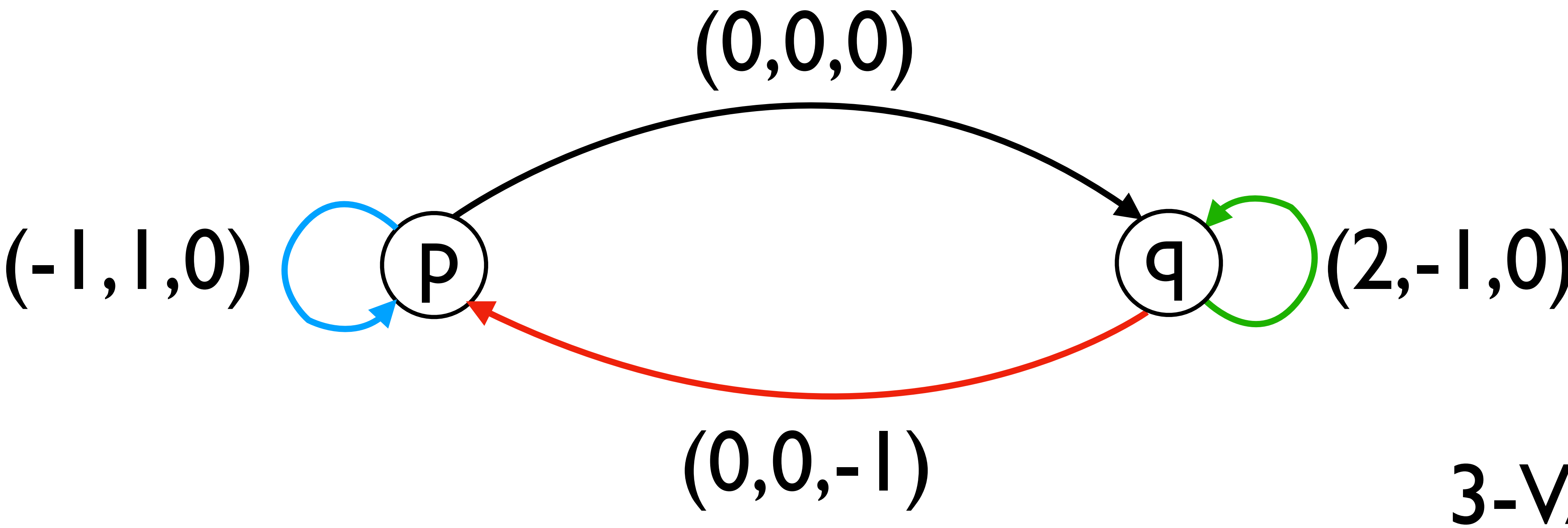


3-VASS

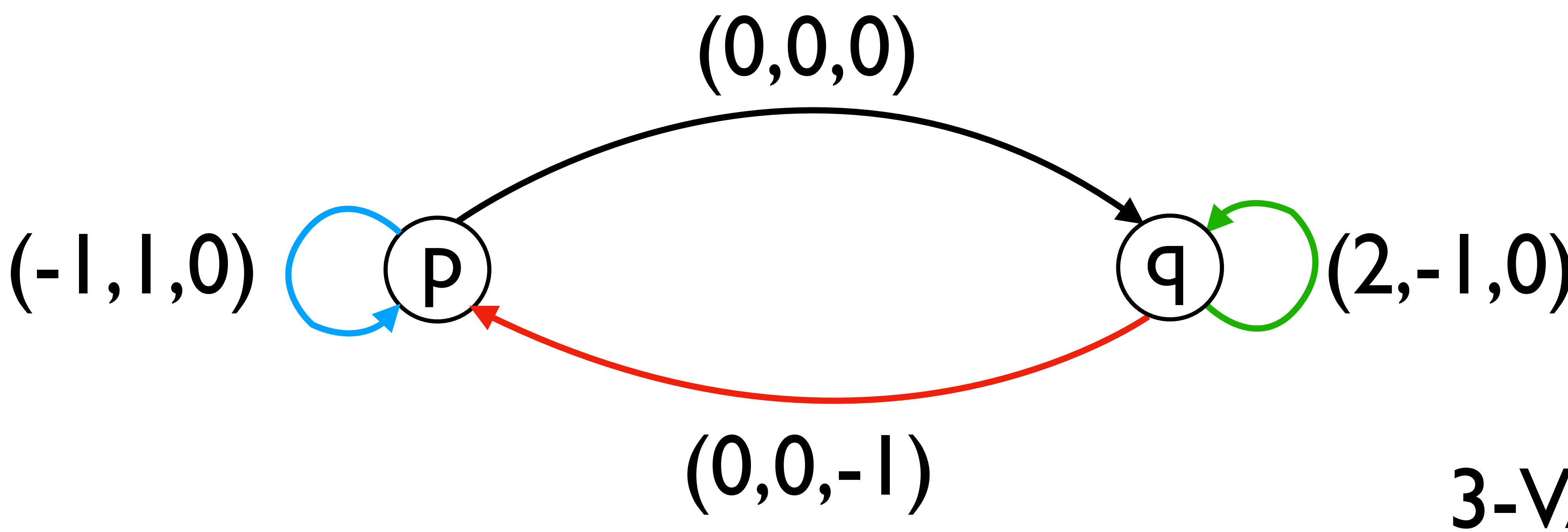
Vector Addition Systems with States (VASS)



3-VASS

p(2,0,7)

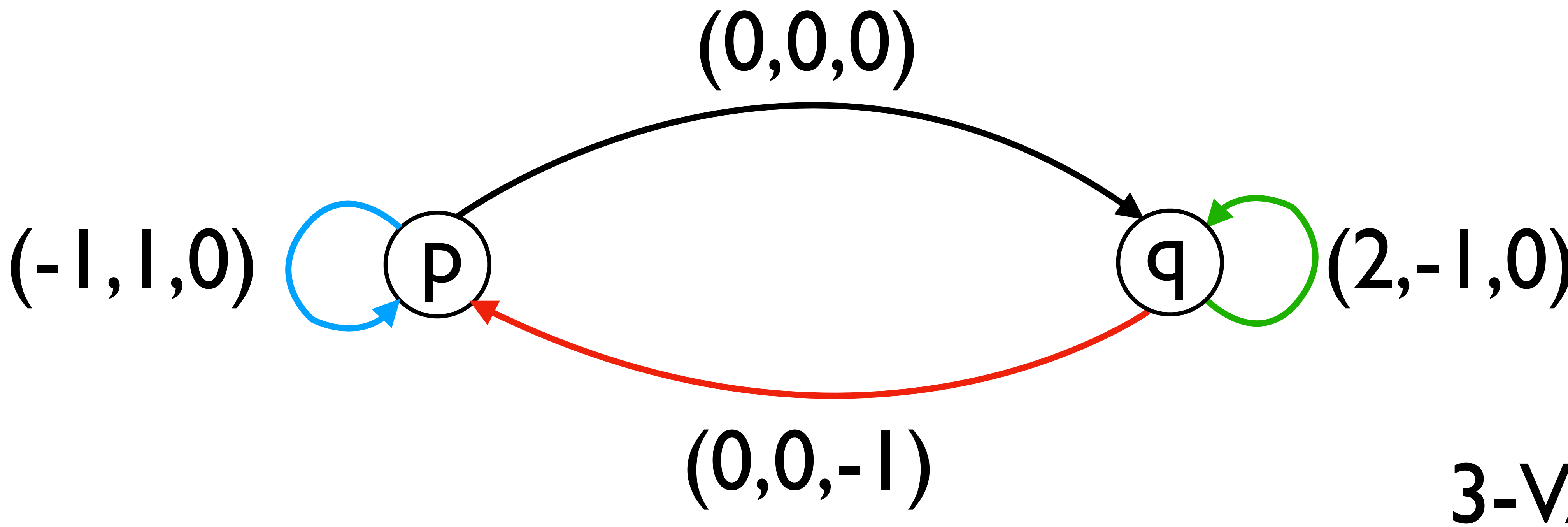
Vector Addition Systems with States (VASS)



3-VASS

p(2,0,7) ⟶ p(1,1,7)

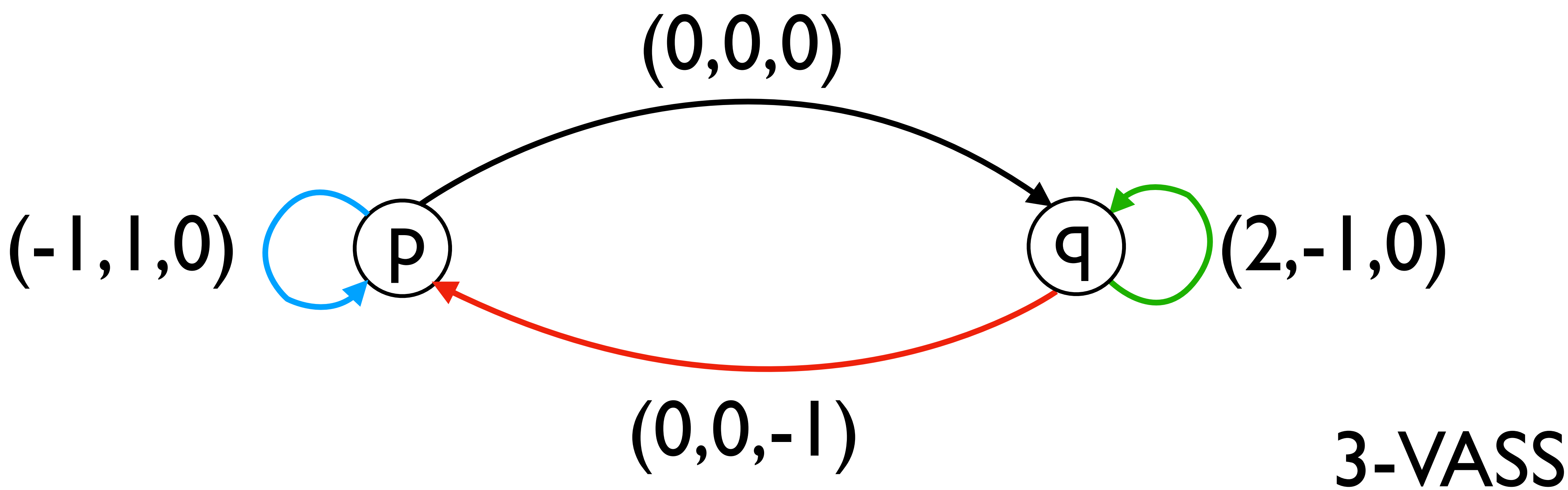
Vector Addition Systems with States (VASS)



3-VASS

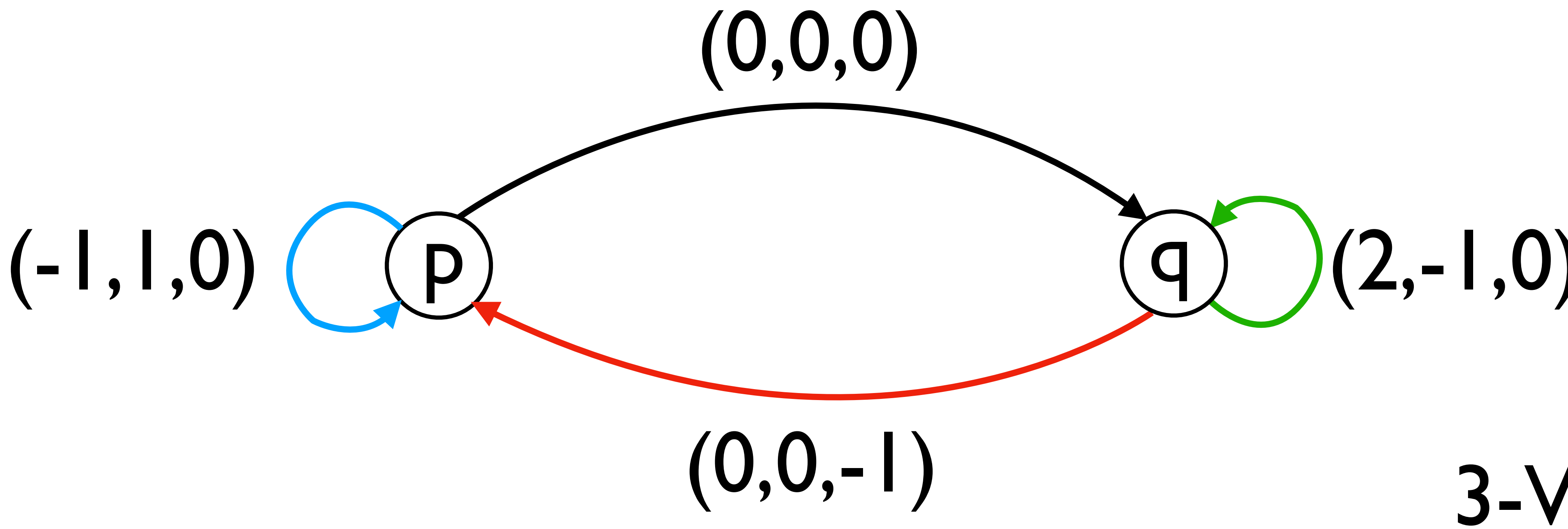
p(2,0,7) → p(1,1,7) → p(0,2,7)

Vector Addition Systems with States (VASS)



$p(2,0,7) \longrightarrow p(1,1,7) \longrightarrow p(0,2,7) \longrightarrow q(0,2,7)$

Vector Addition Systems with States (VASS)



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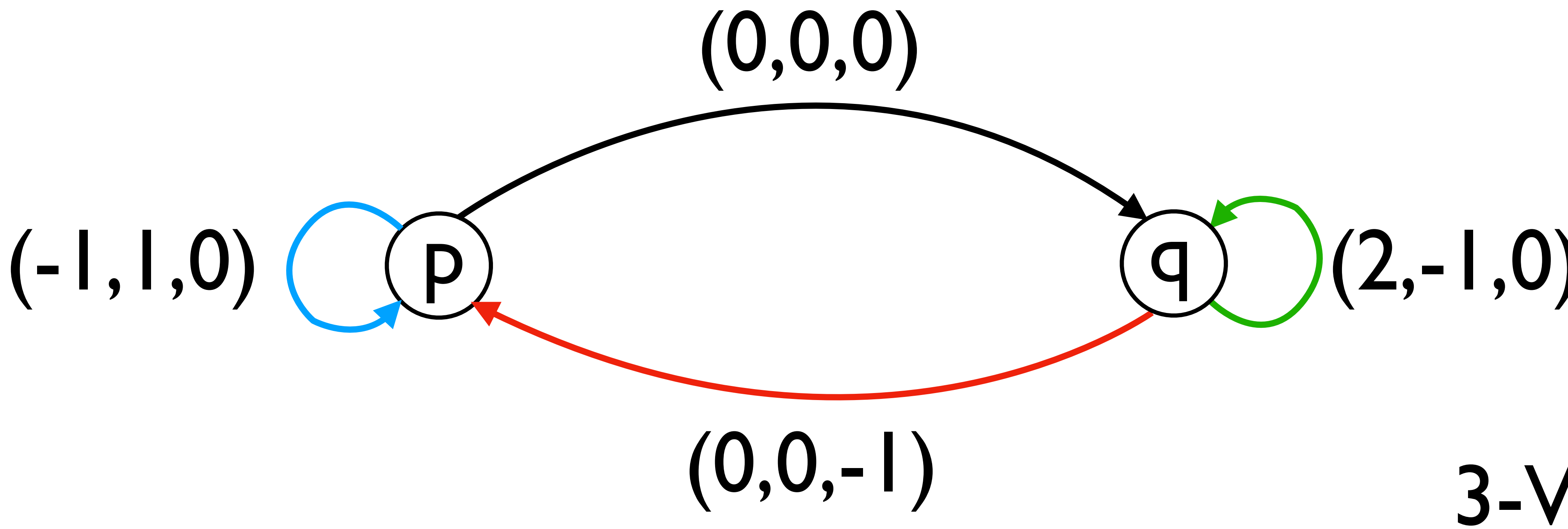

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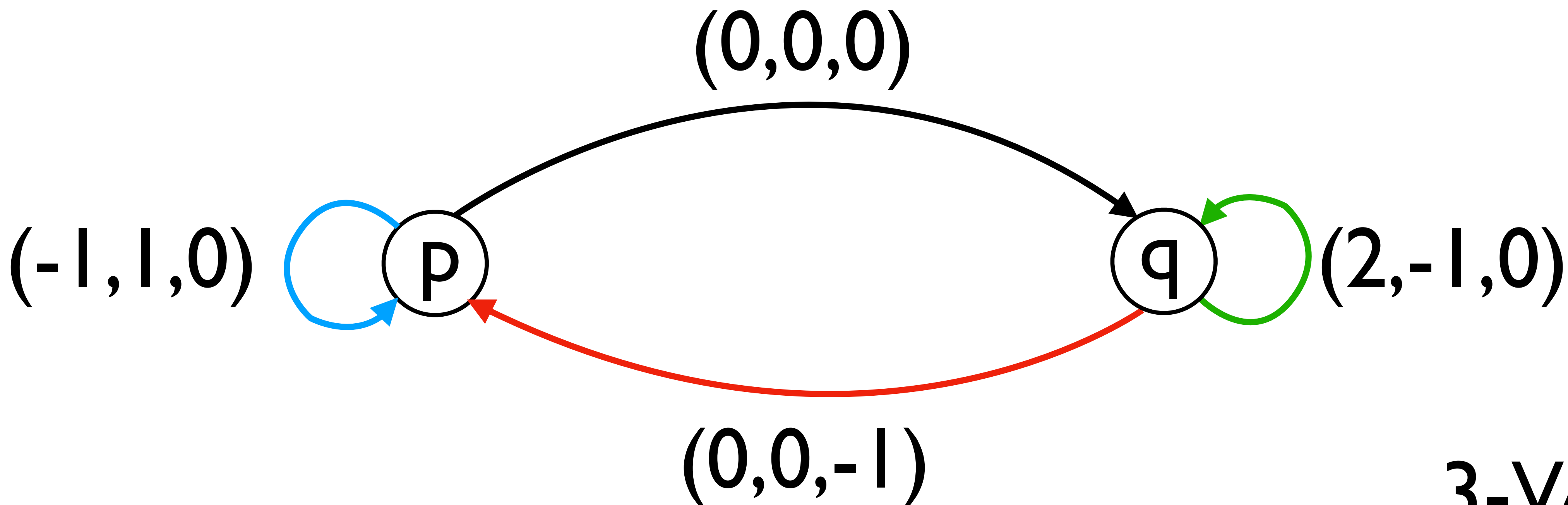
p(2,0,7) → p(1,1,7) → p(0,2,7) → q(0,2,7) → q(2,1,7)

→ q(4,0,7)

Vector Addition Systems with States (VASS)



3-VASS

p(2,0,7) → p(1,1,7) → p(0,2,7) → q(0,2,7) → q(2,1,7)

→ q(4,0,7) → p(4,0,6)

Reachability problem

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Given: a VASS, two its configurations s and t

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Question: is there a run from s to t?

Brief recent history

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Mayr `81: decidability

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Blondin et. al. `15: PSpace-comp. for binary 2-VASS

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For binary 3-VASS we have:
PSpace-hardness and Tower

Our main result

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Theorem

The reachability problem for binary 3-VASSs is in 2-ExpSpace.

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The reachability problem for binary 3-VASSs is in 2-ExpSpace.

Lemma

For each binary 3-VASS if there is a run from s to t then there is a one of at most triply-exponential length.

Why care?

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3-VASSs: the problem to look at for further progress

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first algorithm with complexity lower than
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and 10-fold exponentially big

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standard techniques cannot go below Tower

we invented novel techniques

Techniques

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challenge:

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finite reachability sets of up to Tower size

Techniques

challenge: finite reachability sets of up to Tower size

solution:

Techniques

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solution: approximation good enough for reachability

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challenge: finite reachability sets of up to Tower size

solution: approximation good enough for reachability

sandwiching technique:

Techniques

challenge: finite reachability sets of up to Tower size

solution: approximation good enough for reachability

sandwiching technique: over- and under-approximating a big reachability set by similarly behaving small semilinear sets

Techniques cont.

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For each binary 3-VASS if there is a run from s to t then there is a one of at most triply-exponential length.

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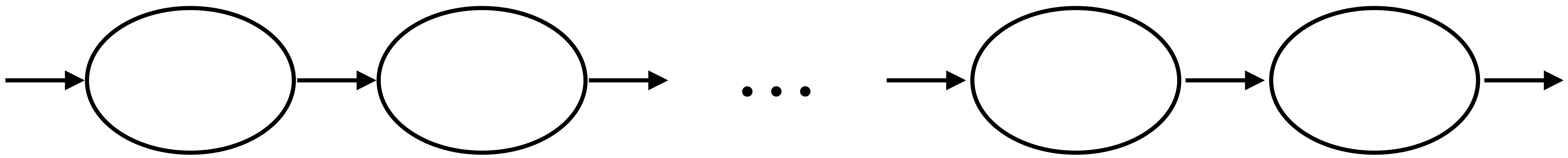

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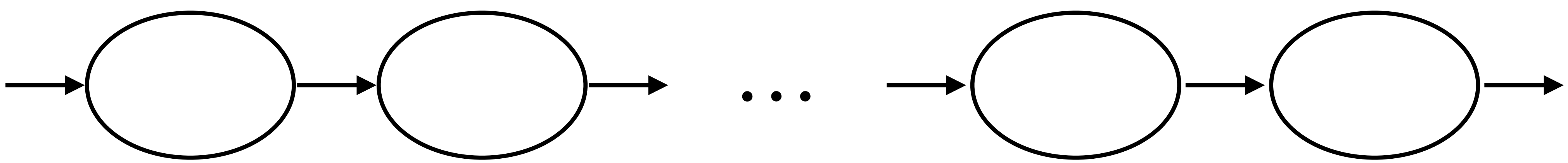

induction on the number of components

Techniques cont.

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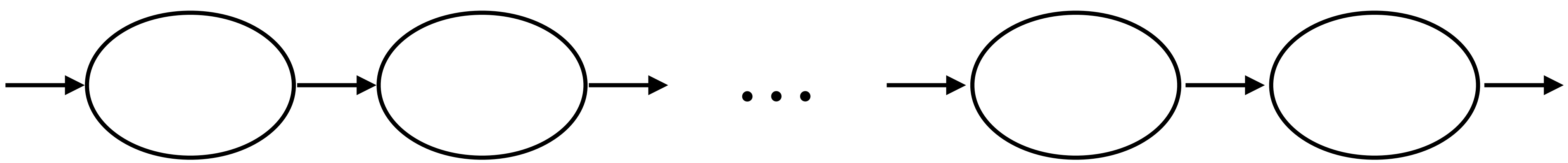

induction on the number of components

adding one component: squaring the exponent

Techniques cont.

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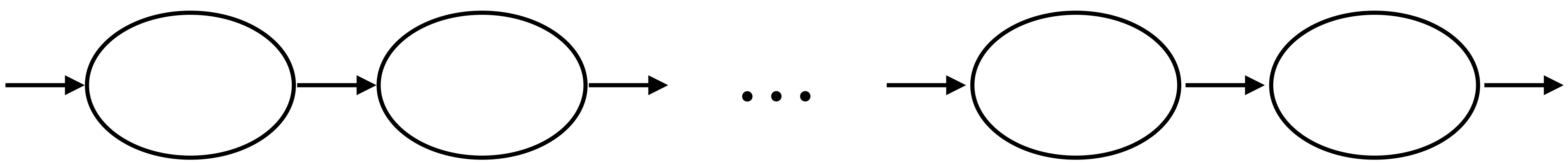

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$$n^{k} \longrightarrow n^{k^2}$$

Techniques cont.

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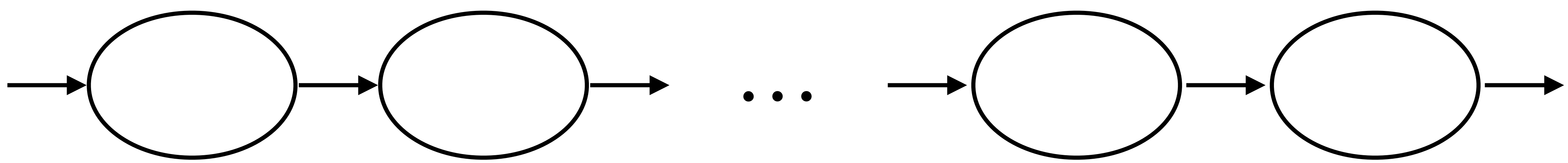

induction on the number of components

adding one component: squaring the exponent

$$n^k \longrightarrow n^{k^2}$$

this gives 2-exp exponent, so 3-exp path length

Techniques cont.

Techniques cont.

induction: adding one component

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easier part: diagonal 3-VASS components

Techniques cont.

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non-diagonal 3-VASS are similar to 2-VASS

Techniques cont.

induction: adding one component

easier part: diagonal 3-VASS components

hard part: non-diagonal 3-VASS components

non-diagonal 3-VASS are similar to 2-VASS

we use knowledge about 2-VASS

Techniques cont.

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finite reachability set of 2-VASS are exponential

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NEW: but can be well-approximated
by semilinear sets of polynomial representation

Techniques cont.

finite reachability set of 2-VASS are exponential

NEW: but can be well-approximated
by semilinear sets of polynomial representation

NEW: we show this approximation
and use it in the induction step

Open questions

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exact for 3-VASS, 4-VASS, d-VASS

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2-VASS + many $\mathbb{Z}$-counters

Open questions

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fixed VASS (PSpace-hard, in Ackermann)

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2-VASS + many $\mathbb{Z}$-counters

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reusing the sandwiching technique

Summary

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novel sandwiching technique

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hard case reduces to 2-VASS

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better understanding of 2-VASS

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Thank you!