

The Reachability Problem for Vector Addition Systems is Not Elementary

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Short history

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Lipton '76: EXPSPACE-hardness

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Mayr '81: decidability

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Kosaraju `82, Lambert `92: simplifications

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Big complexity gap

Main result

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Theorem

The **Reachability Problem** for **Vector Addition Systems** is not elementary.

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Not in time:

$$2^{2^2 \dots 2^n}$$

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Not in time:

$2^{2^2 \dots 2^n}$ ← fixed height

Proof ideas

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- surprising equation on fractions

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- implementing equation in VASes

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- surprising equation on fractions
- implementing equation in VASes
- iterating to obtain tower

Fractional equation

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and all a_i , b_i , a and b are at most exponential in k .

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multiplies **place** at most **2^l** times by **a_l / b_l**

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K doubly exponential

Lifting higher

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- exponential to doubly exponential numbers

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- exponential to doubly exponential numbers
- more tricks needed to iterate the construction

Thank you!