Reachability Problem for Fixed Dimensional Vector Addition Systems with States

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Towards Understanding the Reachability Problem for Vector Addition Systems with States

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Plan

Plan

- definition

Plan

- definition
- state of art

Plan

- definition
- state of art
- motivation

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- definition
- state of art
- motivation
- our results

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- definition
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- 3D example

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- 3D example
- 4D example

Vector Addition Systems with States (VASS)

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Reachability problem

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Reachability problem

Input: finite set of transitions T in $Q \times \mathbb{Z}^d \times Q$

source s, target t in $Q \times \mathbb{N}^d$

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Question: can one reach t starting in s
by finitely many transitions from T
inside the positive quadrant?

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source s, target t in $Q \times N^d$

Question: can one reach t starting in s
by finitely many transitions from T
inside the positive quadrant?

Assume source and target is 0 vector

State of art

State of art

General VASSes

State of art

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Ackermann (Leroux, Schmitz `19)

State of art

General VASSes

Ackermann (Leroux, Schmitz `19)

Tower-hard (CLLLM `19)

State of art

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Fixed dimension d-VASSes

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Fixed dimension d-VASSes

F_{d+4} (Leroux, Schmitz `19)

State of art

General VASSes

Ackermann (Leroux, Schmitz `19)

Tower-hard (CLLLM `19)

Fixed dimension d-VASSes

F_{d+4} (Leroux, Schmitz `19)

(d-13)-ExpSpace-hard (CLLLM `19)

Small dimension

Small dimension

	unary	binary
1-VASS		
2-VASS		

Small dimension

	unary	binary
1-VASS	NL	
2-VASS		

Small dimension

	unary	binary
1-VASS	NL	NP (HKQW `10)
2-VASS		

Small dimension

	unary	binary
1-VASS	NL	NP (HKQW `10)
2-VASS		PSpace (BFGHM `15)

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← short paths

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linear path schemes

Small dimension

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← short paths

linear path schemes

$u_0\ (v_1)^*\ u_1\ (v_2)^* \ldots u_{k-1}\ (v_k)^*\ u_k$

Small dimension

	unary	binary
1-VASS	NL	NP (HKQW `10)
2-VASS	NL (ELT `16)	PSpace (BFGHM `15)

← short paths

linear path schemes

$u_0\ (v_1)^*\ u_1\ (v_2)^* \ldots u_{k-1}\ (v_k)^*\ u_k$

Hopcroft-Pansiot example

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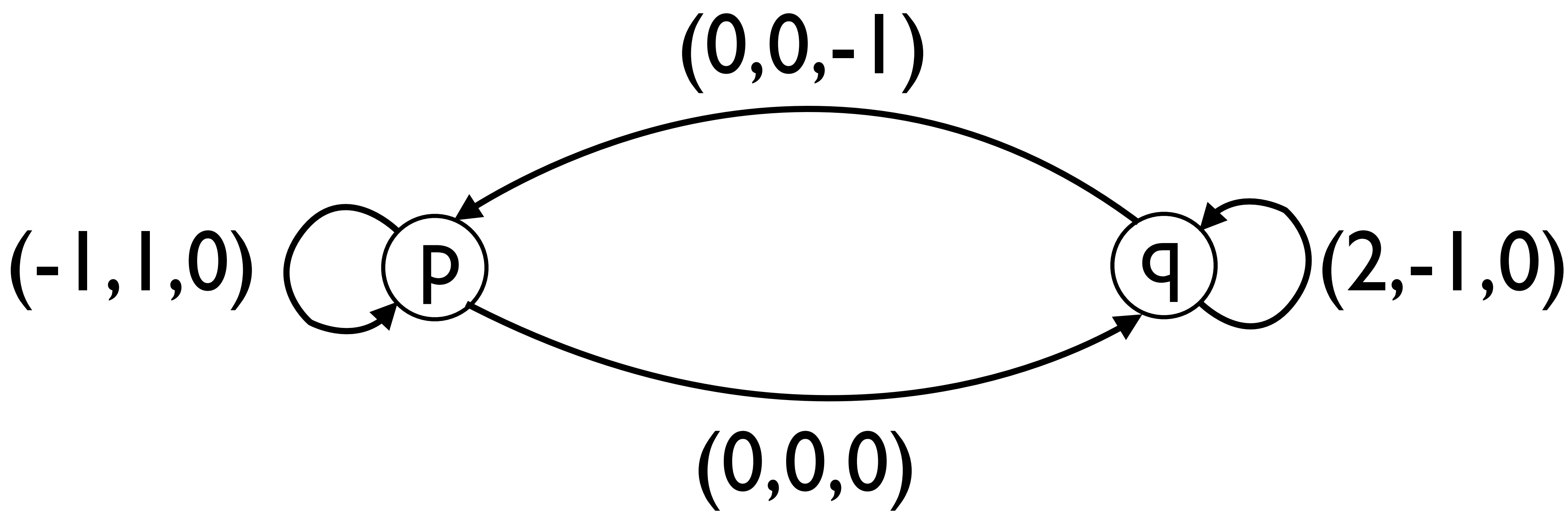
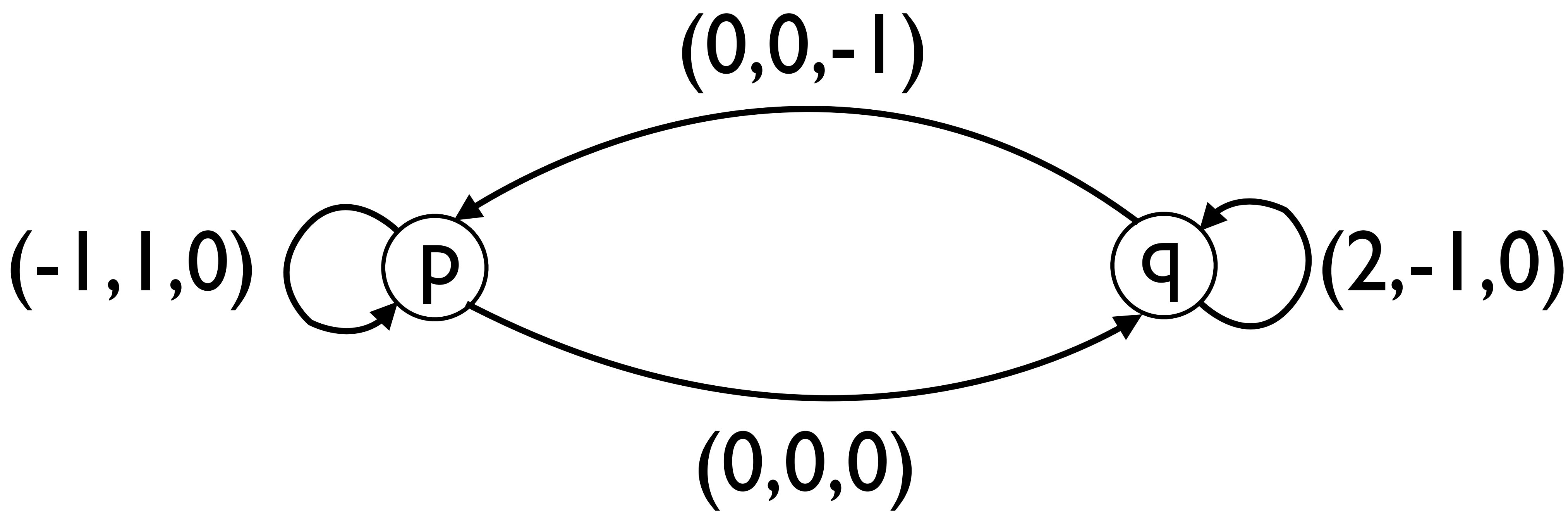
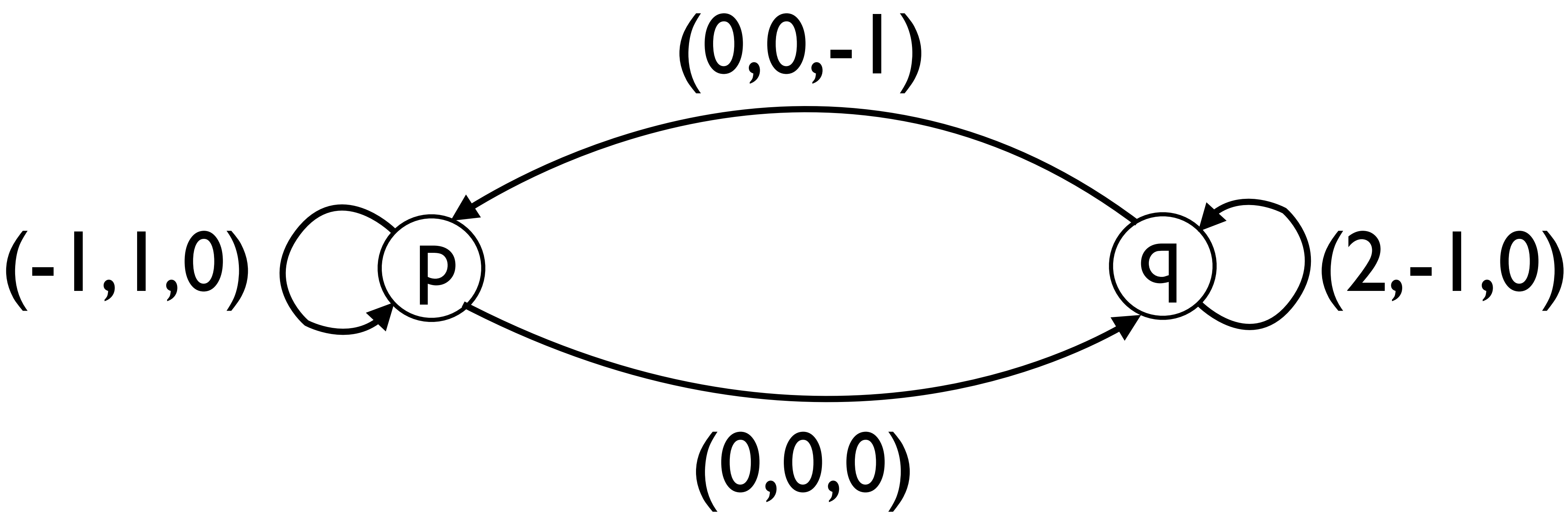


Hopcroft-Pansiot example



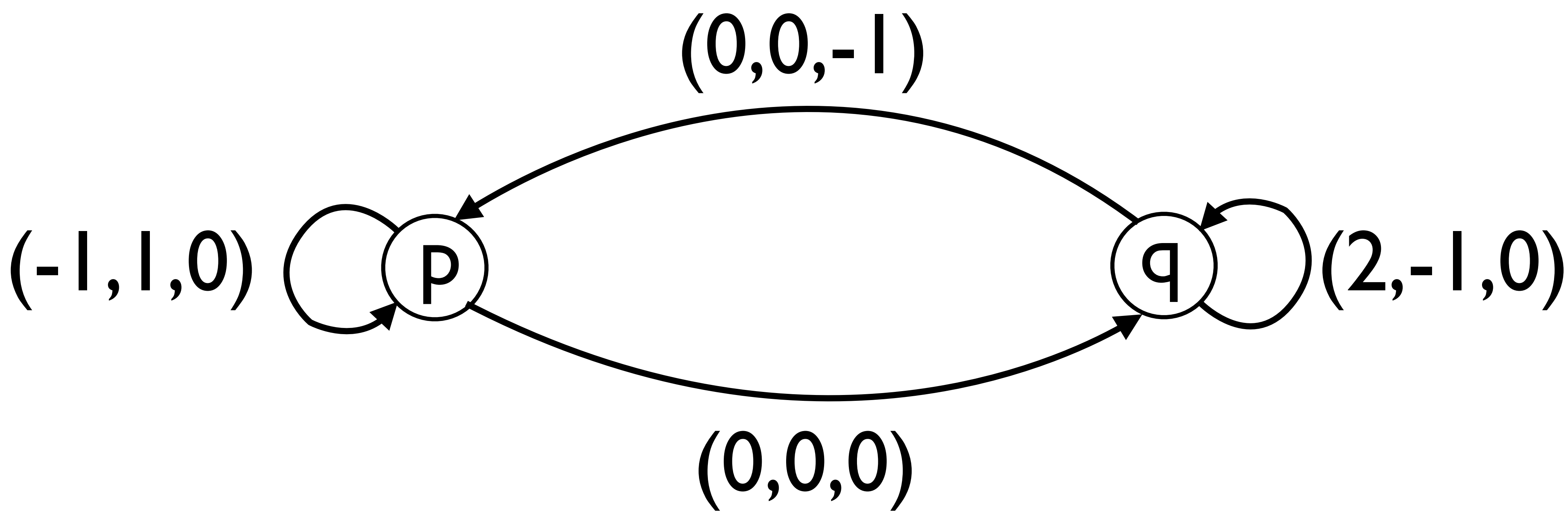
p(k,0,n)

Hopcroft-Pansiot example



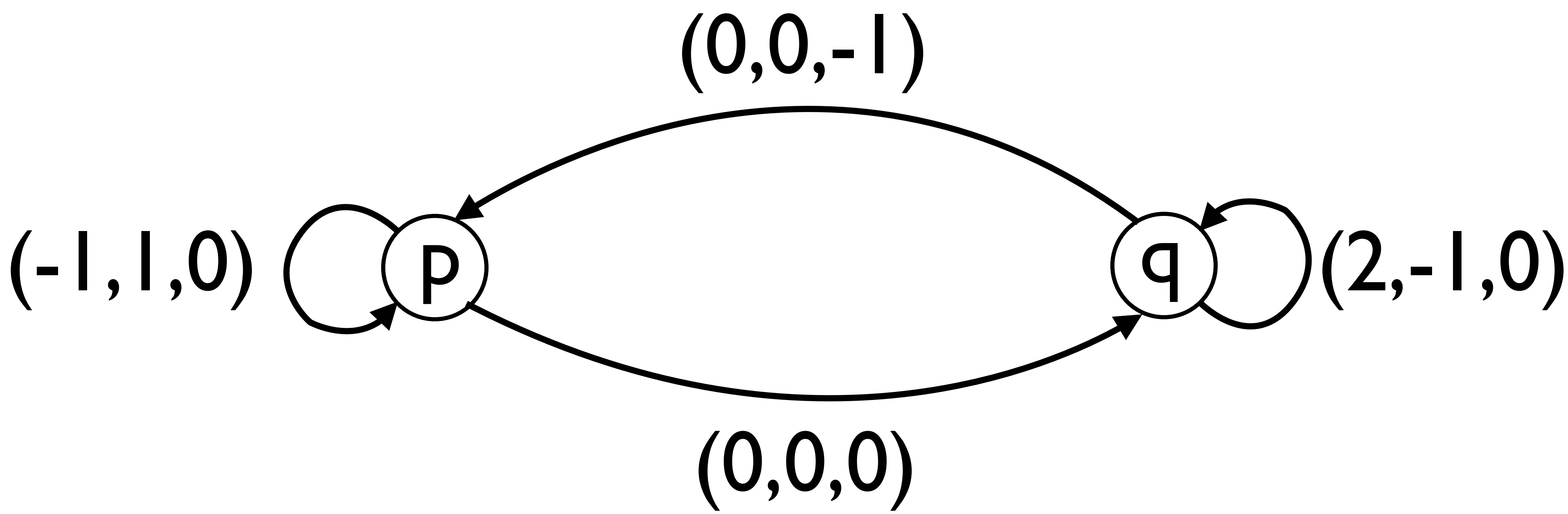
p(k,0,n) ⟶ p(0,k,n)

Hopcroft-Pansiot example



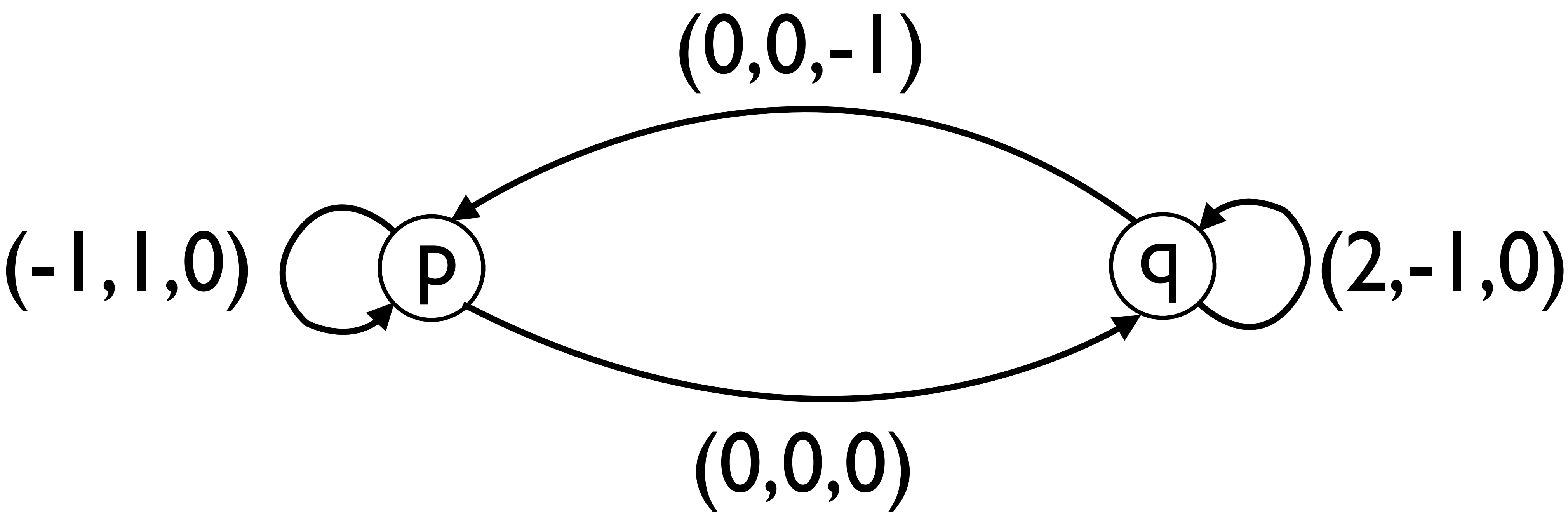
p(k,0,n) ⟶ p(0,k,n) ⟶ q(0,k,n)

Hopcroft-Pansiot example



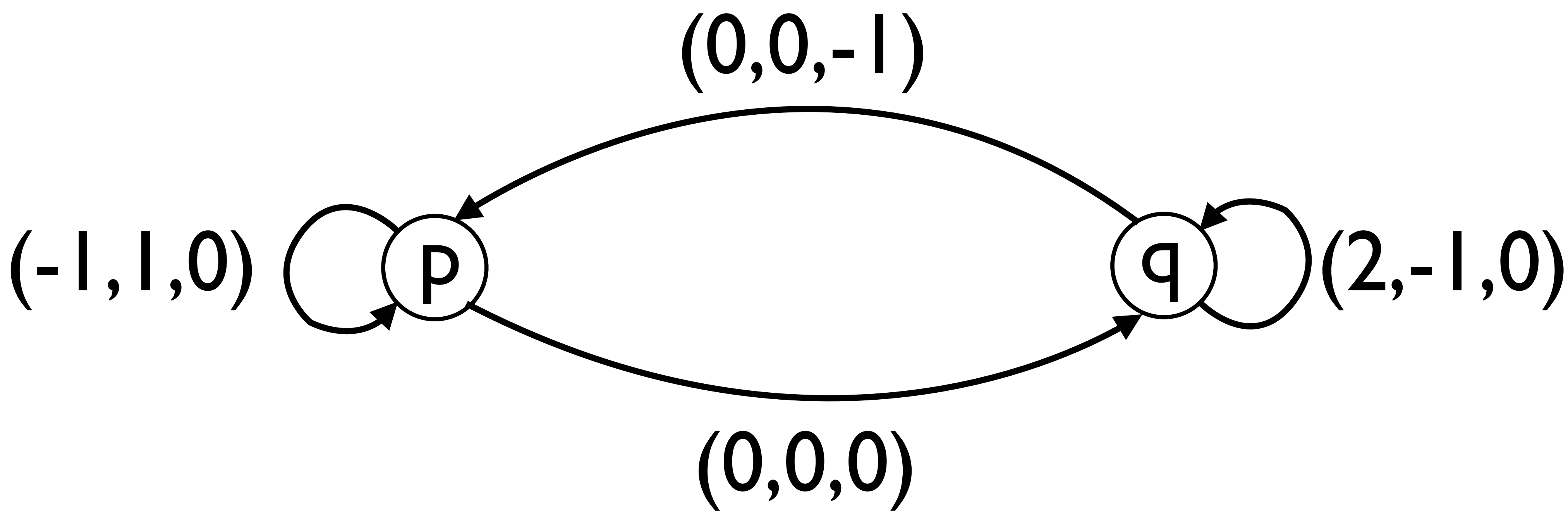
p(k,0,n) ⟶ p(0,k,n) ⟶ q(0,k,n) ⟶ q(2k,0,n)

Hopcroft-Pansiot example



p(k,0,n) ⟶ p(0,k,n) ⟶ q(0,k,n) ⟶ q(2k,0,n) ⟶ p(2k,0,n-1)

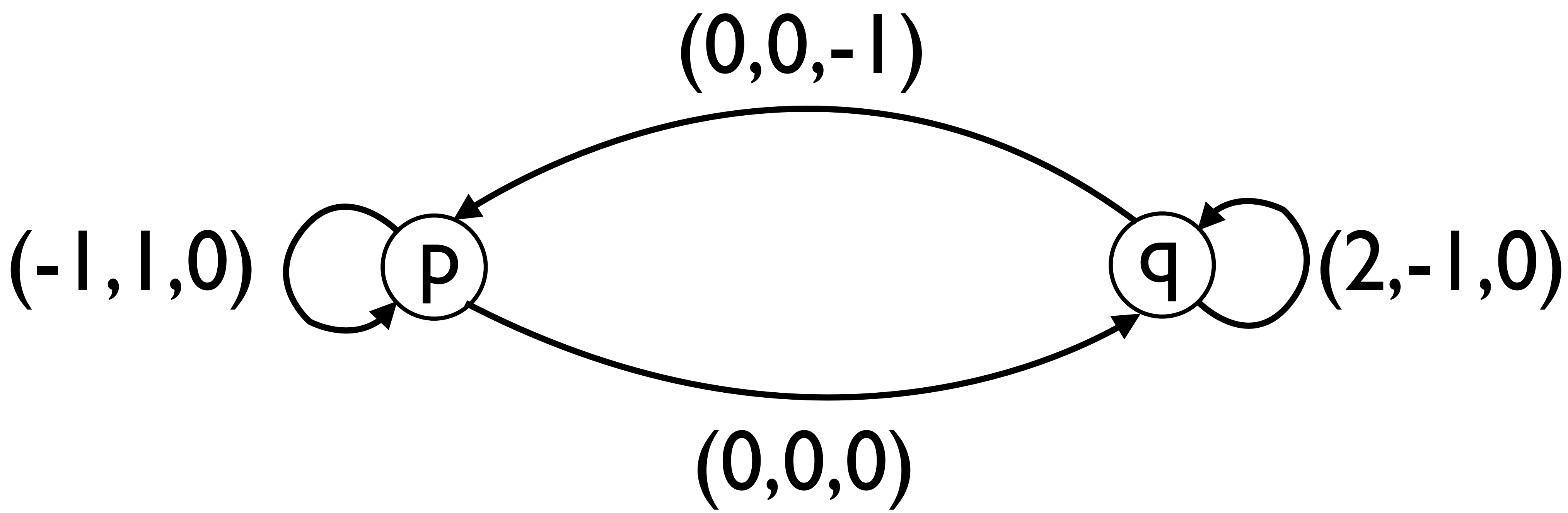
Hopcroft-Pansiot example



$p(k,0,n) \longrightarrow p(0,k,n) \longrightarrow q(0,k,n) \longrightarrow q(2k,0,n) \longrightarrow p(2k,0,n-1)$

$p(1,0,n)$

Hopcroft-Pansiot example



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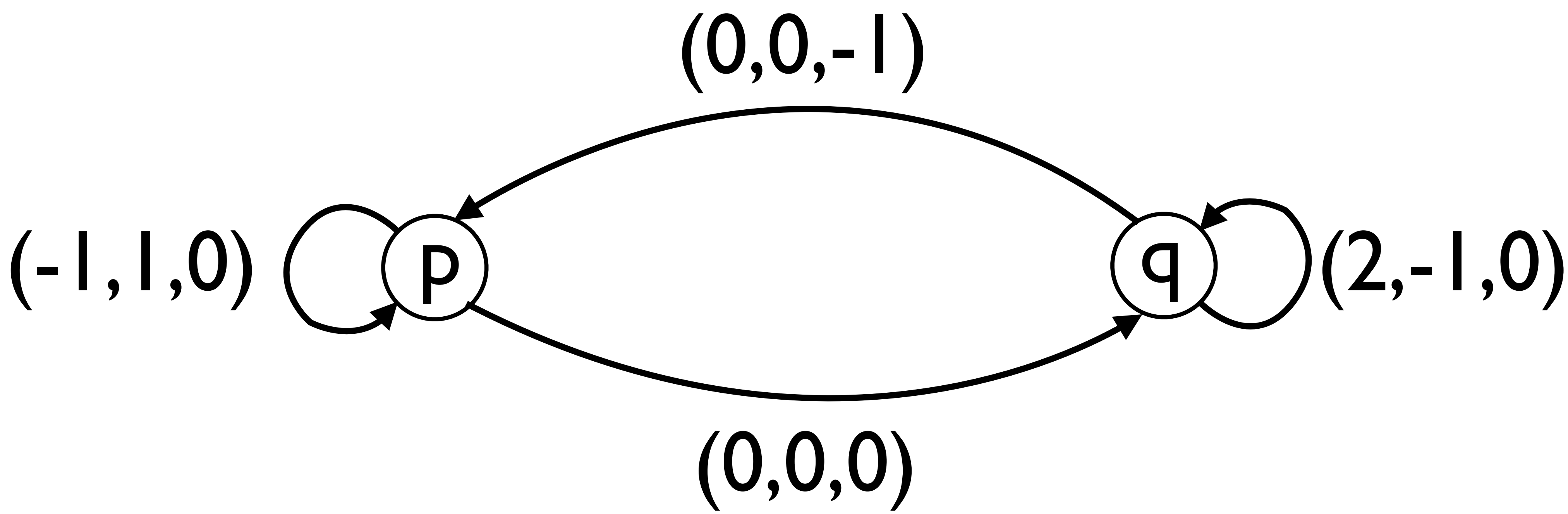
Hopcroft-Pansiot example



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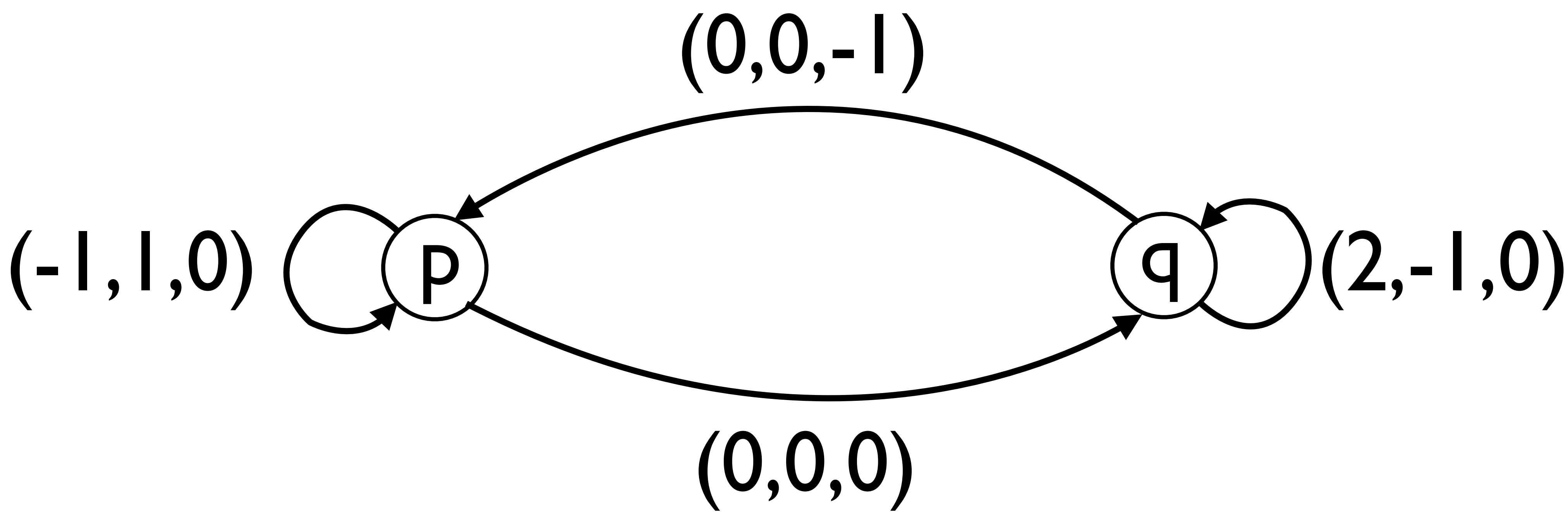
Hopcroft-Pansiot example



$$p(k,0,n) \longrightarrow p(0,k,n) \longrightarrow q(0,k,n) \longrightarrow q(2k,0,n) \longrightarrow p(2k,0,n-1)$$

$$p(1,0,n) \longrightarrow p(2,0,n-1) \longrightarrow \quad \ldots$$

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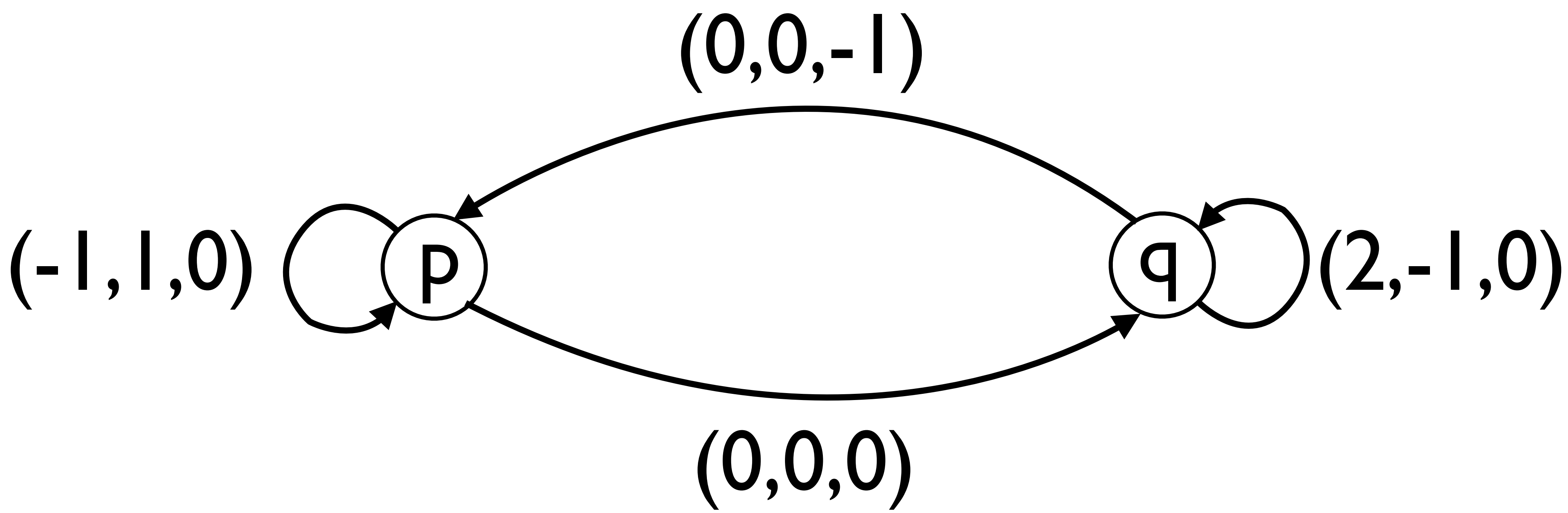

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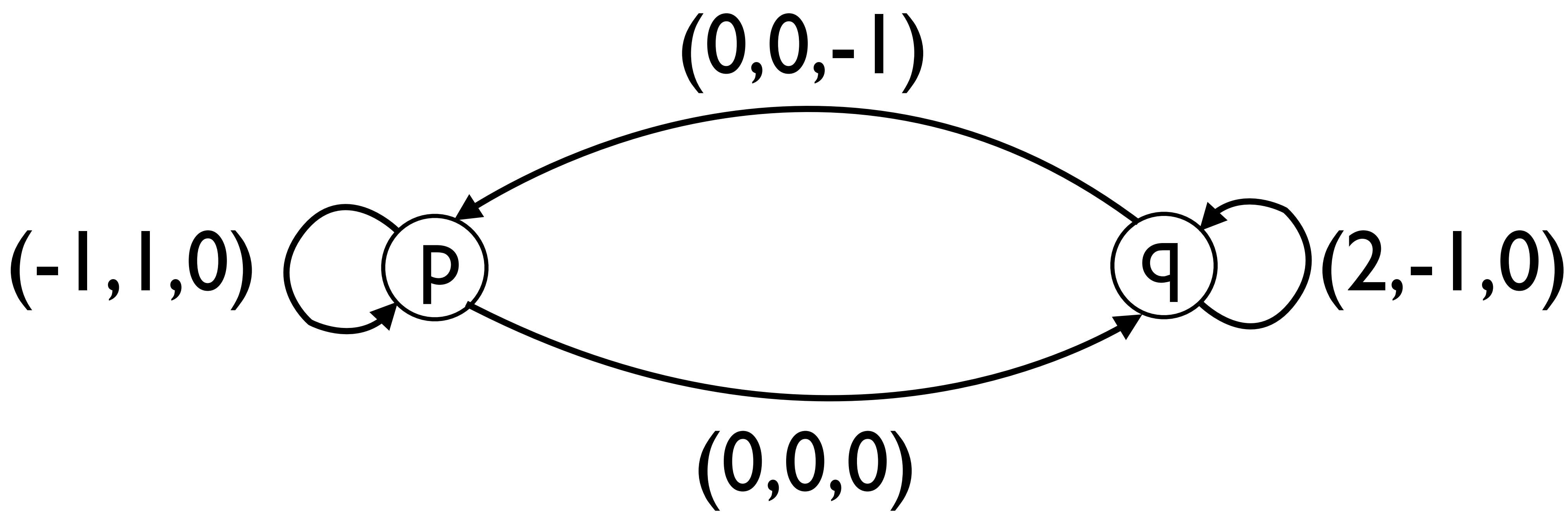

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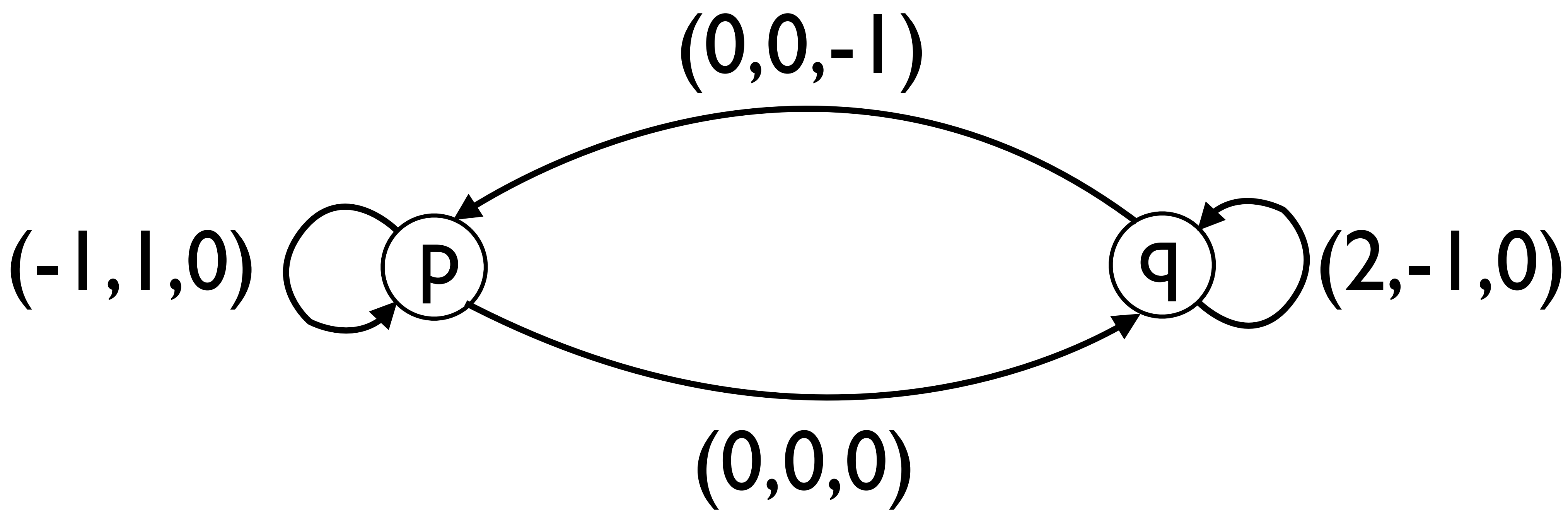

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Each $p(x, y, 0)$ for $x+y= 2^n$ is reachable

Hopcroft-Pansiot example

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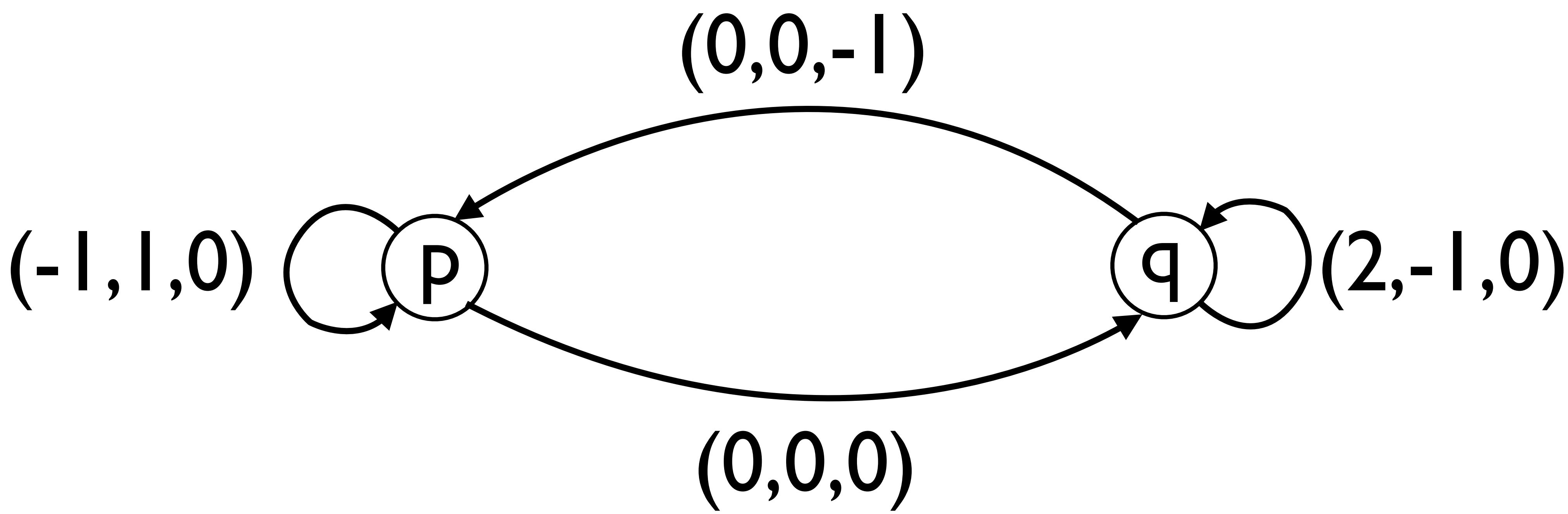

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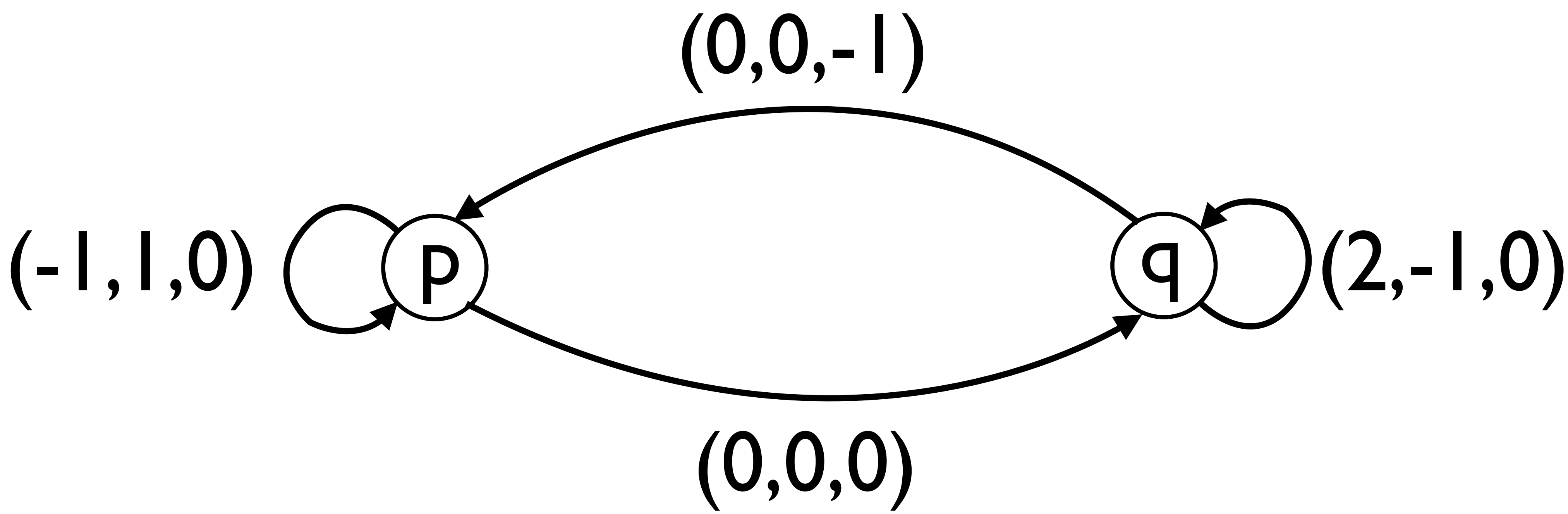

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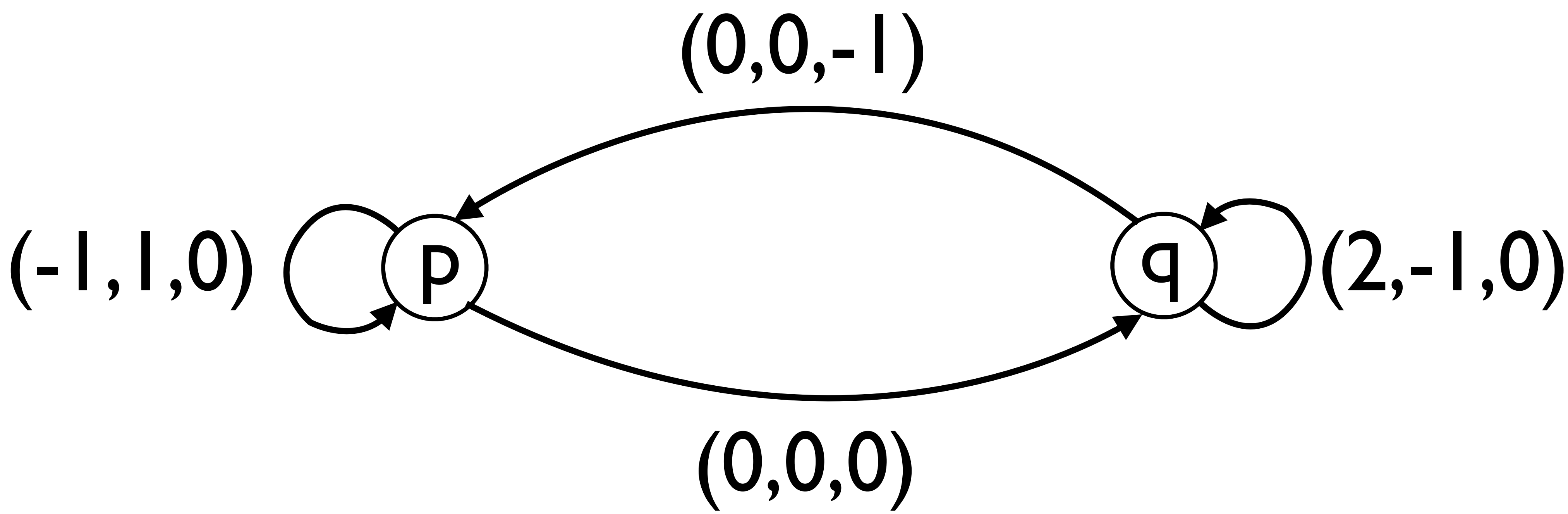

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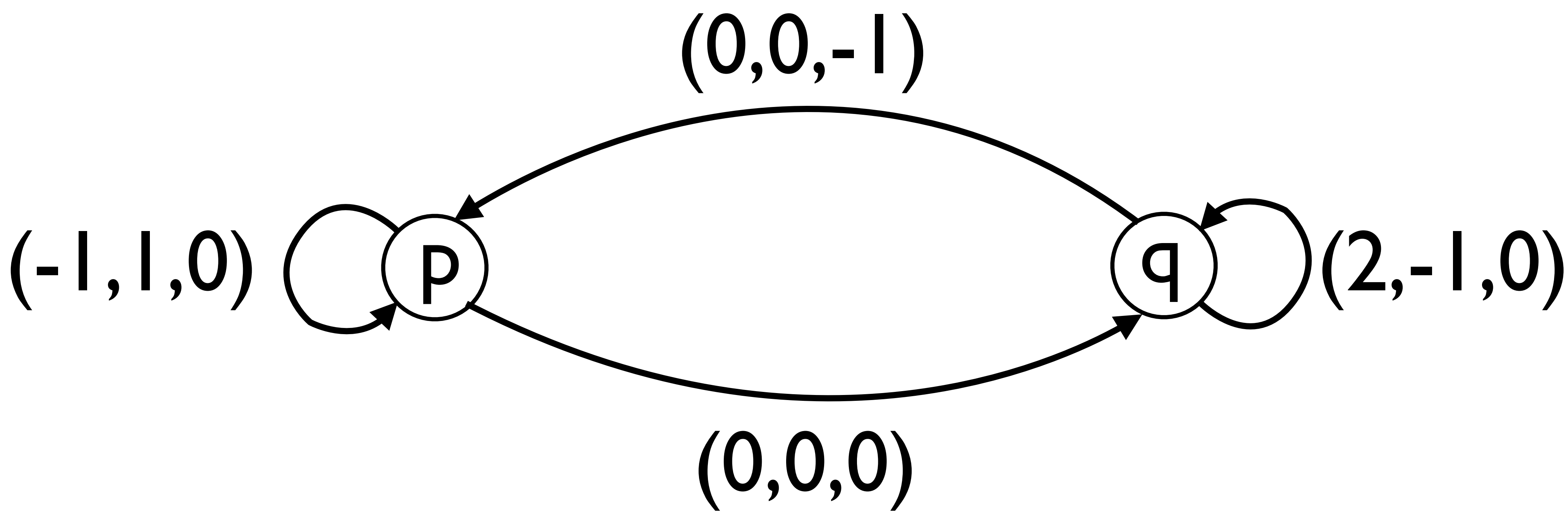
Non-semilinear reachability set

Hopcroft-Pansiot example



Each p(x, y, 0) for $x+y= 2^n$ is reachable

Non-semilinear reachability set

$\{(x,y,n): x+y \leq 2^n\}$

Motivation

Motivation

Reachability problem for finite reachability sets

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KLM decomposition useless?

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Reachability problem for finite reachability sets

KLM decomposition useless?

Reachability for finite 3-VASSes (PSpace? Tower?)

Conjectures

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Conjecture 1

In each dimension configuration reachable by a LPS is reachable by a polynomial length path

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In each dimension configuration reachable by a LPS is reachable by a polynomial length path

$u_0\ (v_1)^*\ u_1\ (v_2)^* \ldots u_{k-1}\ (v_k)^*\ u_k$

Conjectures

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In each dimension configuration reachable by a LPS is reachable by a polynomial length path

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Conjecture II

In each dimension reachability by a LPS is in PTime

Conjectures

Conjecture I

In each dimension configuration reachable by a LPS is reachable by a polynomial length path

$u_0 \, (v_1)^* \, u_1 \, (v_2)^* \ldots u_{k-1} \, (v_k)^* \, u_k$

Conjecture II

In each dimension reachability by a LPS is in PTime

Conjecture III

In each dimension shortest path is at most exponential (in binary)

Results

Results

Theorem 1

There is a 3-VASS with configuration reachable by a LPS and with exponential shortest path

Results

Theorem I

There is a 3-VASS with configuration reachable by a LPS and with exponential shortest path

Theorem II

Reachability by a LPS in dimension 7 is NP-complete

Results

Theorem I

There is a 3-VASS with configuration reachable by a LPS and with exponential shortest path

Theorem II

Reachability by a LPS in dimension 7 is NP-complete

Theorem III

There is a 4-VASS with shortest path of doubly exponential length

3D example

Theorem 1

There is a 3-VASS with configuration reachable by a LPS and with exponential shortest path

3D example

Theorem 1

There is a 3-VASS with configuration reachable by a LPS and with exponential shortest path

$$(2 / 1) \cdot (3 / 2) \cdot \ldots \cdot (k / k\text{-}1) = k$$

3D example

Theorem 1

There is a 3-VASS with configuration reachable by a LPS and with exponential shortest path

$$(2\ /\ 1)\ \cdot\ (3\ /\ 2)\ \cdot \dots \cdot\ (k\ /\ k\text{-}1) = k$$

Idea: enforce that run visits a configuration p(N, N, 0) with N divisible by all primes in {1, 2, …, k}.

3D example

Theorem 1

There is a 3-VASS with configuration reachable by a LPS and with exponential shortest path

3D example

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3D example

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start = end = (0,0,0)

3D example

Theorem 1

There is a 3-VASS with configuration reachable by a LPS and with exponential shortest path

$$(2 / 1) \cdot (3 / 2) \cdot \ldots \cdot (k / k-1) = k$$

start = end = (0,0,0)

(1,1,0) (1,1,0)* (-1,0,1)*(k,0,-(k-1))… (-1,0,1)*(3,0,-2)…

3D example

Theorem 1

There is a 3-VASS with configuration reachable by a LPS and with exponential shortest path

$$(2 / 1) \cdot (3 / 2) \cdot \ldots \cdot (k / k-1) = k$$

$$\text{start} = \text{end} = (0,0,0)$$

$$(1,1,0)\ (1,1,0)^*\ (-1,0,1)^*(k,0,-(k-1))\ldots\ (-1,0,1)^*(3,0,-2)\ldots$$

$$\ldots(-1,0,1)^*(2,0,-1)(-k,-1,0)$$

3D example

Theorem 1

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start = end = (0,0,0)

(1,1,0) (1,1,0)* (-1,0,1)*(k,0,-(k-1))… (-1,0,1)*(3,0,-2)…

…(-1,0,1)*(2,0,-1)(-k,-1,0)

(N,N,0) after green part with exponential N

4D example

4D example

Theorem III

There is a 4-VASS with shortest path of doubly exponential length

4D example

Theorem III

There is a 4-VASS with shortest path of doubly exponential length

Idea: enforce exact multiplications by fraction powers

4D example

Theorem III

There is a 4-VASS with shortest path of doubly exponential length

Idea: enforce exact multiplications by fraction powers

Needed: lemma about fraction powers

Fractional equation

Fractional equation

Lemma

Fractional equation

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For every k there are k fractions

$1 < a_1 / b_1 \leq \ldots \leq a_k / b_k$

such that

Fractional equation

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$$(a_1 / b_1)^{2^1} \cdot (a_2 / b_2)^{2^2} \cdot \ldots \cdot (a_k / b_k)^{2^k} = a / b$$

Fractional equation

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such that

$$(a_1 / b_1)^{2^1} \cdot (a_2 / b_2)^{2^2} \cdot \ldots \cdot (a_k / b_k)^{2^k} = a / b$$

and all a_i, b_i, a and b are at most exponential in k.

VASS building blocks

VASS building blocks

$$s(0,0,0,0) \longrightarrow s(K,K,0,0)$$

VASS building blocks

$s(0,0,0,0) \longrightarrow s(K,K,0,0)$ (+1,0,0,+1) in s

VASS building blocks

$s(0, 0, 0, 0) \longrightarrow s(K, K, 0, 0)$ $(+1,0,0,+1)$ in s

$t(K, Ka/b, 0, 0) \longrightarrow t(0, 0, 0, 0)$

VASS building blocks

$s(0, 0, 0, 0) \longrightarrow s(K, K, 0, 0)$ (+1,0,0,+1) in s

$t(K, Ka/b, 0, 0) \longrightarrow t(0, 0, 0, 0)$ (-a,-b,0,0) in t

VASS building blocks

$s(0, 0, 0, 0) \longrightarrow s(K, K, 0, 0)$ (+1,0,0,+1) in s

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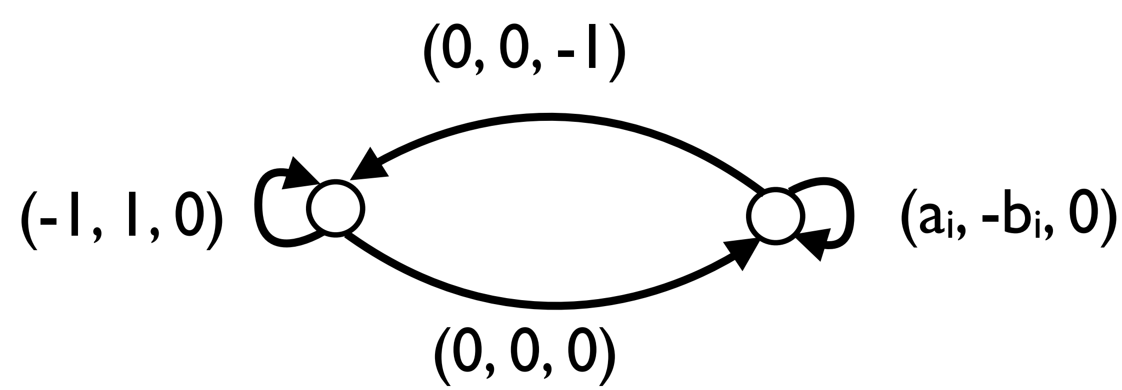

VASS building blocks

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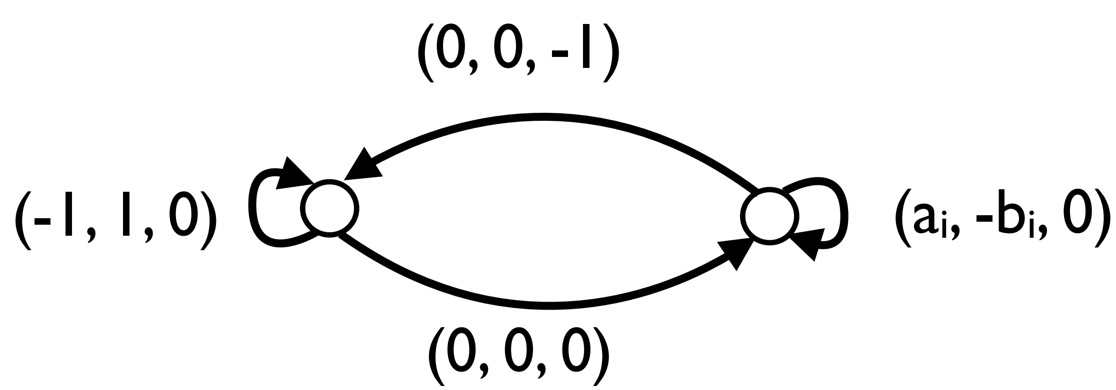
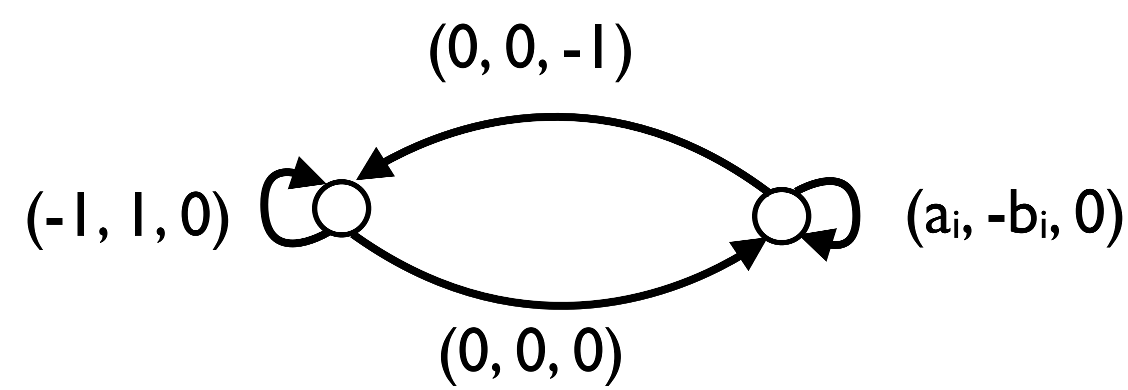


$(Cb_i^n, 0, n) \longrightarrow (Ca_ib_i^{n-1}, 0, n-1) \longrightarrow \ldots \longrightarrow (Ca_i^n, 0, 0)$

VASS building blocks

$s(0, 0, 0, 0) \longrightarrow s(K, K, 0, 0)$ $(+1,0,0,+1)$ in s

$t(K, Ka/b, 0, 0) \longrightarrow t(0, 0, 0, 0)$ $(-a,-b,0,0)$ in t

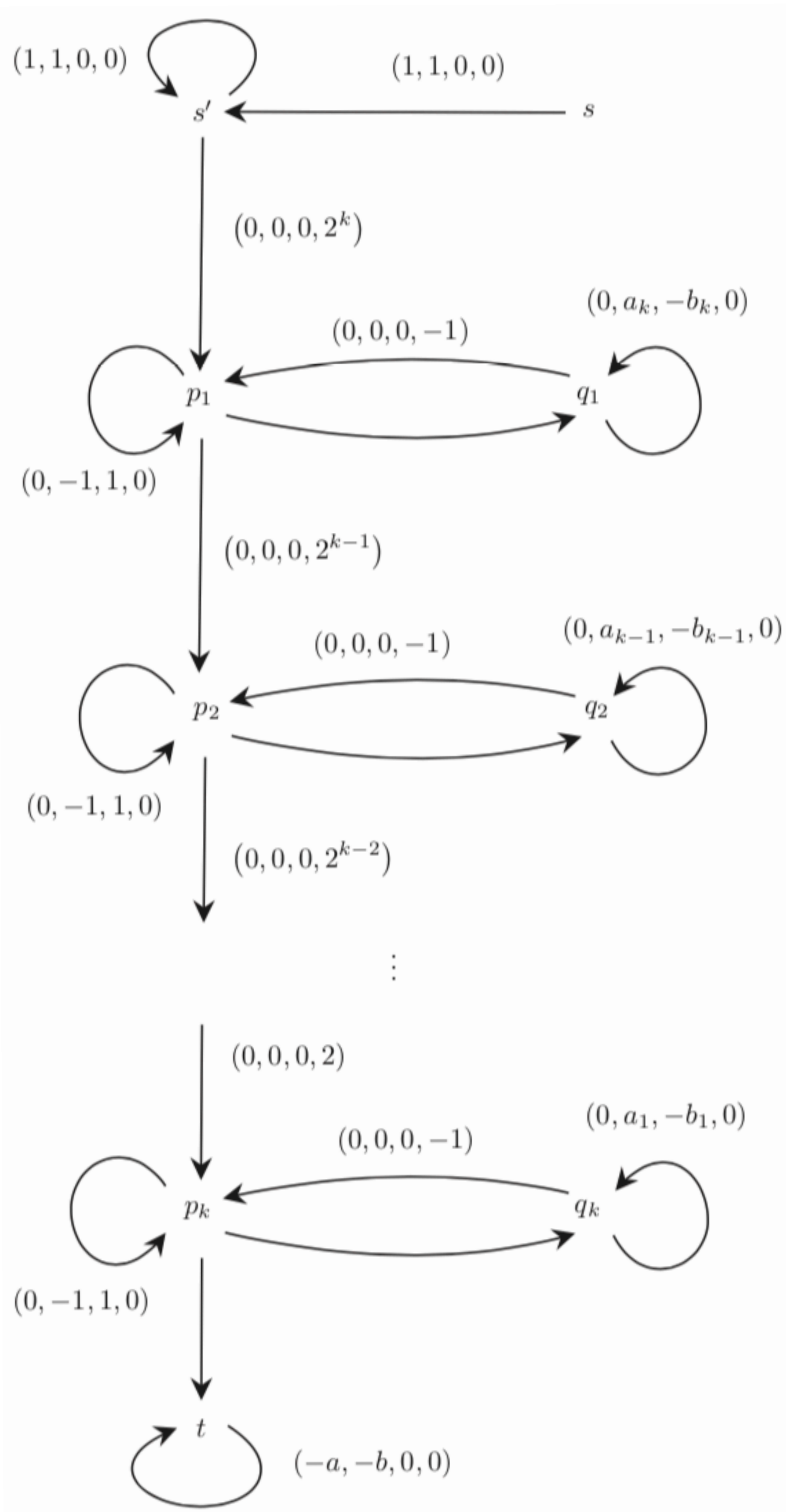


$$(Cb_i^n, 0, n) \longrightarrow (Ca_ib_i^{n-1}, 0, n-1) \longrightarrow \ldots \longrightarrow (Ca_i^n, 0, 0)$$

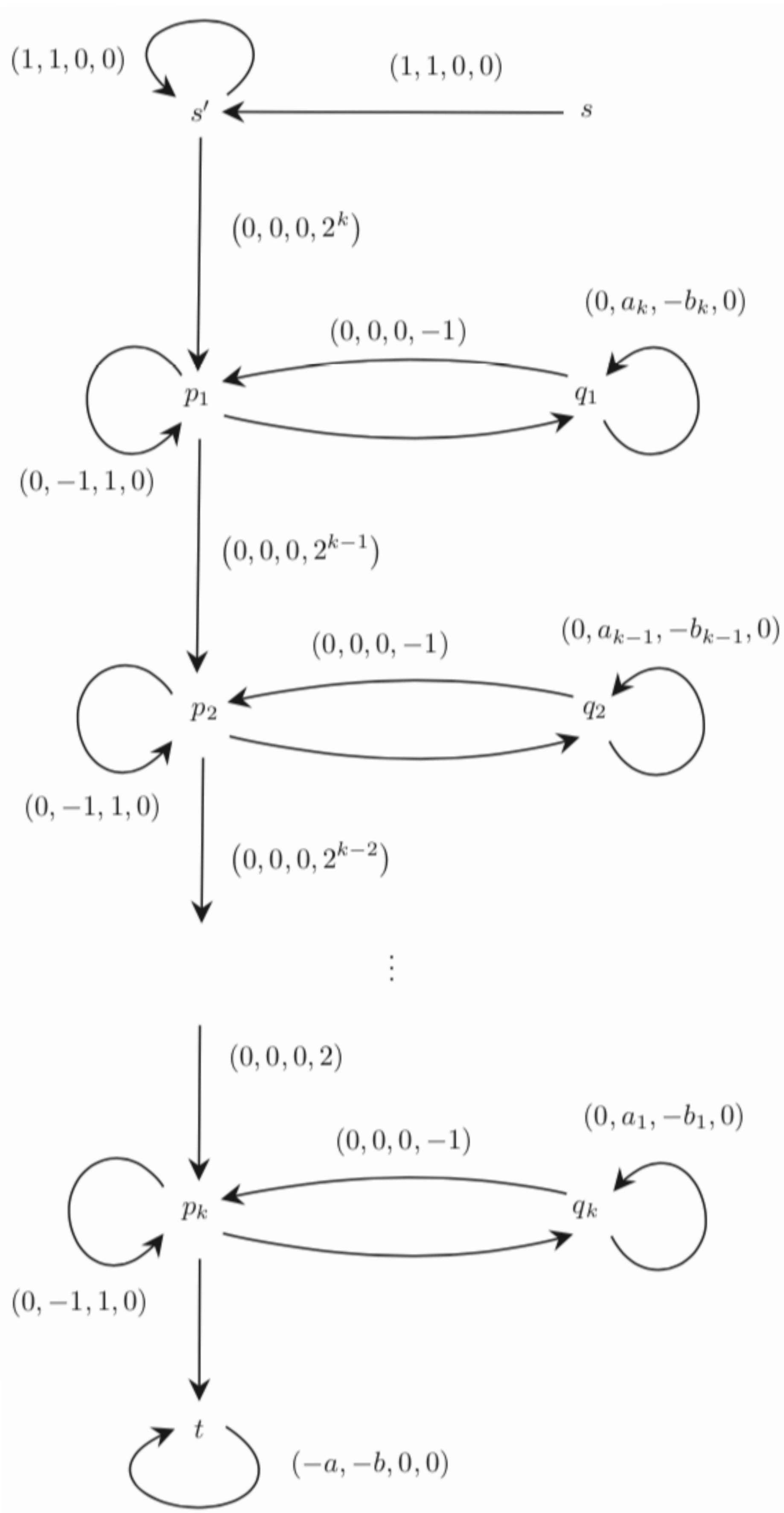
one can multiply by at most n times by at most a_i / b_i

VASS implementation

VASS implementation

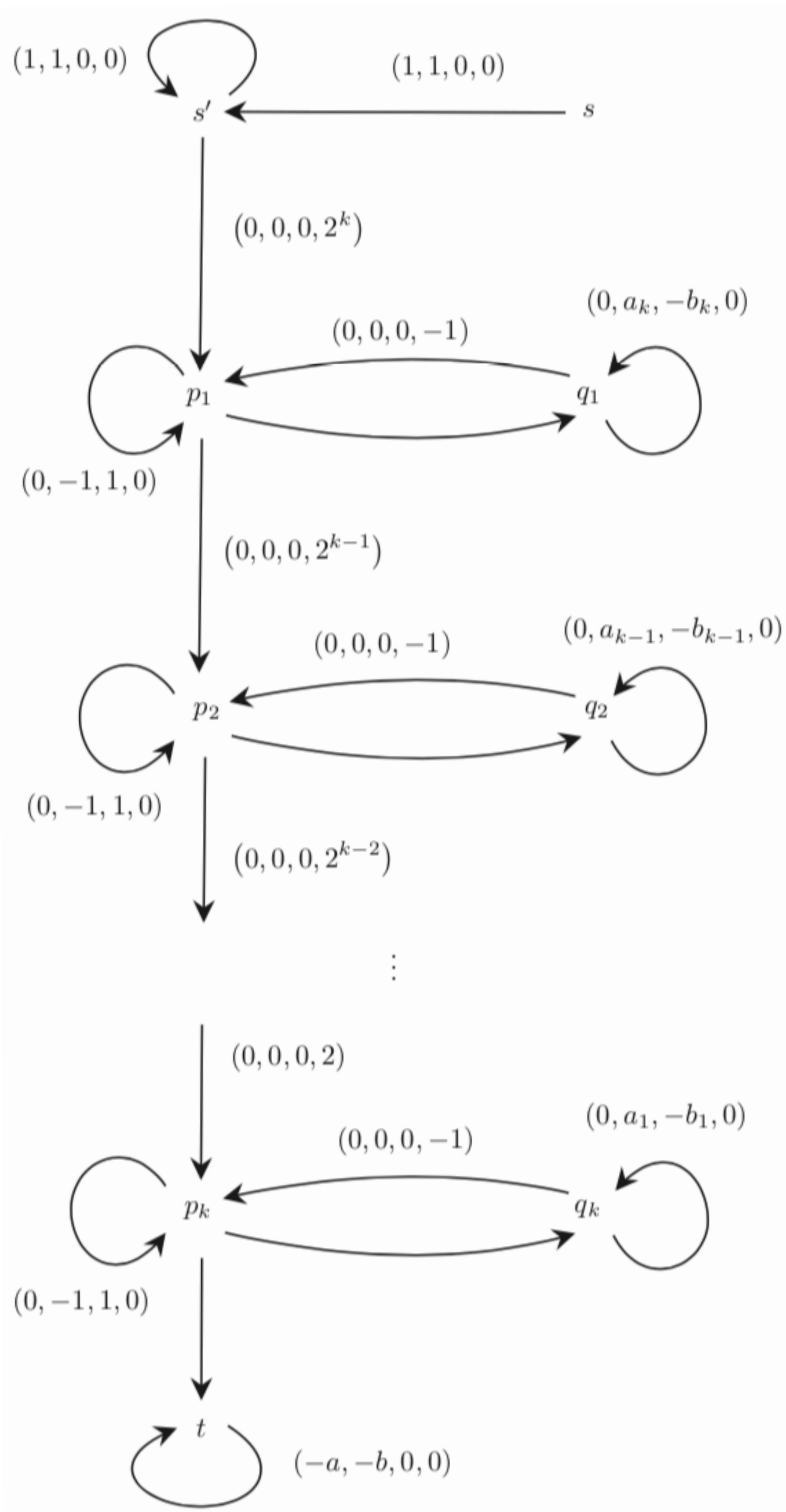


VASS implementation



can I reach t(0,0,0,0) from s(0,0,0,0)?

VASS implementation



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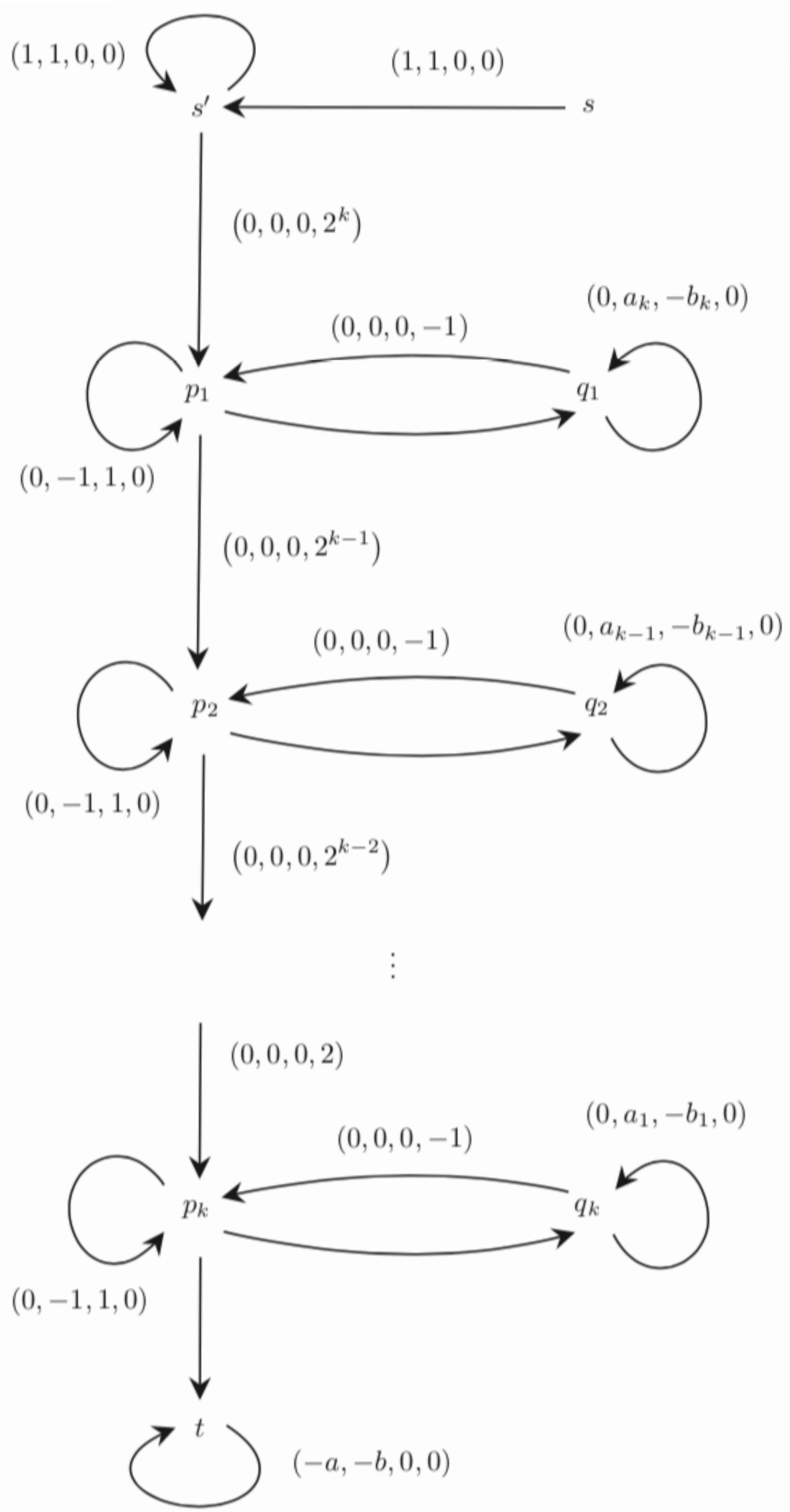
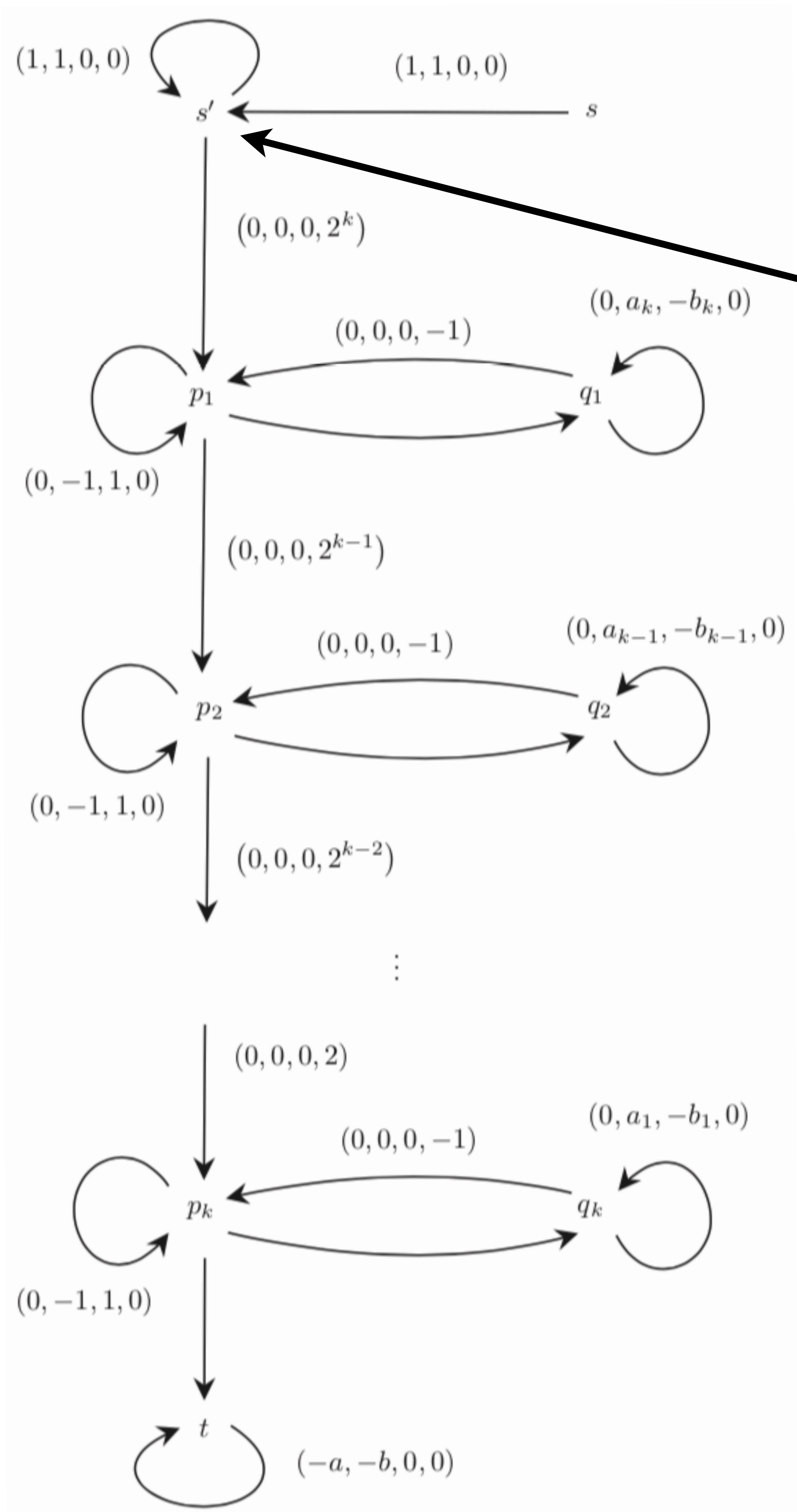


$$(a_1 / b_1)^{2^1} \cdot (a_2 / b_2)^{2^2} \cdot \ldots \cdot (a_k / b_k)^{2^k} = a / b$$

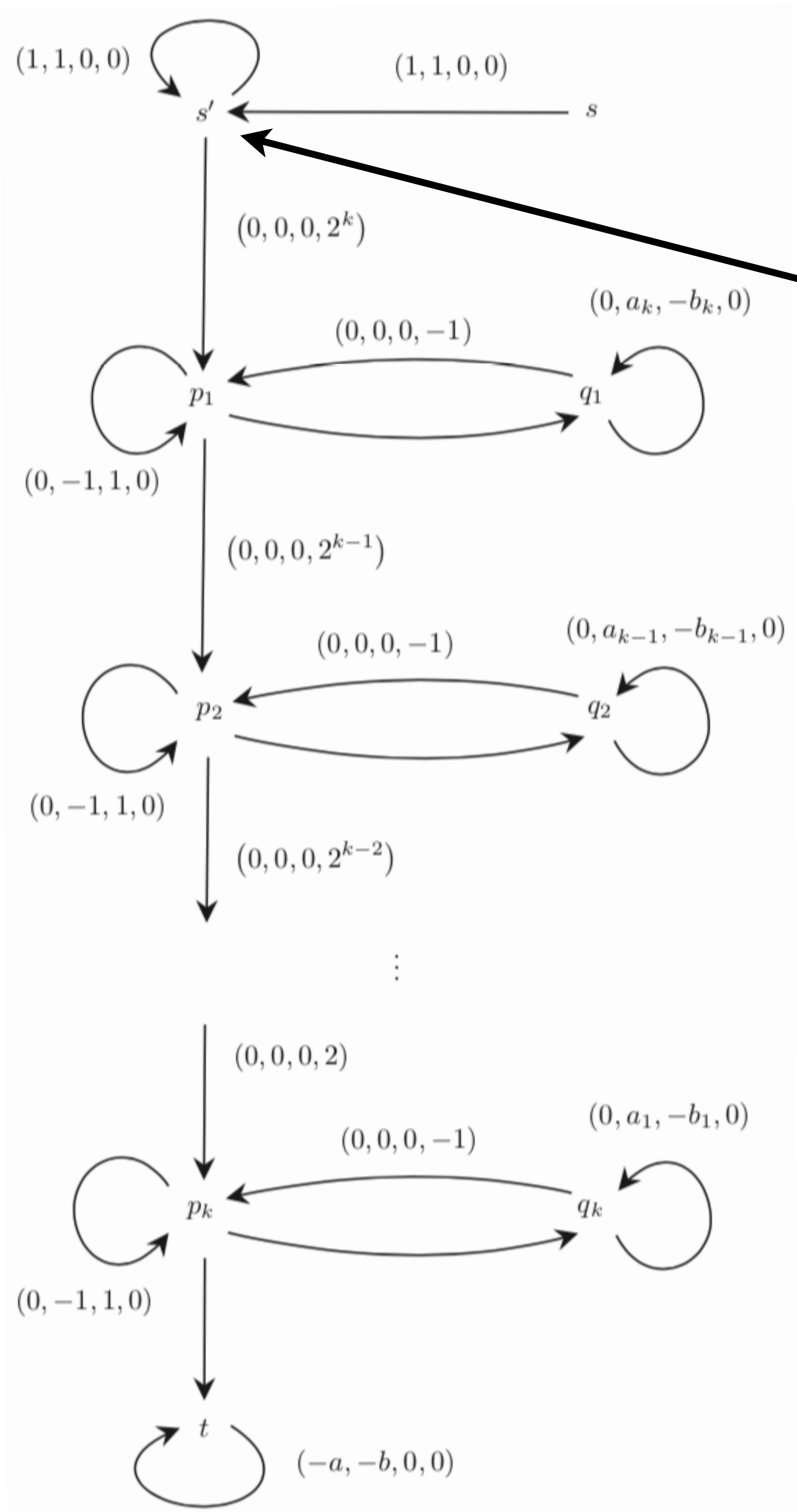
VASS implementation



$(a_1 / b_1)^{2^1} \cdot (a_2 / b_2)^{2^2} \cdot \ldots \cdot (a_k / b_k)^{2^k} = a / b$

s'(**K**,**K**,0,0)

VASS implementation



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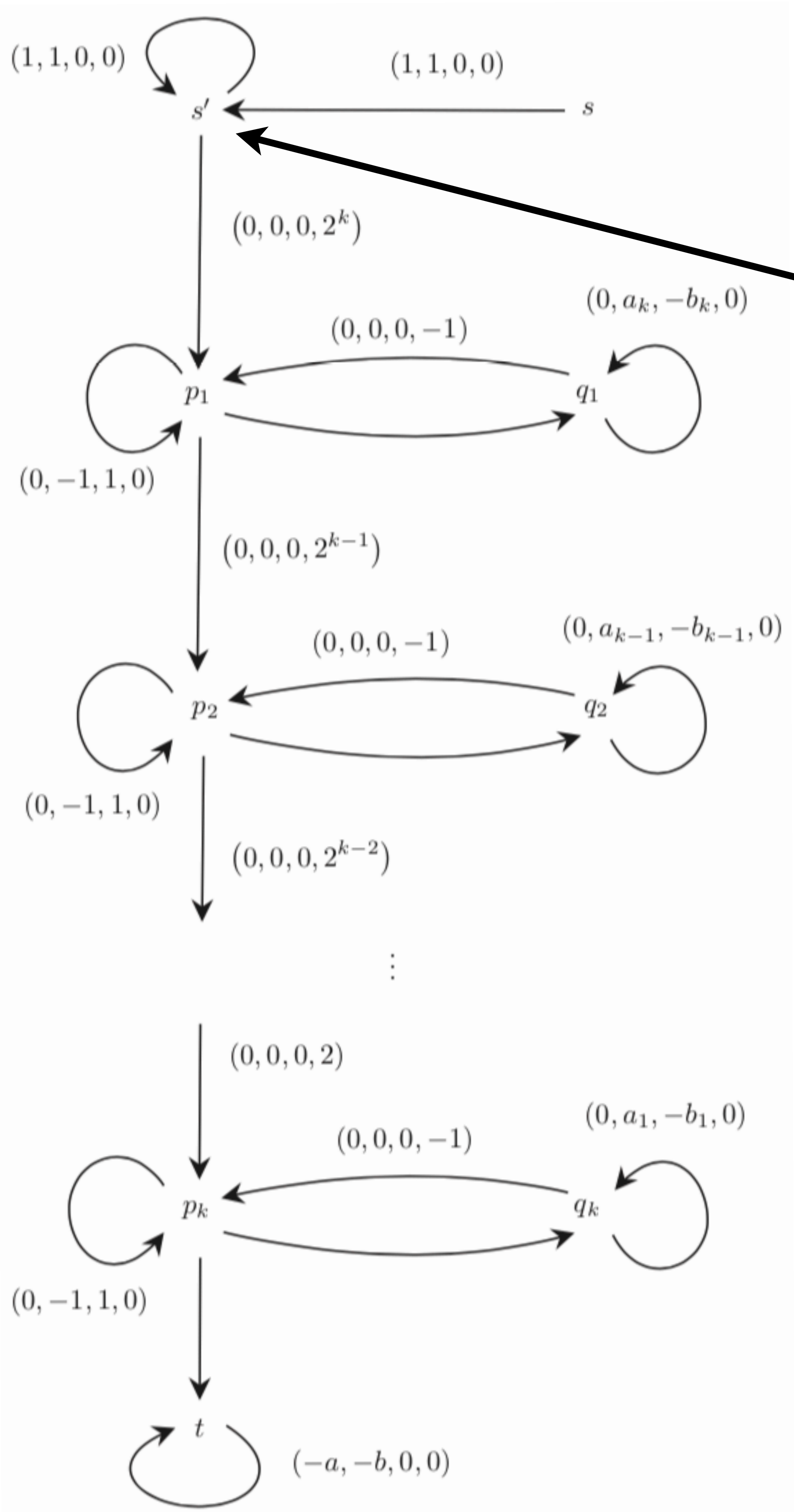
s'(**K**,**K**,0,0)

multiplies **K** at most 2^k times by $\leq a_k / b_k$

VASS implementation



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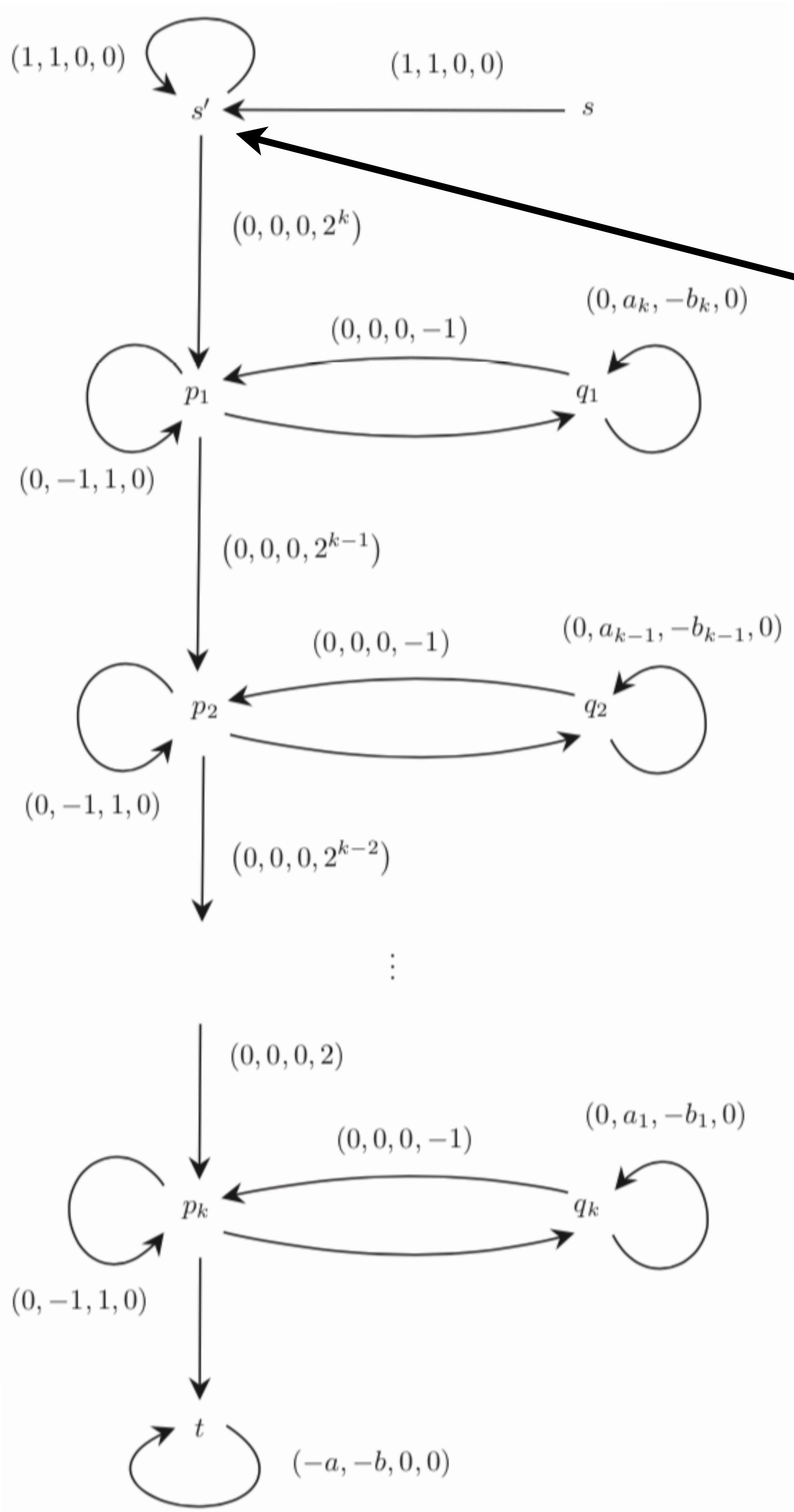
VASS implementation



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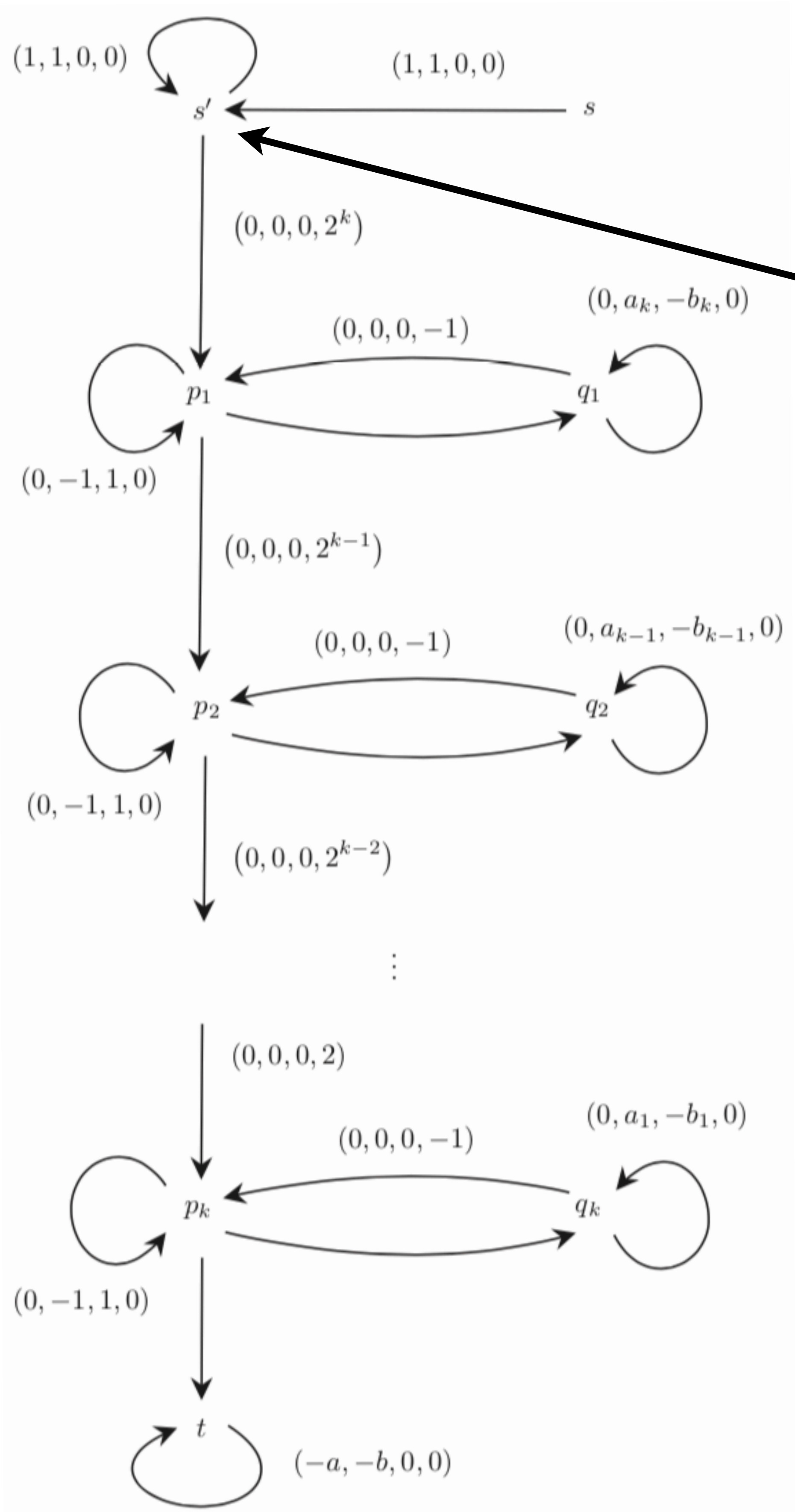
multiplies **K** at most 2^k times by $\leq a_k / b_k$

...

multiplies **it** at most 2^1 times by $\leq a_1 / b_1$

VASS implementation



$(a_1 / b_1)^{2^1} \cdot (a_2 / b_2)^{2^2} \cdot \ldots \cdot (a_k / b_k)^{2^k} = a / b$

s'(**K**,**K**,0,0)

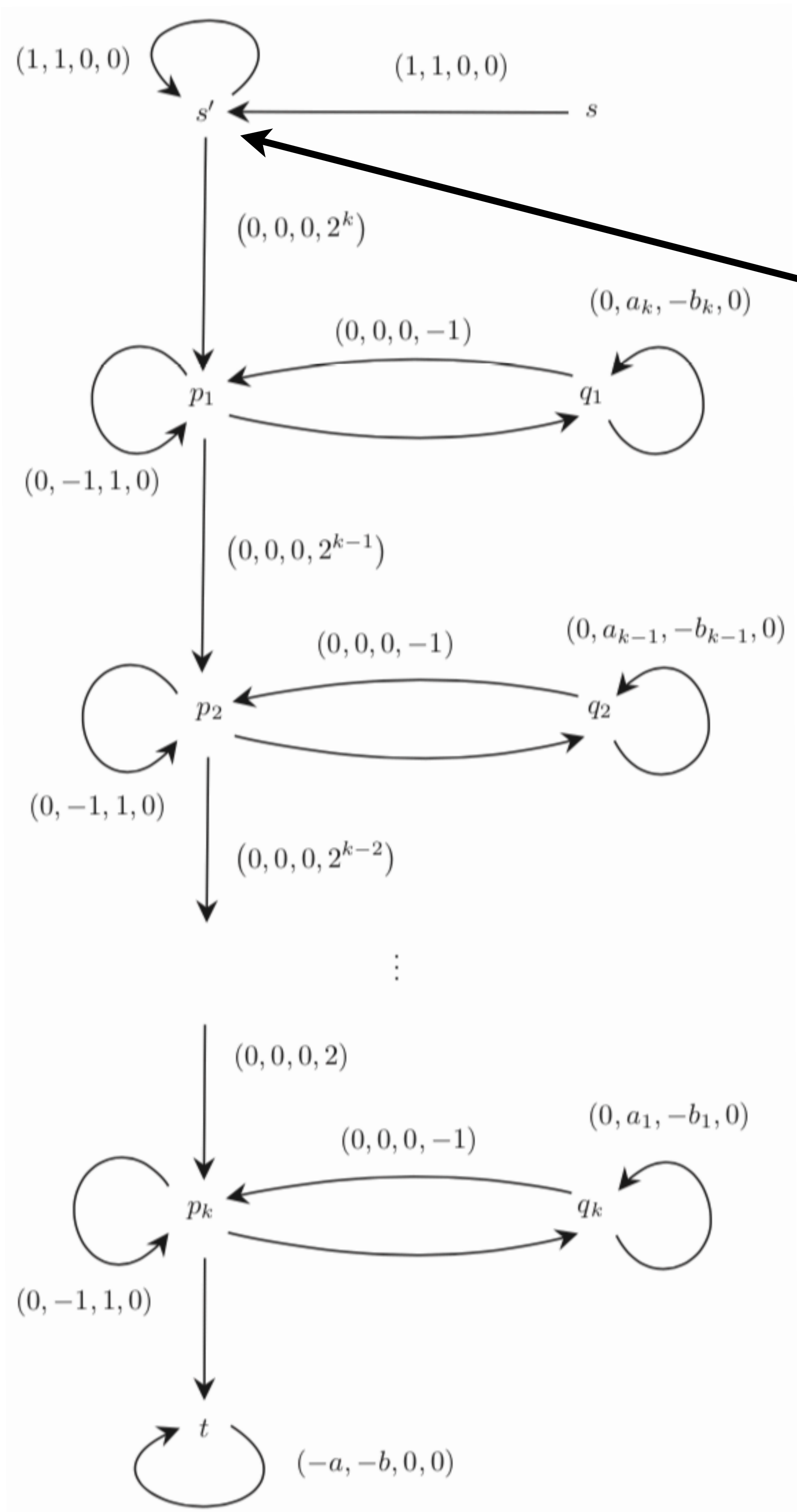
multiplies **K** at most 2^k times by $\leq a_k / b_k$

...

multiplies **it** at most 2^1 times by $\leq a_1 / b_1$

t(**K**,≤**K**a/b,≥0,≥0)

VASS implementation



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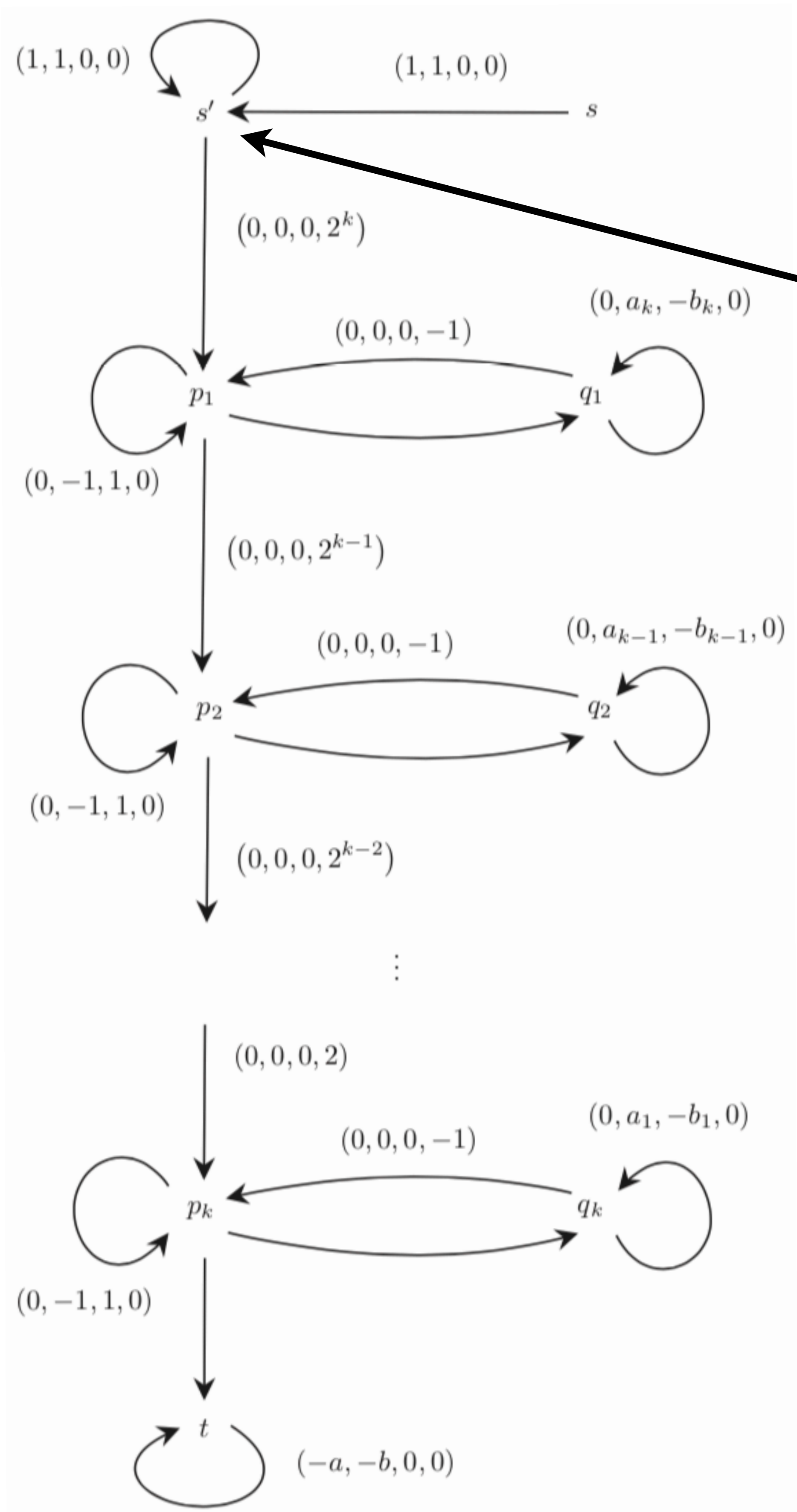
...

multiplies **it** at most 2^1 times by $\leq a_1 / b_1$

subtracts from **it** **K** a/b   t(**K**,≤**K**a/b,≥0,≥0)

VASS implementation



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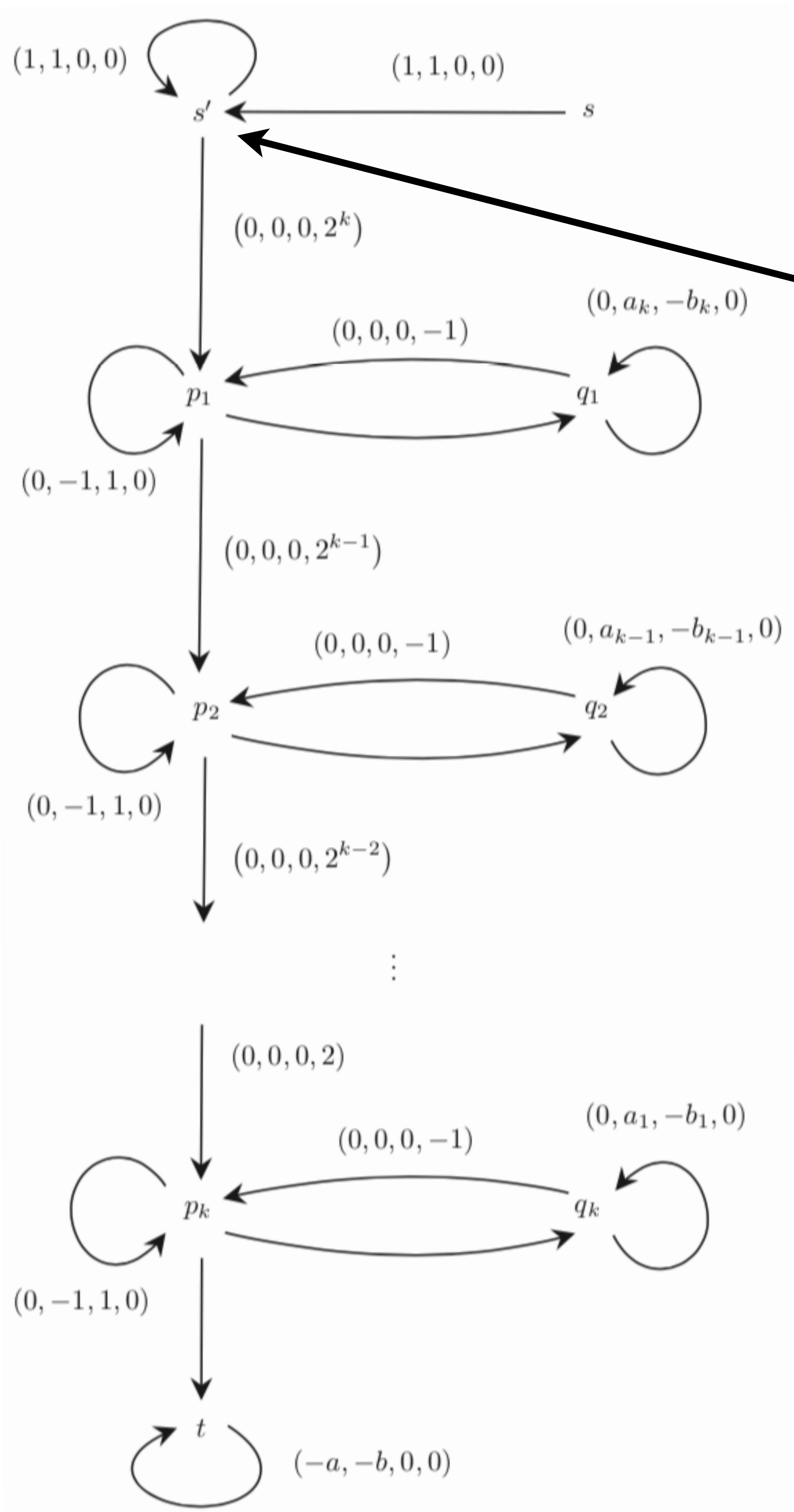
subtracts from **it** **K** a/b   t(**K**,≤**K**a/b,≥0,≥0)

K divisible by $b_k{}^{2^k}$

VASS implementation



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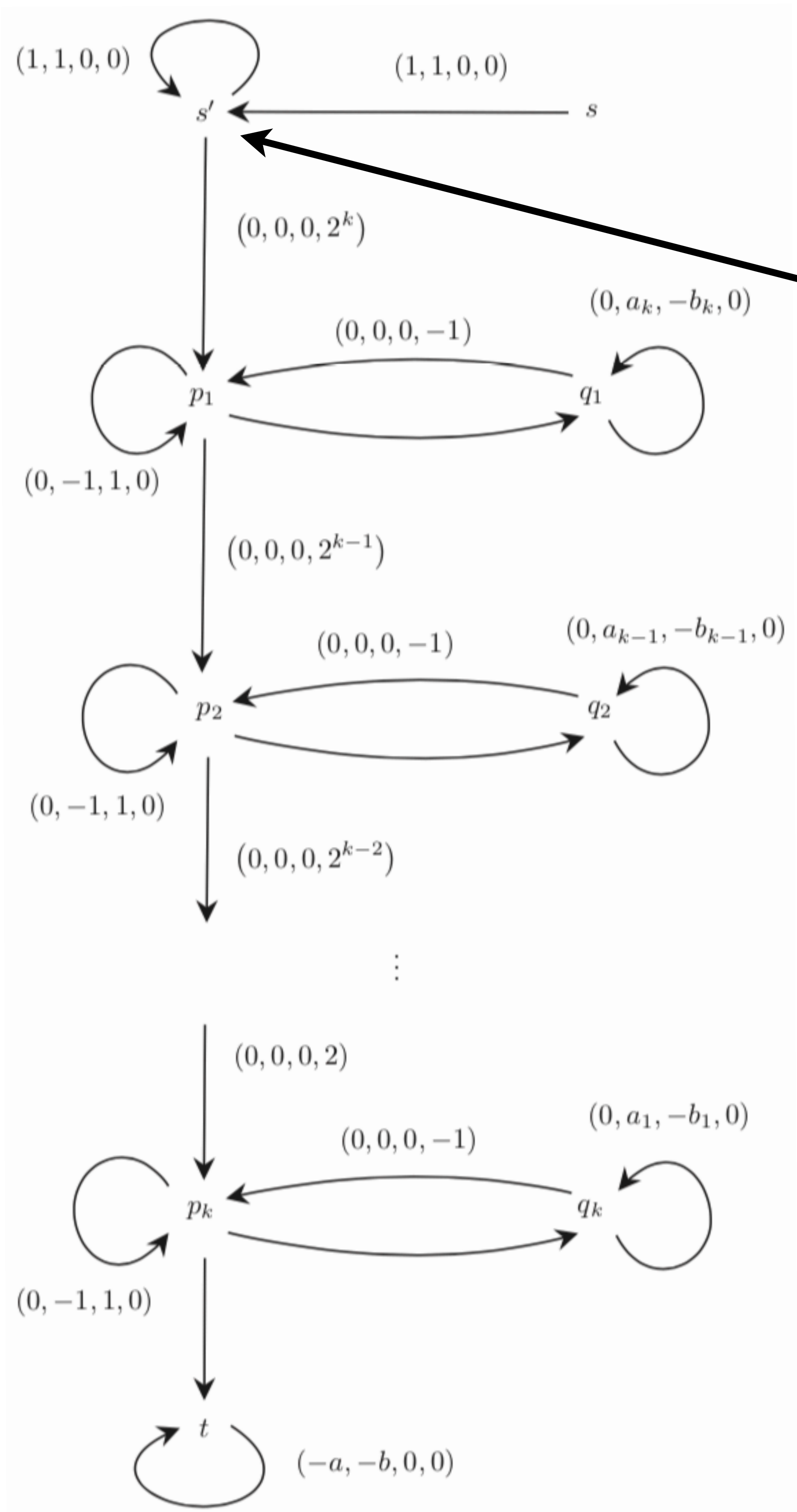
VASS implementation



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subtracts from **it** **K** a/b   t(**K**,≤**K**a/b,≥0,≥0)

K doubly exponential

Open problem

Open problem

Reachability for finite 3-VASSes

Open problem

Reachability for finite 3-VASSes

PSpace?

Open problem

Reachability for finite 3-VASSes

PSpace? Tower?

Open problem

Reachability for finite 3-VASSes

PSpace? Tower?

Conjecture: PSpace

Thank you!