The Reachability Problem for Vector Addition Systems

Wojciech Czerwiński

Aalborg 2022

Plan

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- basic notions and the problem

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- basic notions and the problem
- short history

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- interesting examples

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- decidability (idea)

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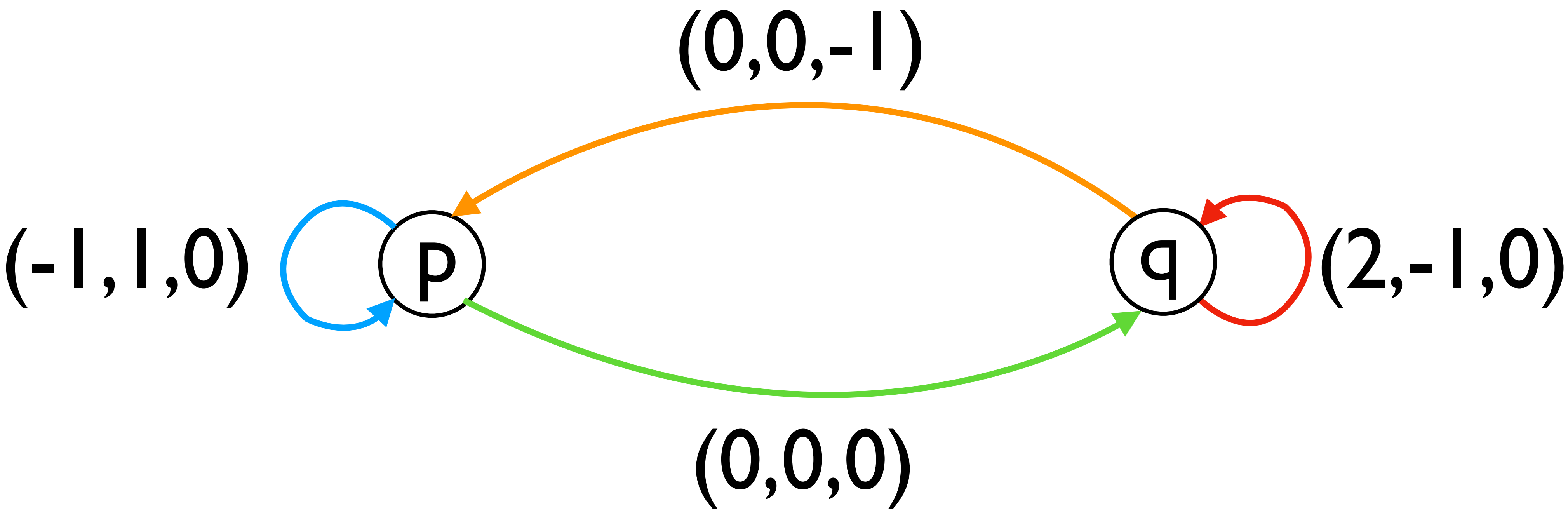
- open problems
- goal: intuitions

Vector Addition Systems with States

Vector Addition Systems with States



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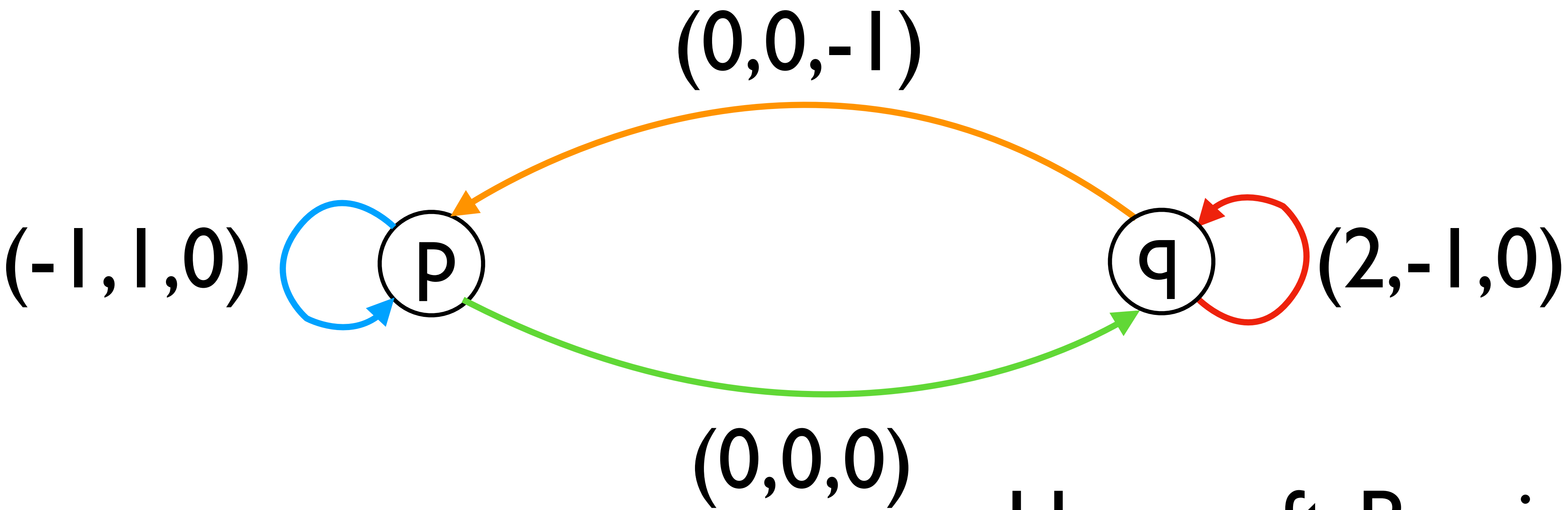

Hopcroft-Pansiot `78

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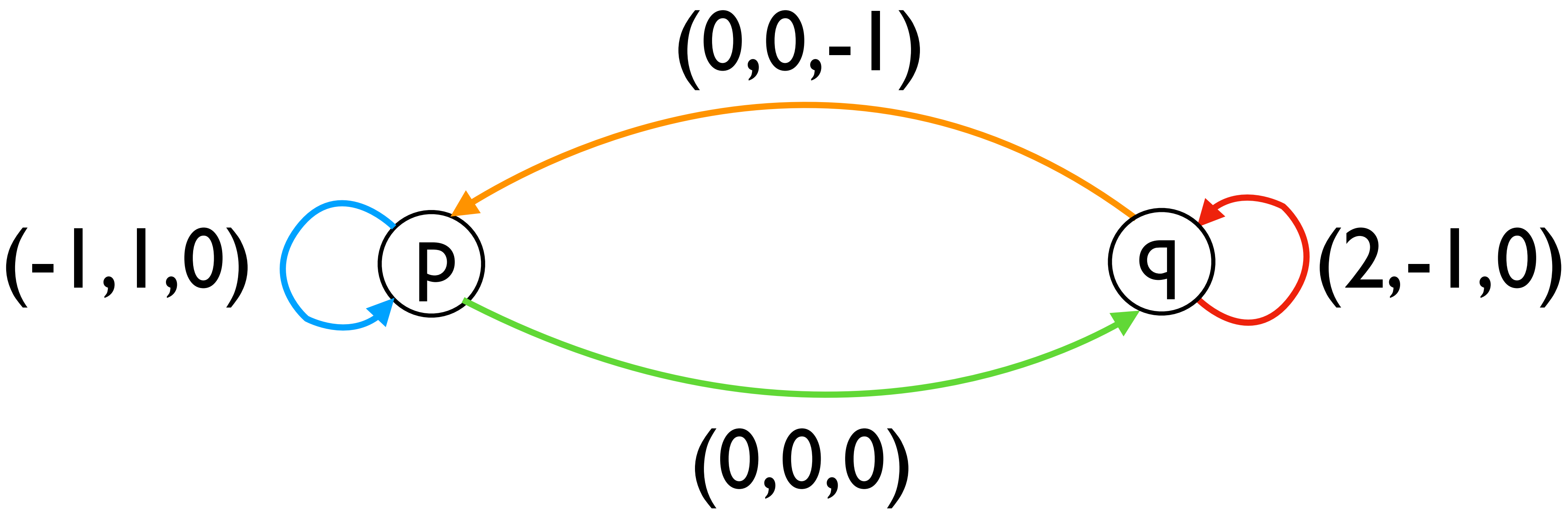

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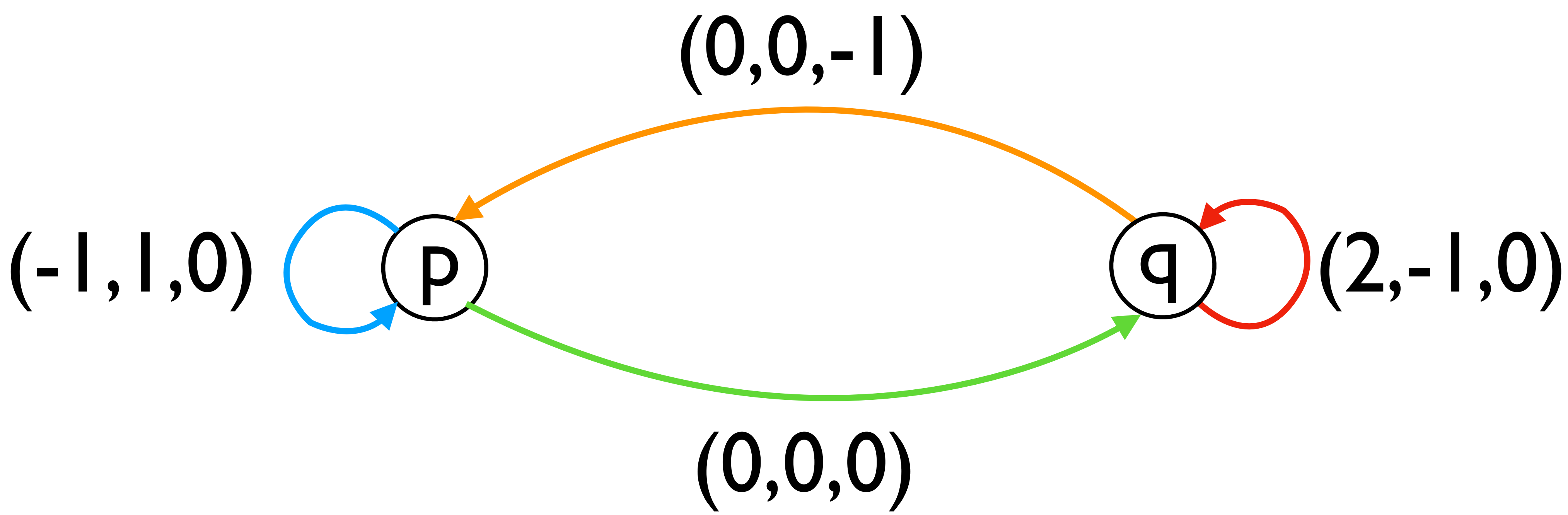
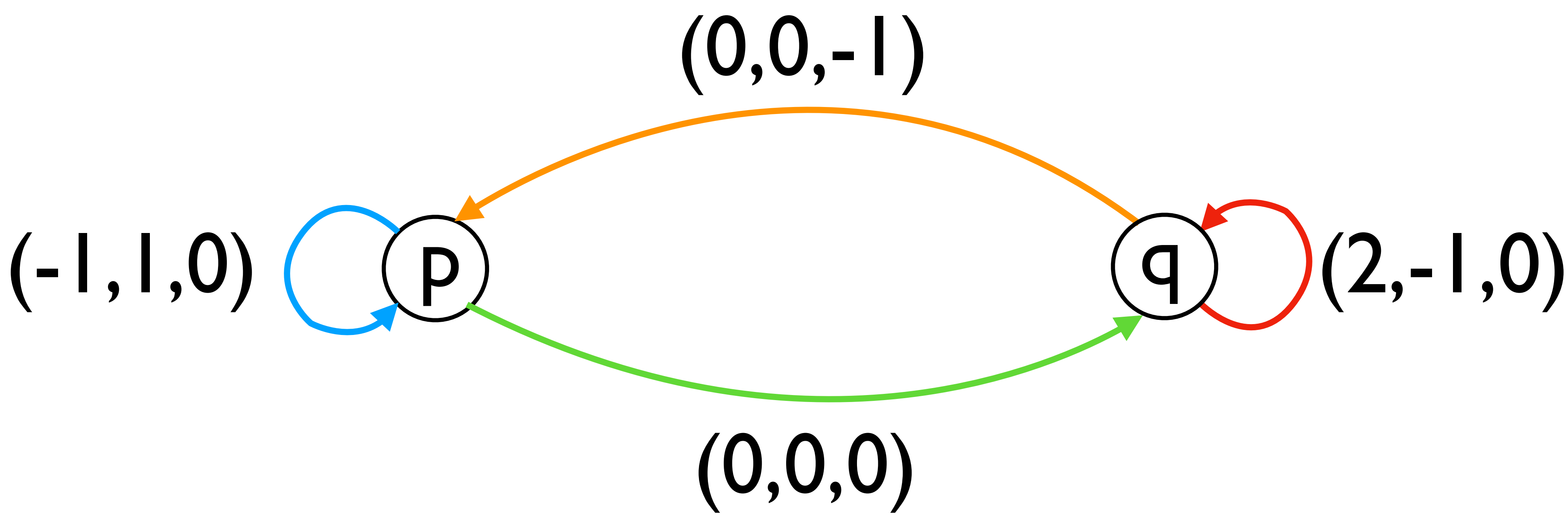


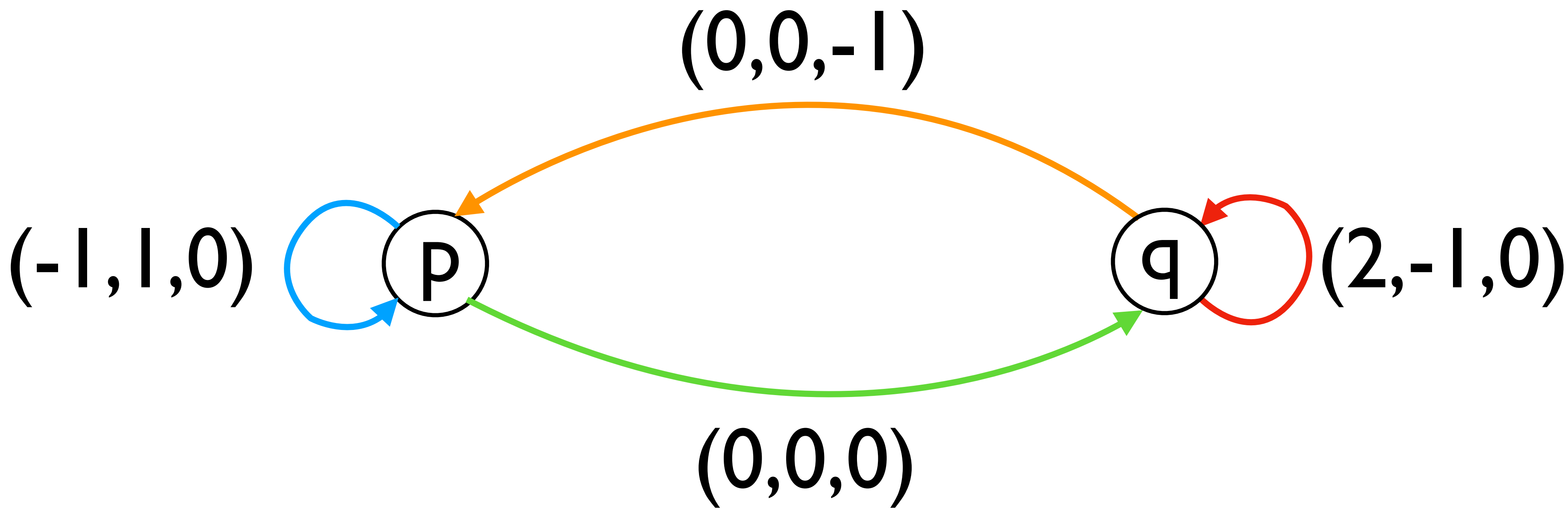
p(2,0,7)

Vector Addition Systems with States



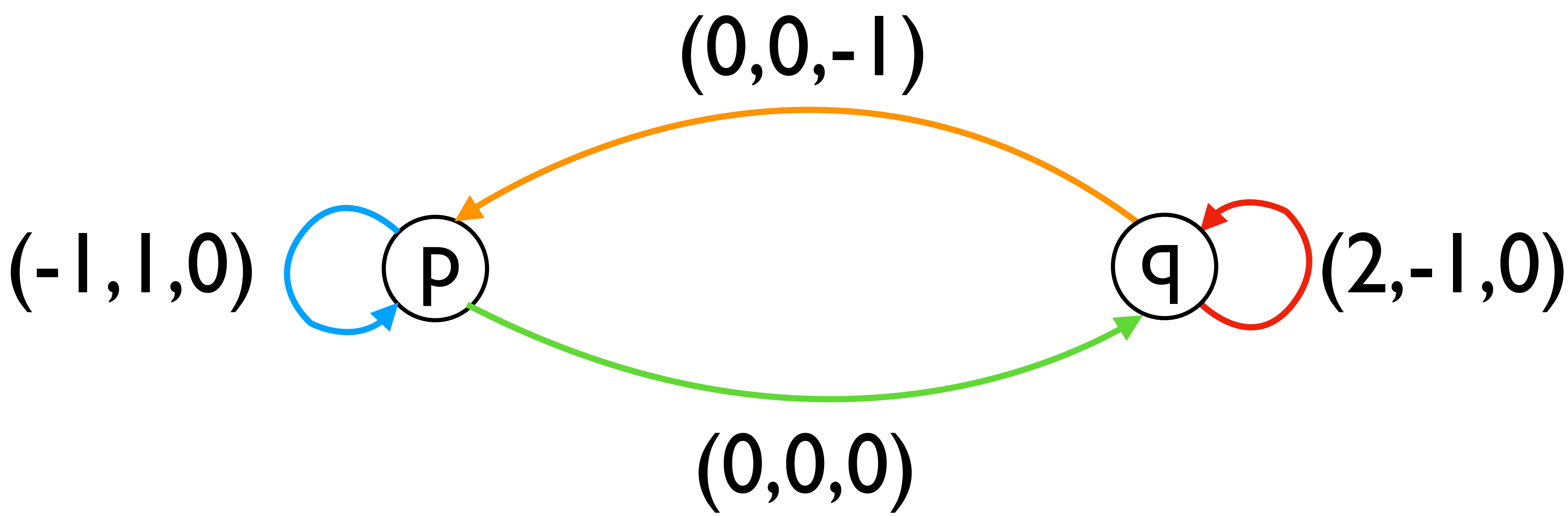
p(2,0,7) → p(1,1,7)

Vector Addition Systems with States



p(2,0,7) → p(1,1,7) → p(0,2,7)

Vector Addition Systems with States



p(2,0,7) → p(1,1,7) → p(0,2,7) → q(0,2,7)

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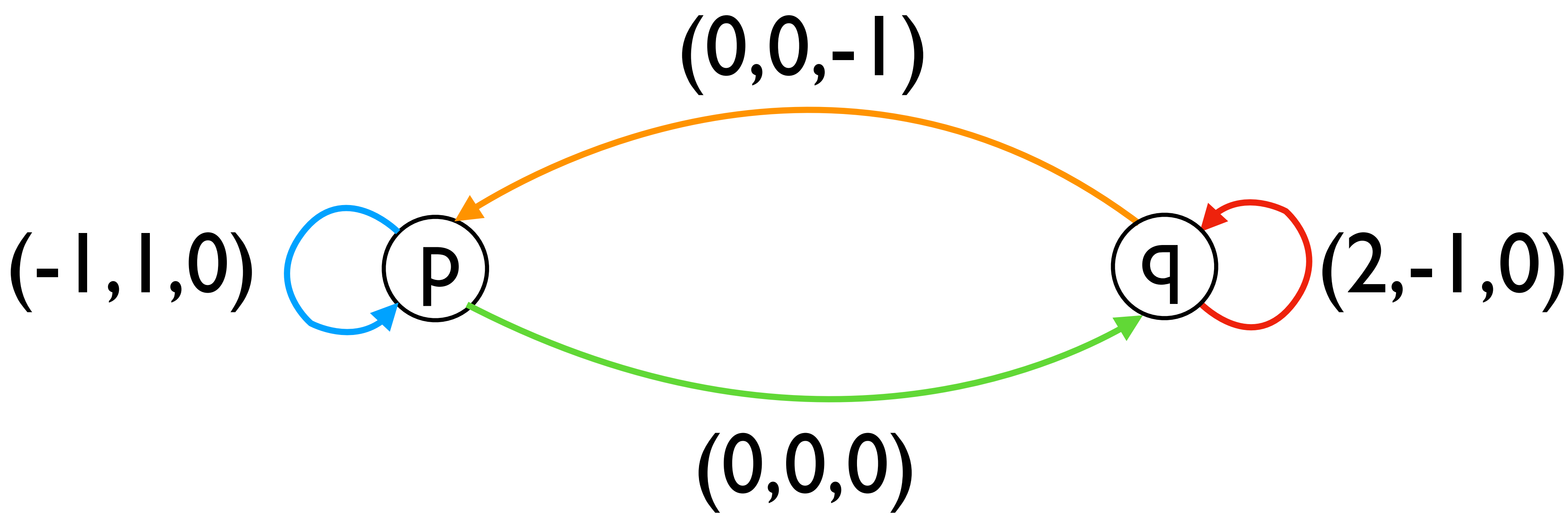

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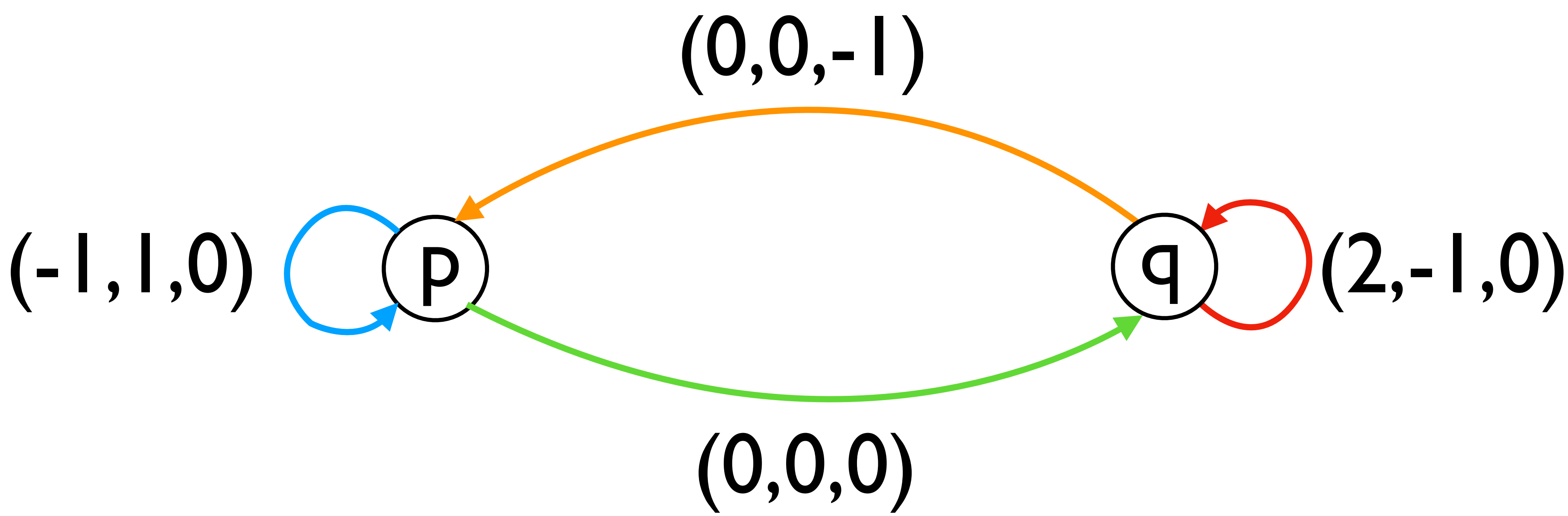
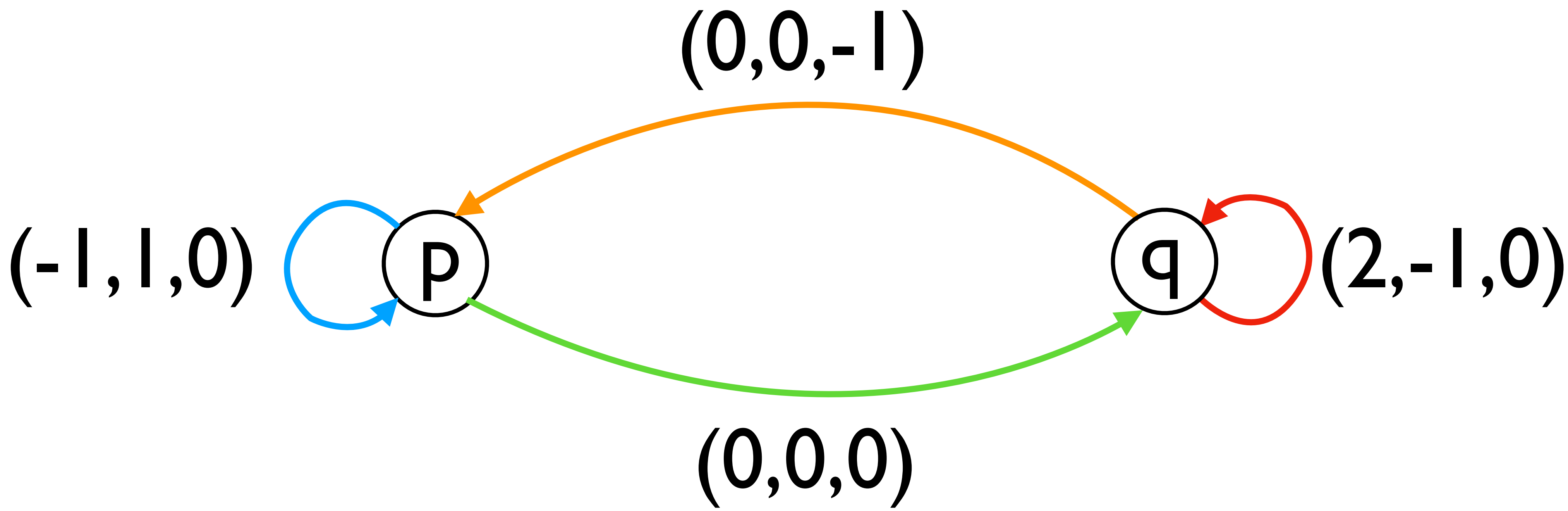


p(2,0,7) → p(1,1,7) → p(0,2,7) → q(0,2,7) → q(2,1,7)

→ q(4,0,7)

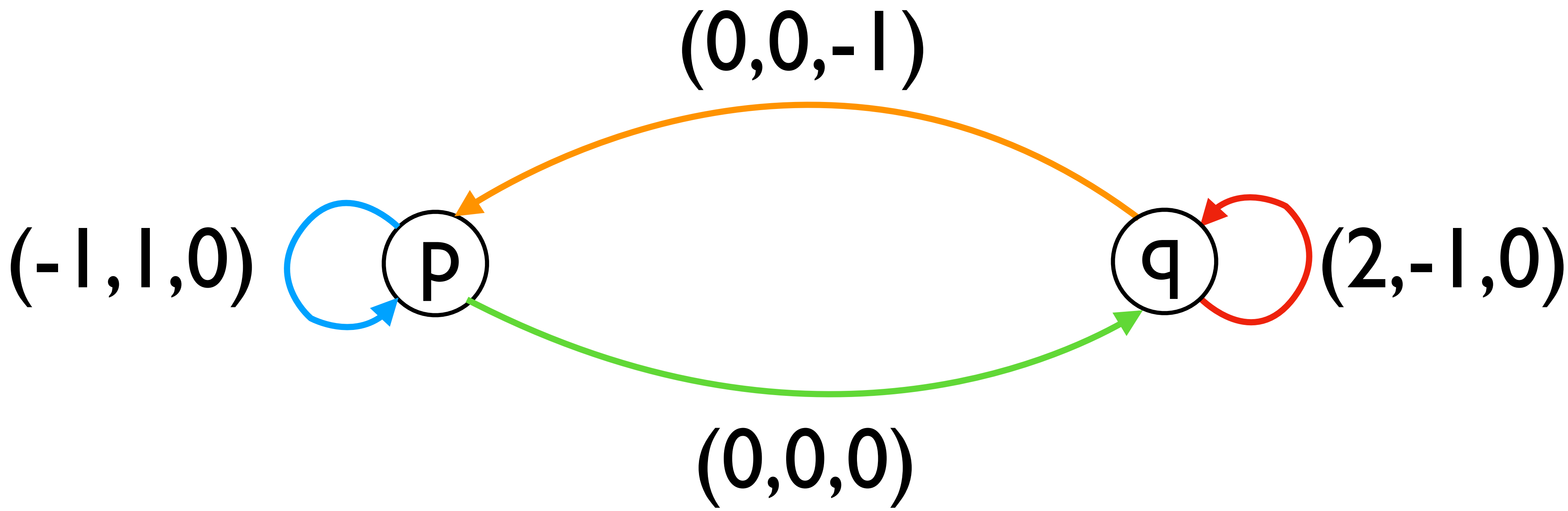
Vector Addition Systems with States



p(2,0,7) → p(1,1,7) → p(0,2,7) → q(0,2,7) → q(2,1,7)

→ q(4,0,7) → p(4,0,6)

Vector Addition Systems with States



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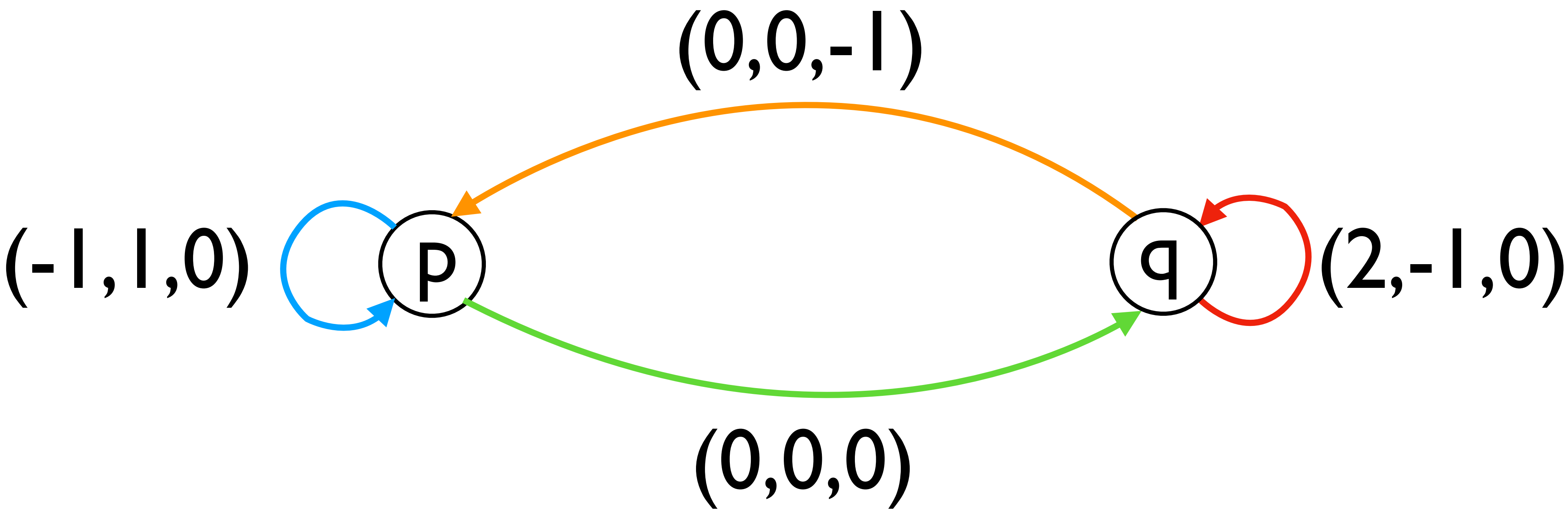
Petri nets

Vector Addition Systems with States



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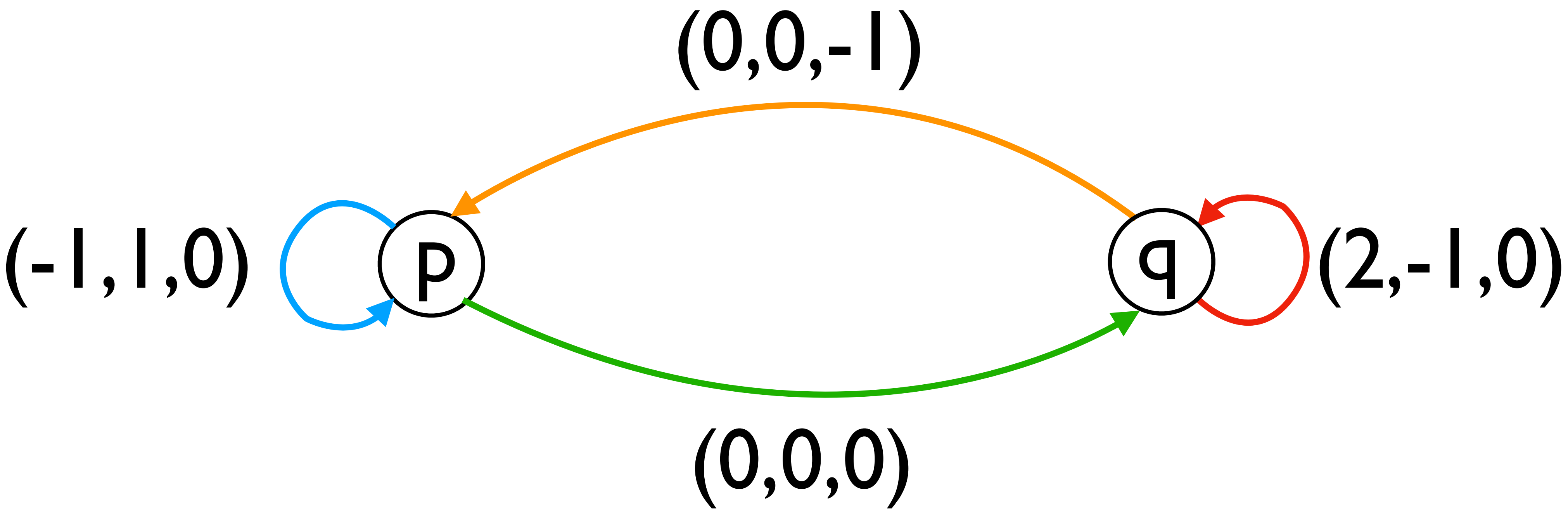

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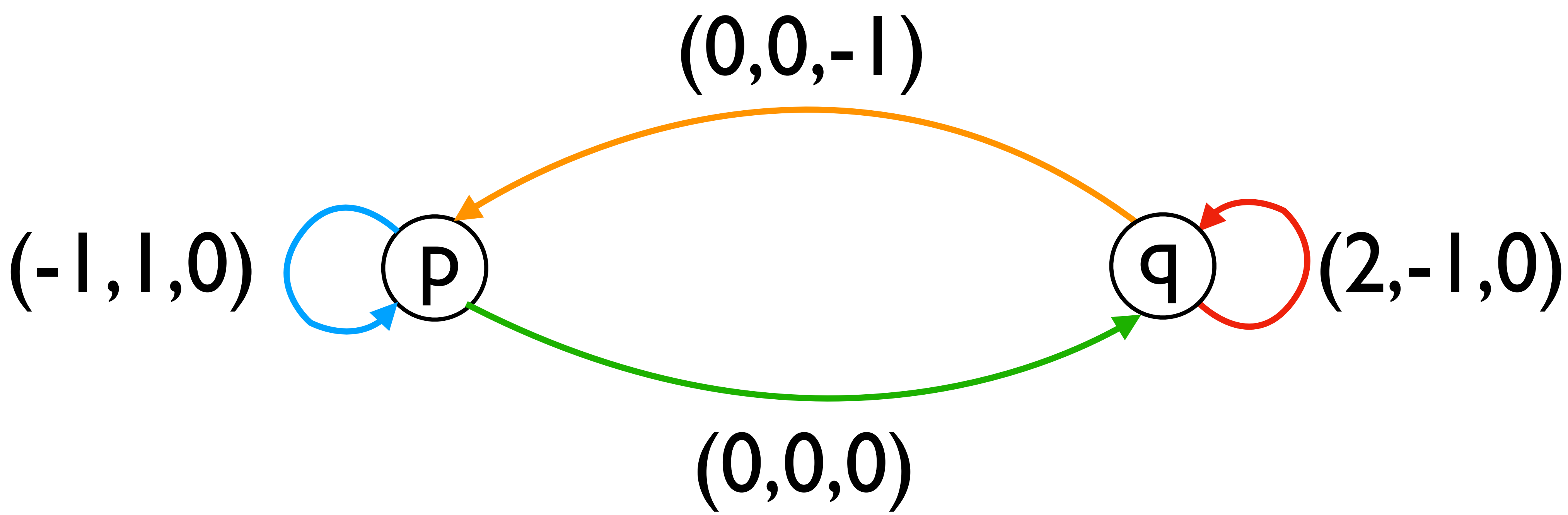
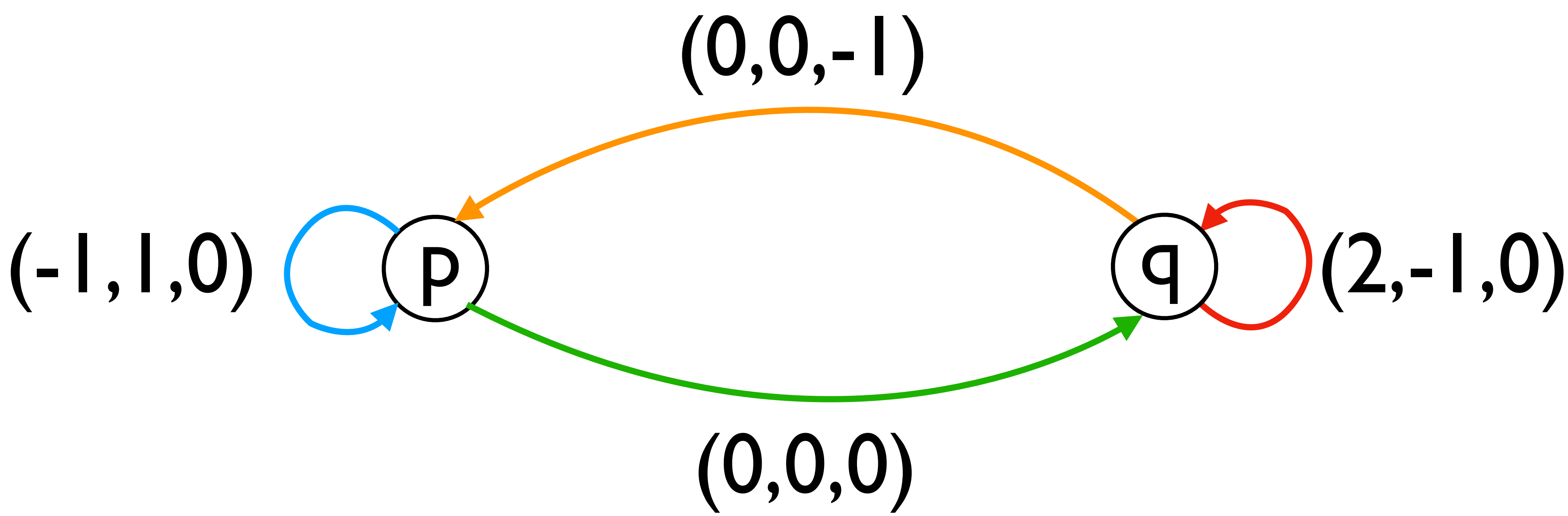


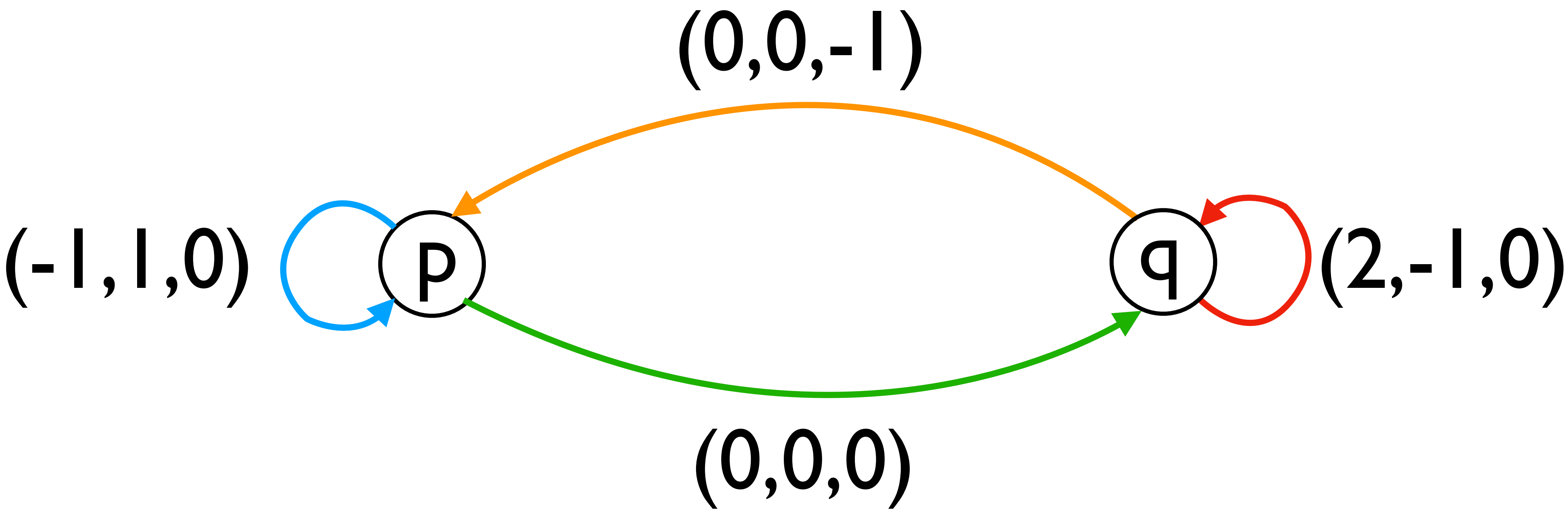
p(k,0,n)

Vector Addition Systems with States



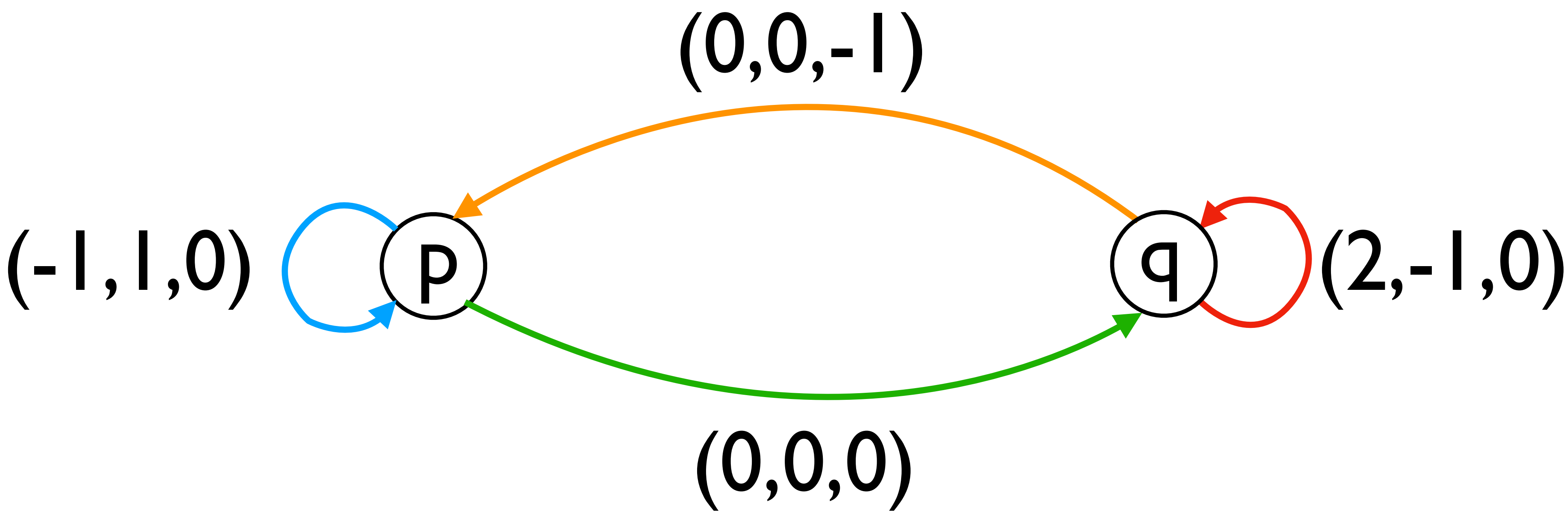
p(k,0,n) ⟶ p(0,k,n)

Vector Addition Systems with States



$p(k,0,n) \longrightarrow p(0,k,n) \longrightarrow q(0,k,n)$

Vector Addition Systems with States



p(k,0,n) → p(0,k,n) → q(0,k,n) → q(2k,0,n)

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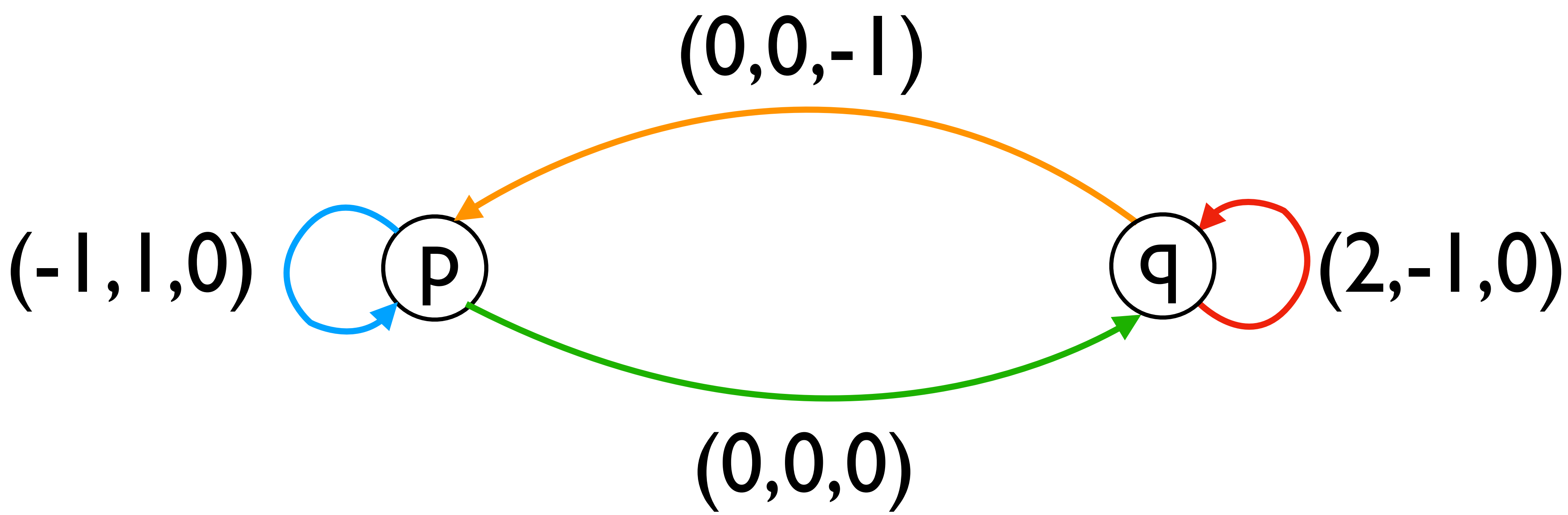

p(k,0,n) → p(0,k,n) → q(0,k,n) → q(2k,0,n) → p(2k,0,n-1)

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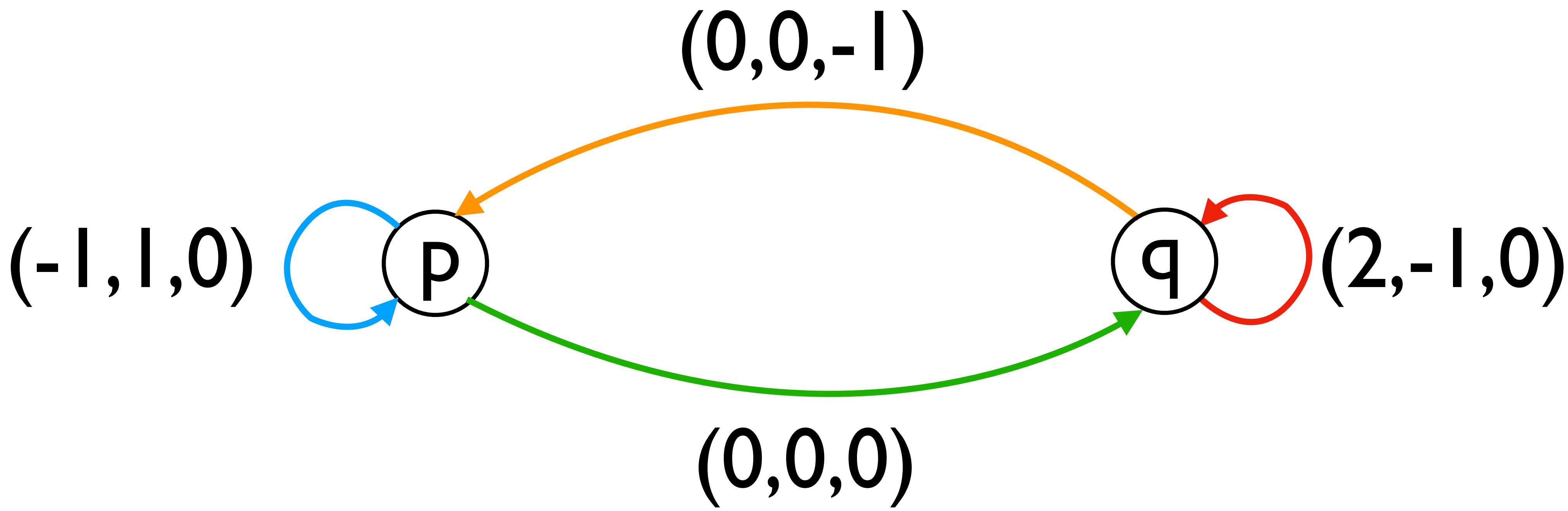

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p(1,0,n)

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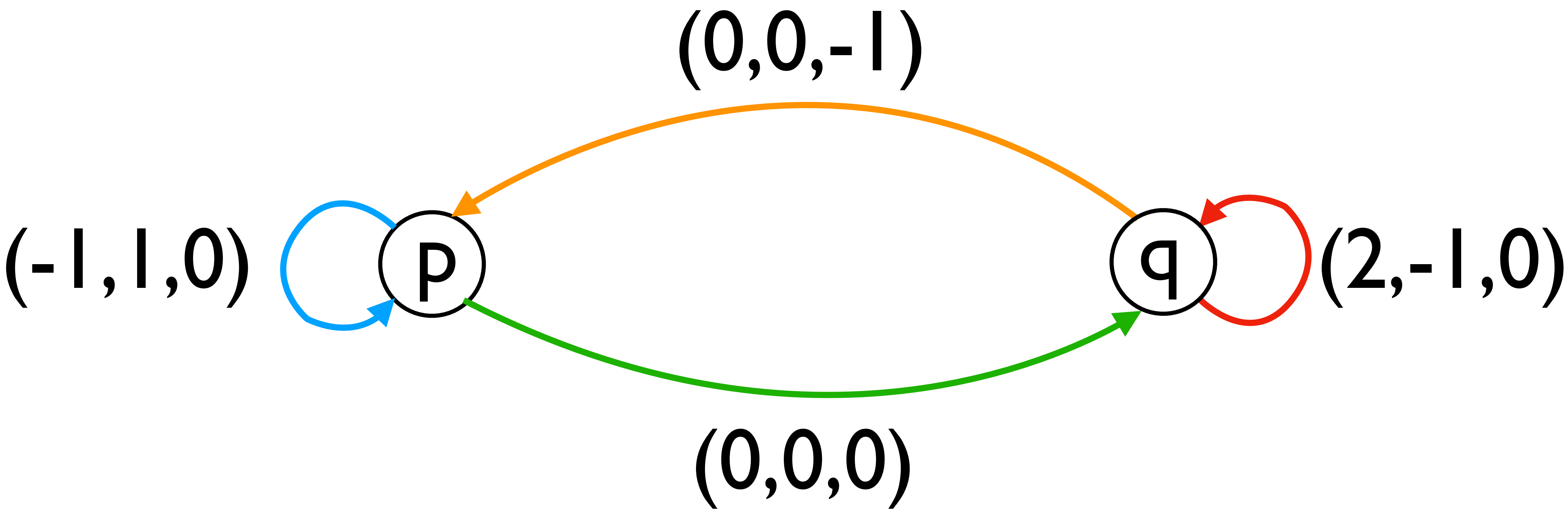

$p(k,0,n) \rightarrow p(0,k,n) \rightarrow q(0,k,n) \rightarrow q(2k,0,n) \rightarrow p(2k,0,n-1)$

$p(1,0,n) \longrightarrow p(2,0,n-1)$

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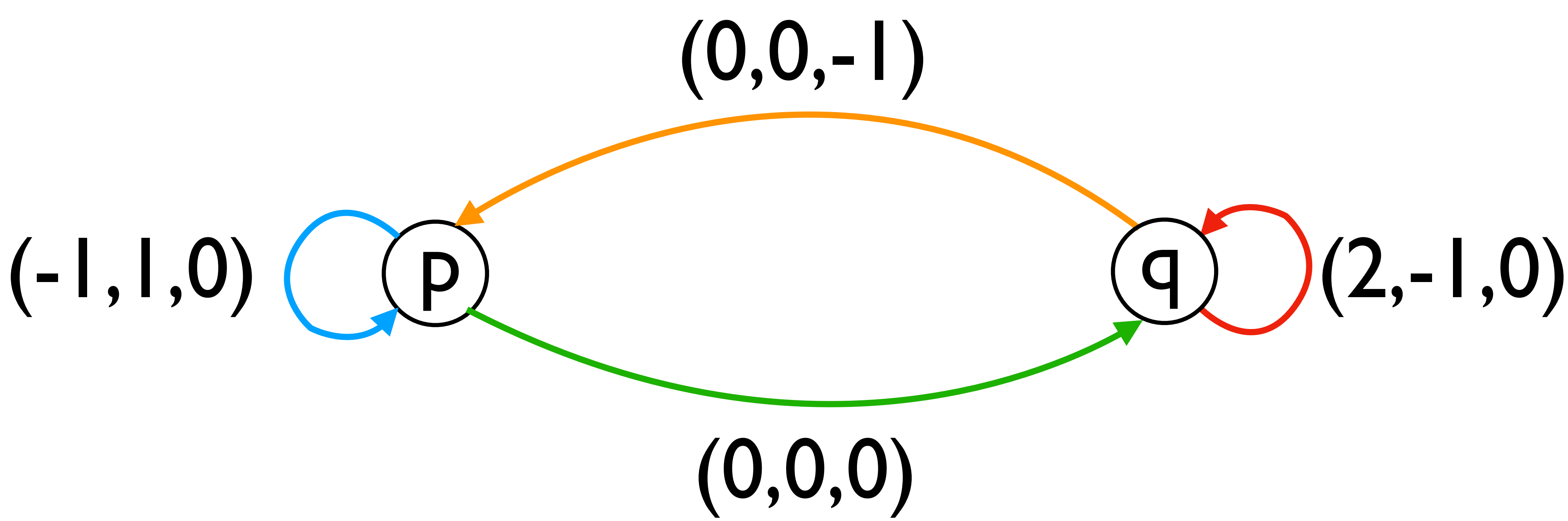

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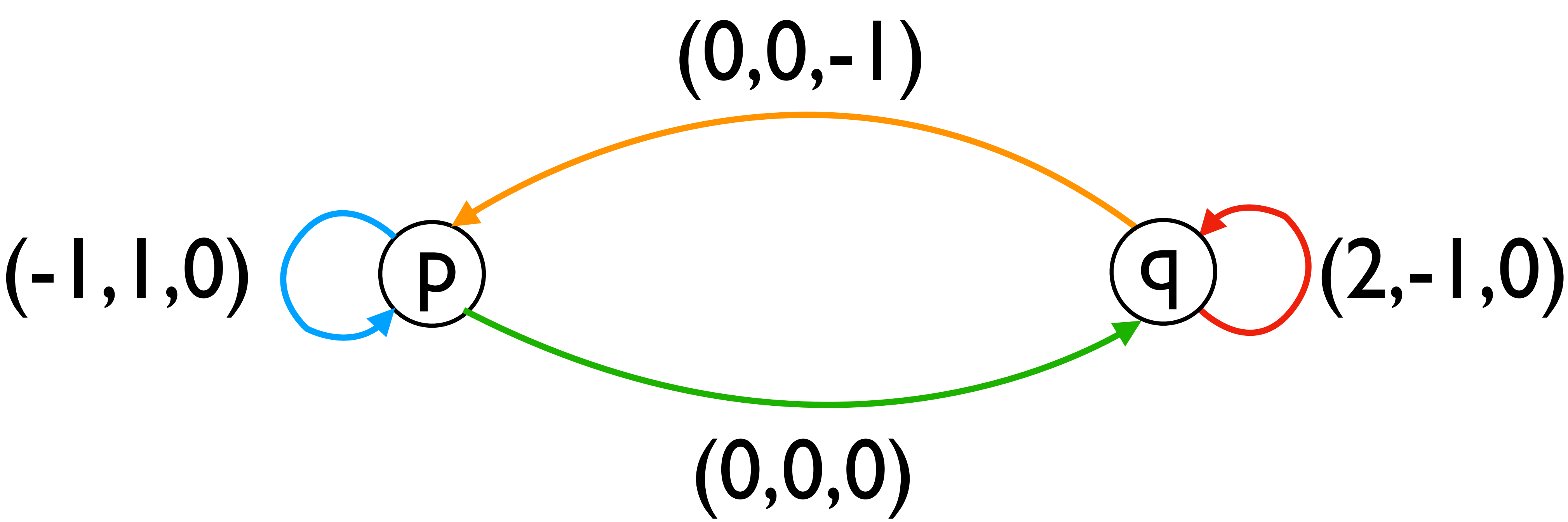
p(1,0,n) → p(2,0,n-1) …

Vector Addition Systems with States



$$p(k,0,n) \rightarrow p(0,k,n) \rightarrow q(0,k,n) \rightarrow q(2k,0,n) \rightarrow p(2k,0,n-1)$$

$$p(1,0,n) \rightarrow p(2,0,n-1) \;\ldots\; \rightarrow p(2^n,0,0)$$

Reachability problem

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Given: a VASS, two its configurations s and t

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Question: is there a run from s to t?

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coverability problem

Short history

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Lipton `76: ExpSpace-hardness of coverability

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Rackoff `78: coverability in ExpSpace

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doubly-exponential length paths

Short history

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Blondin at el. `15: reachability PSpace-complete for 2-VASSes

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Leroux, Schmitz `19: Ackermann

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Conjecture: reachability in ExpSpace

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Cz., Lasota, Lazic, Leroux, Mazowiecki `19:
Tower-hardness

Short history

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Leroux & Cz., Orlikowski`21: Ackermann-hardness

Functions F_k

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$\text{Ack}(n) = F_\omega(n) = F_n(n)$

Hard examples

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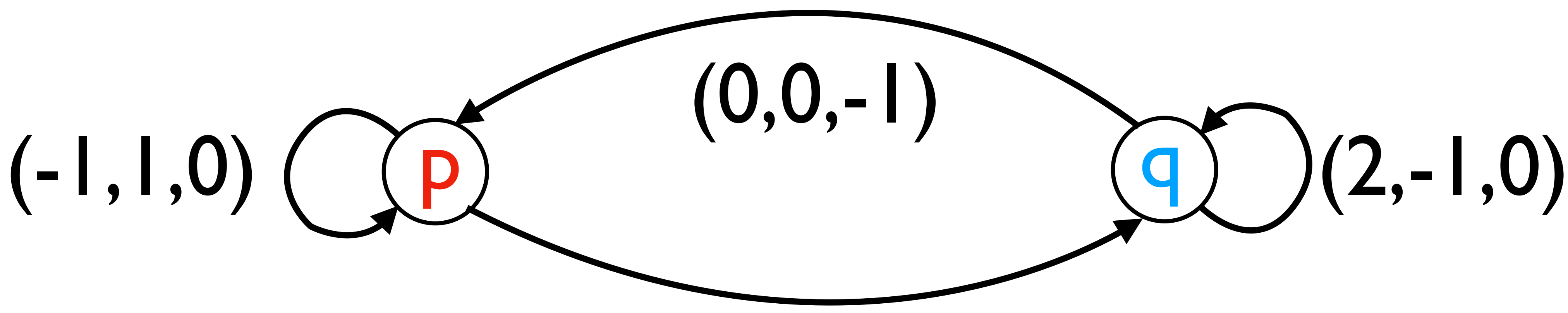

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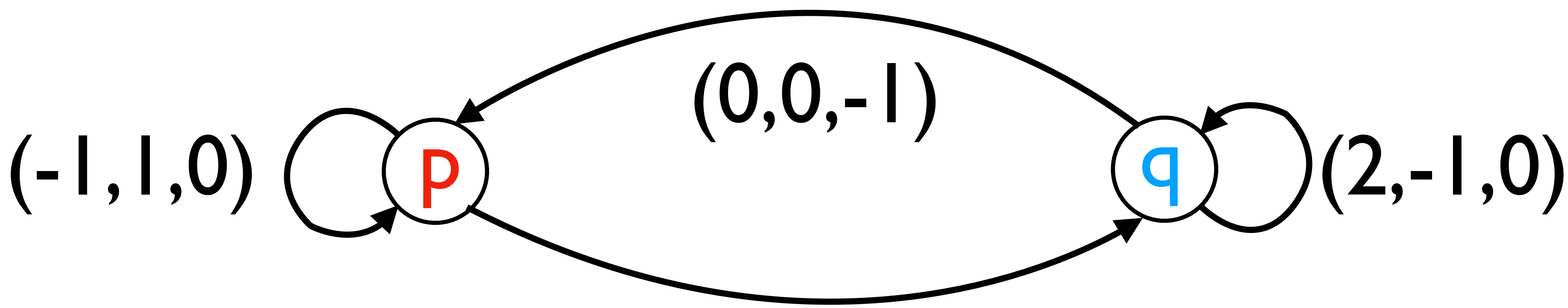

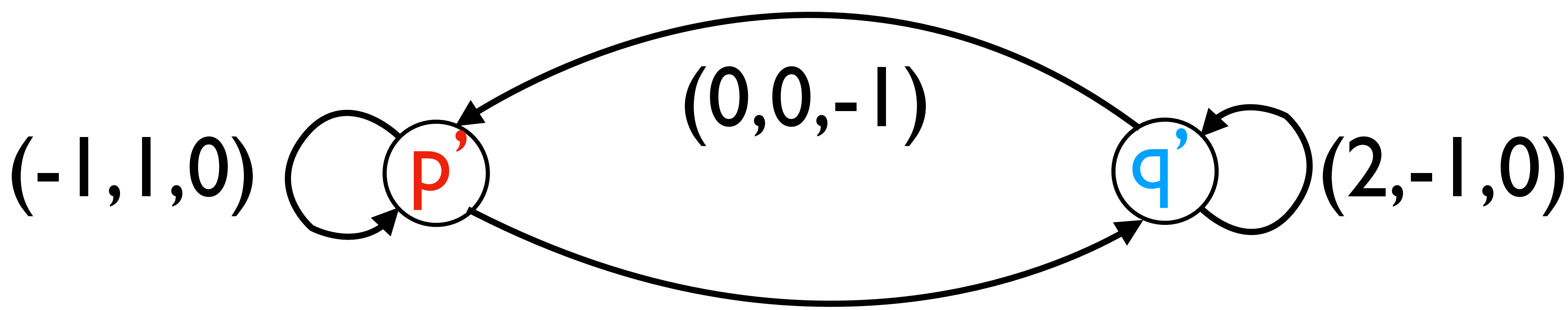

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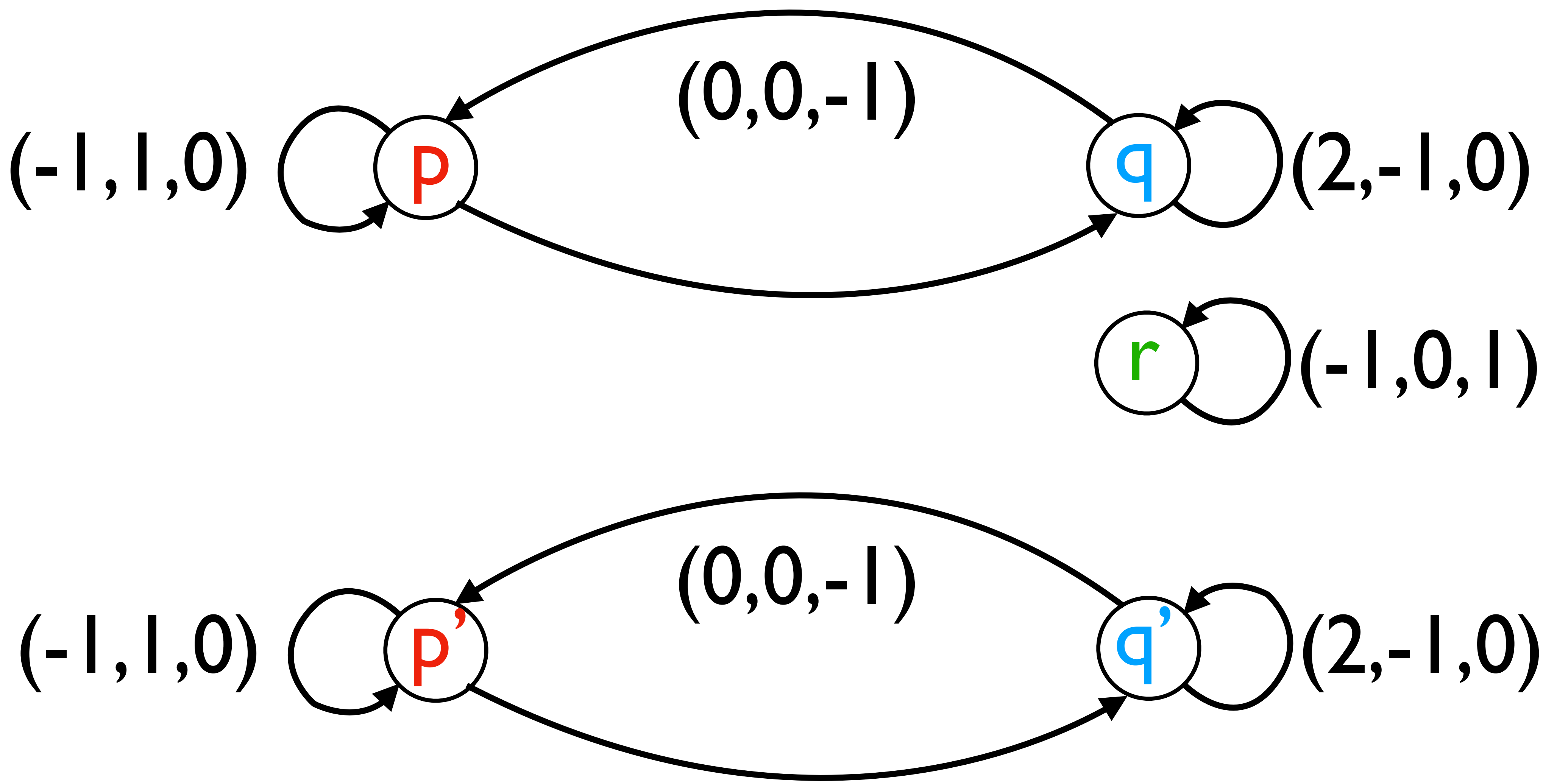

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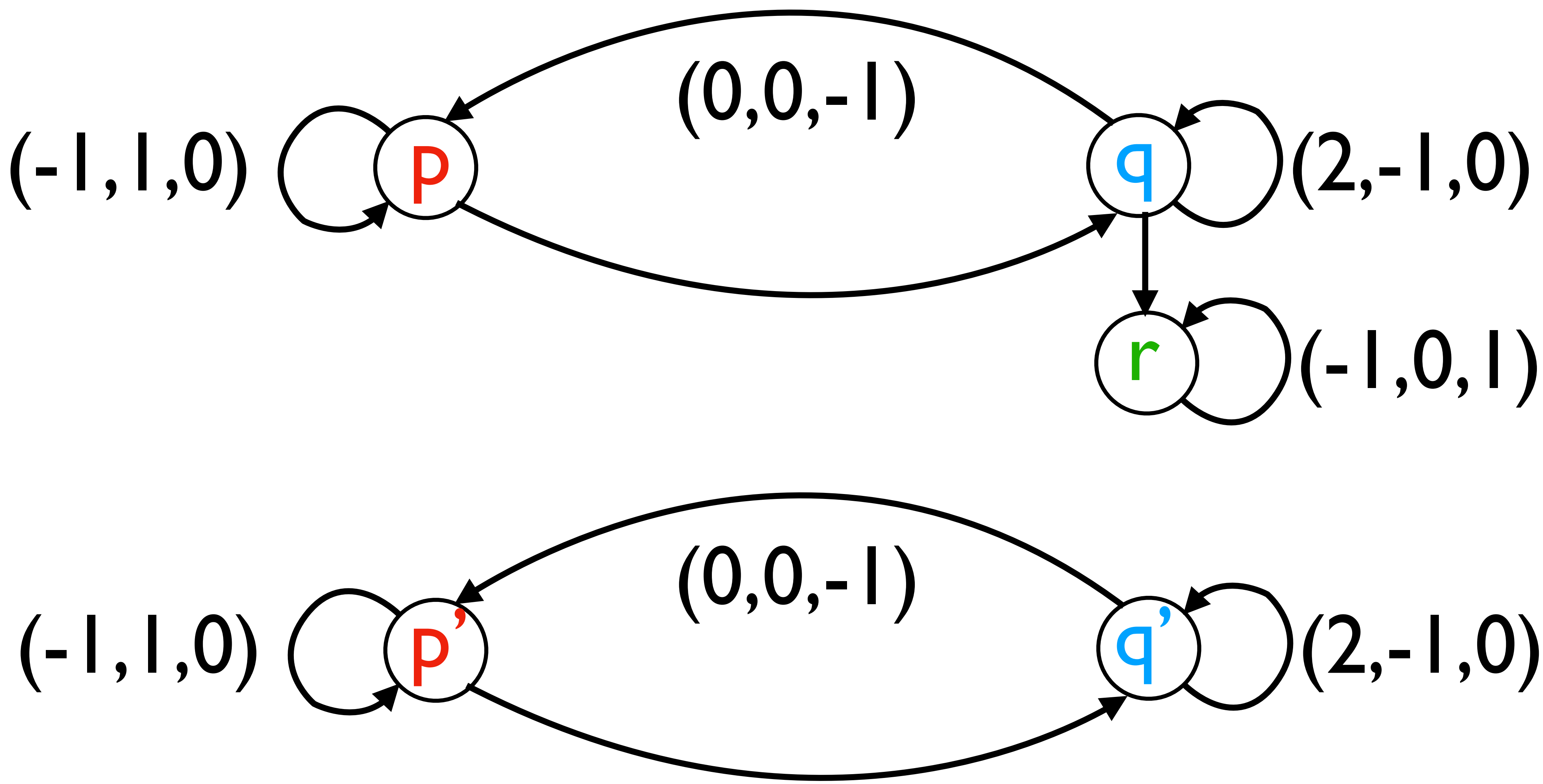

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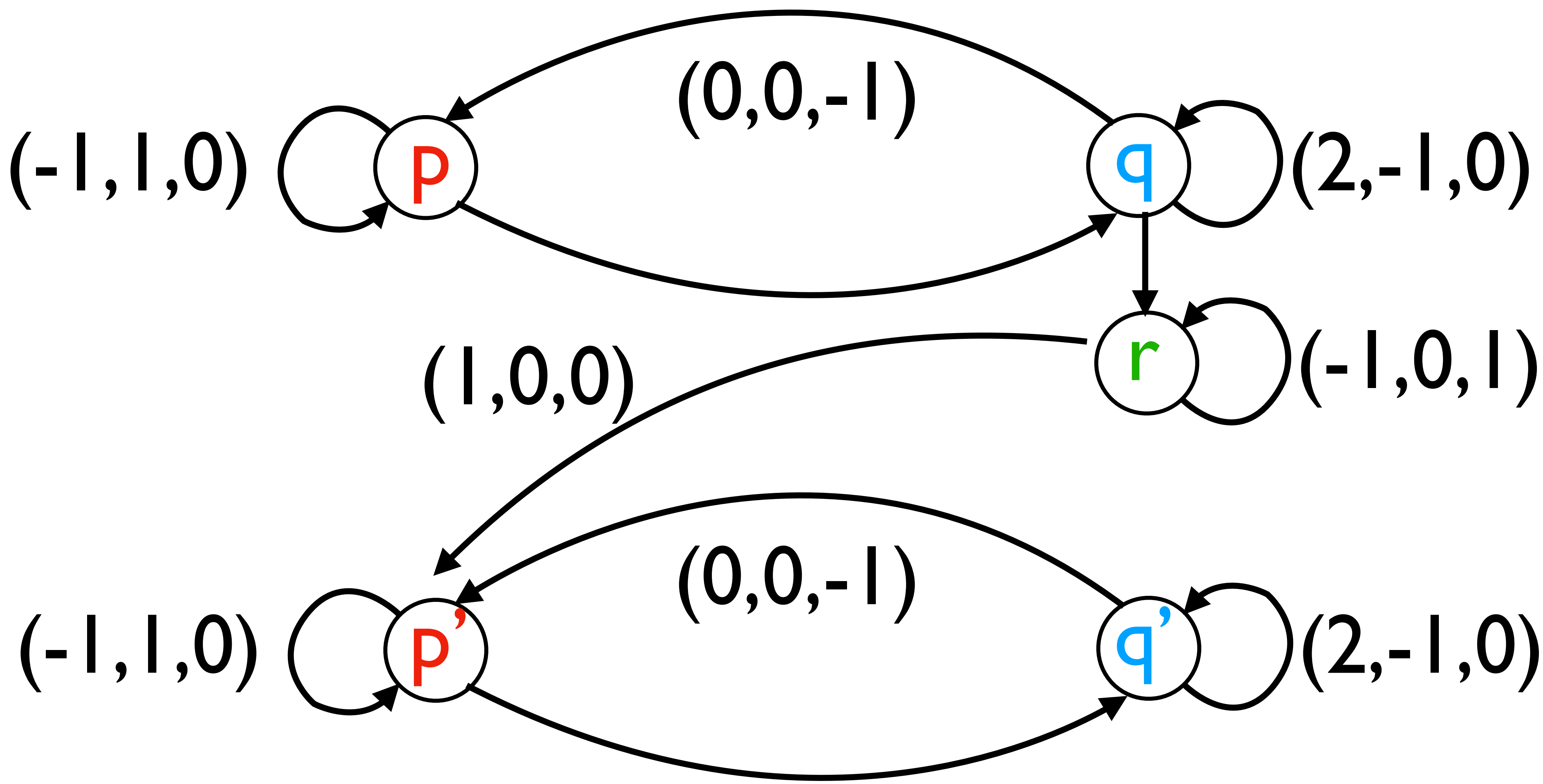

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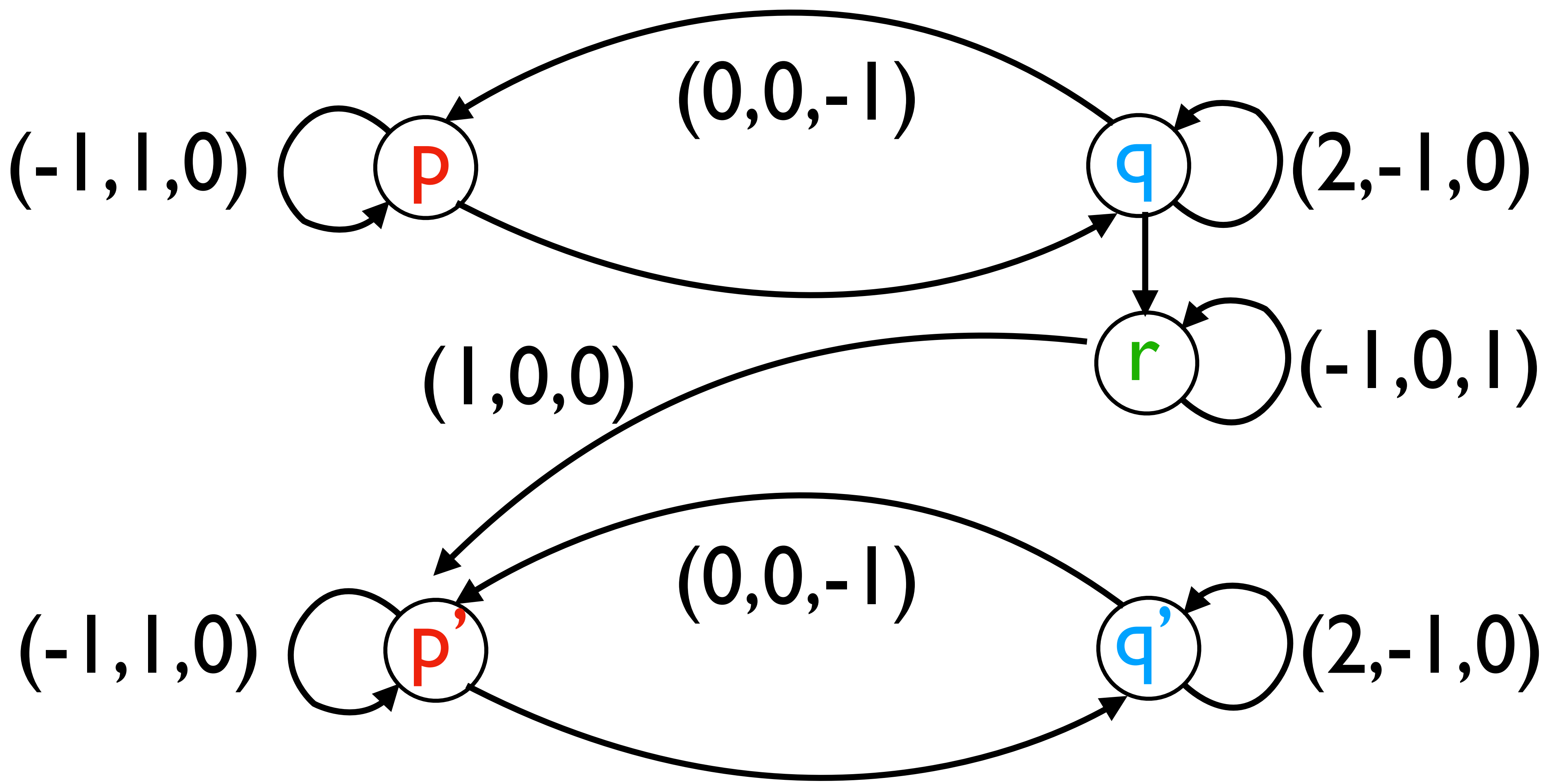
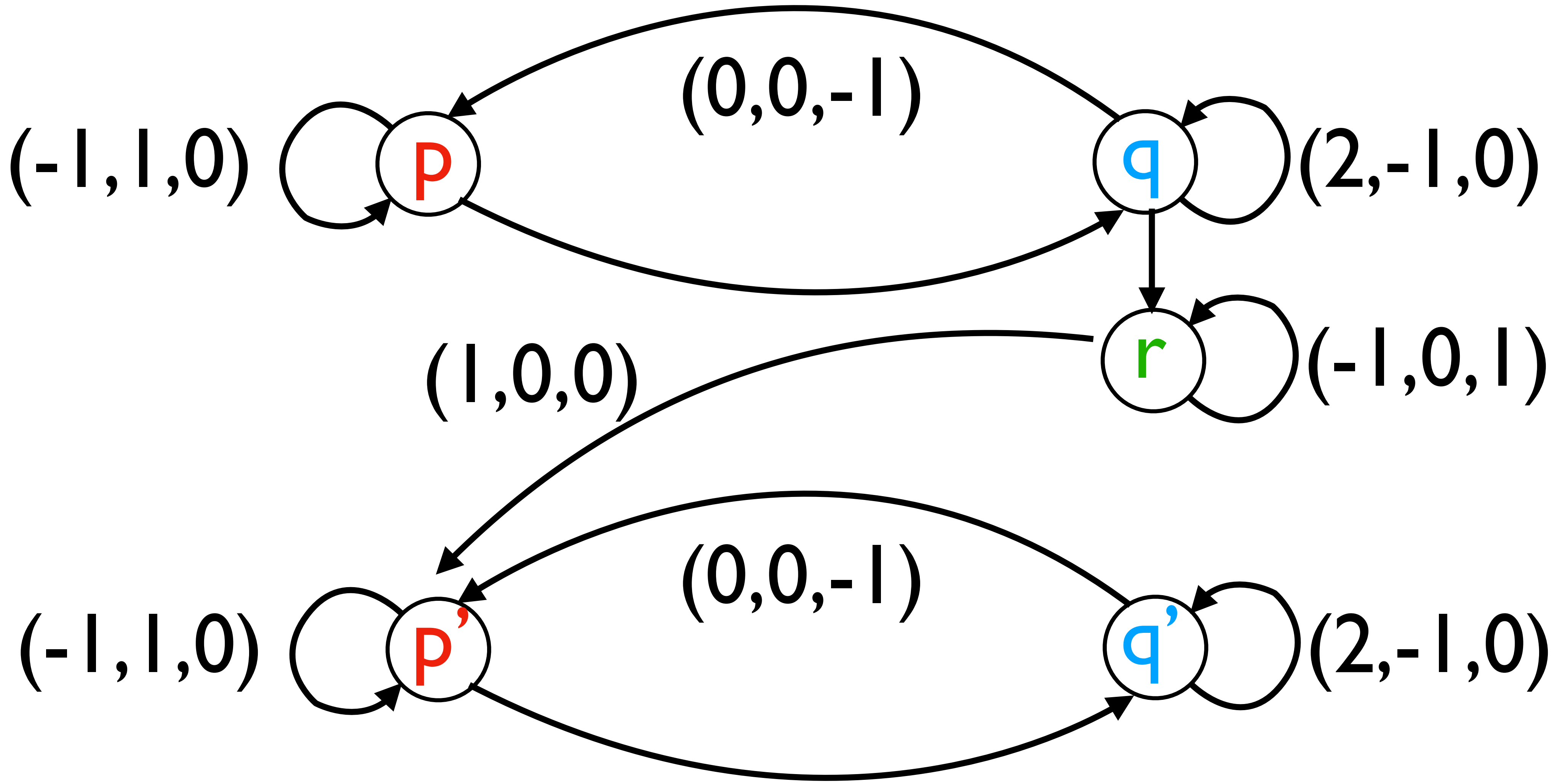


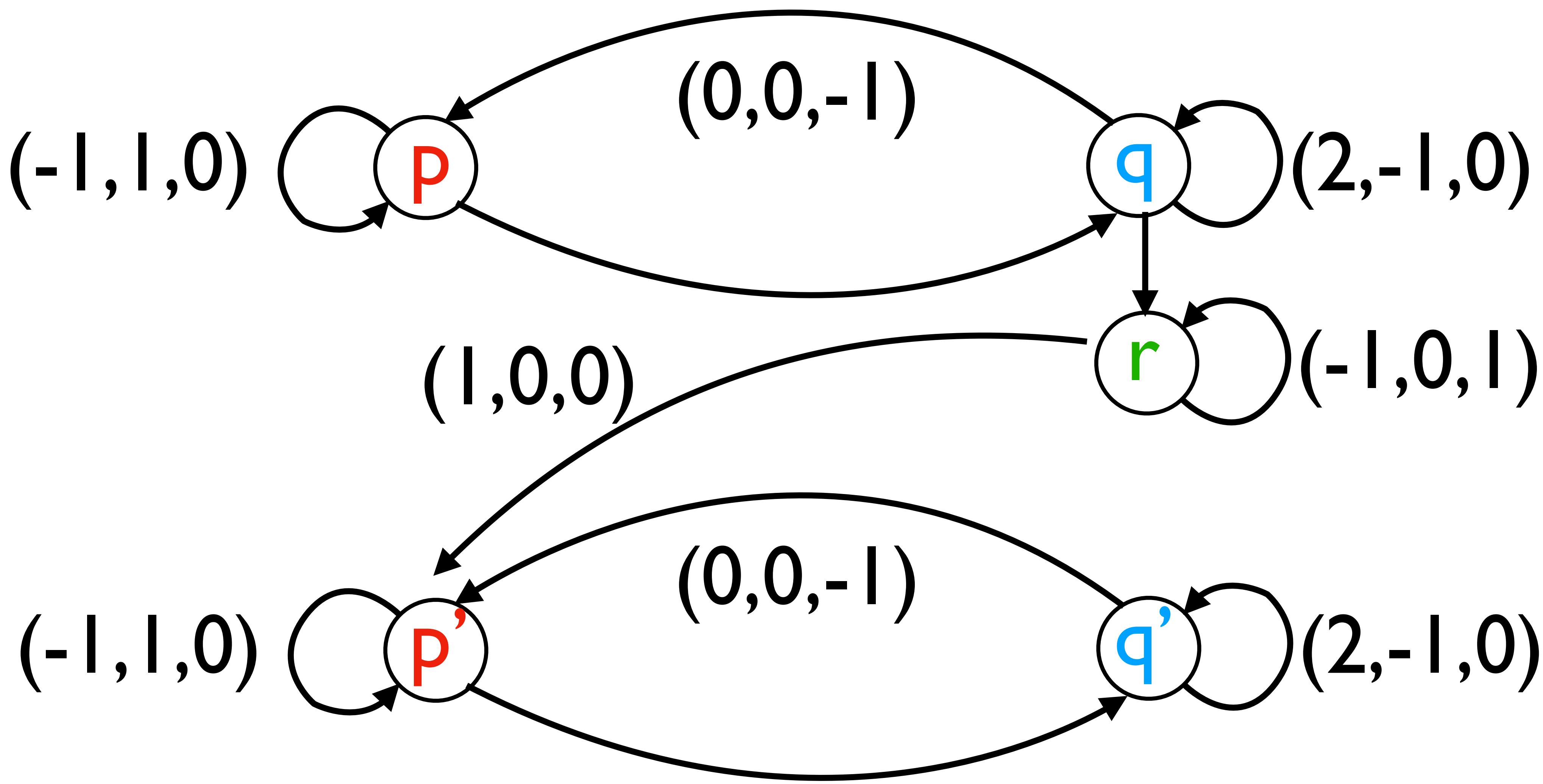
p(1,0,n)

Hard examples



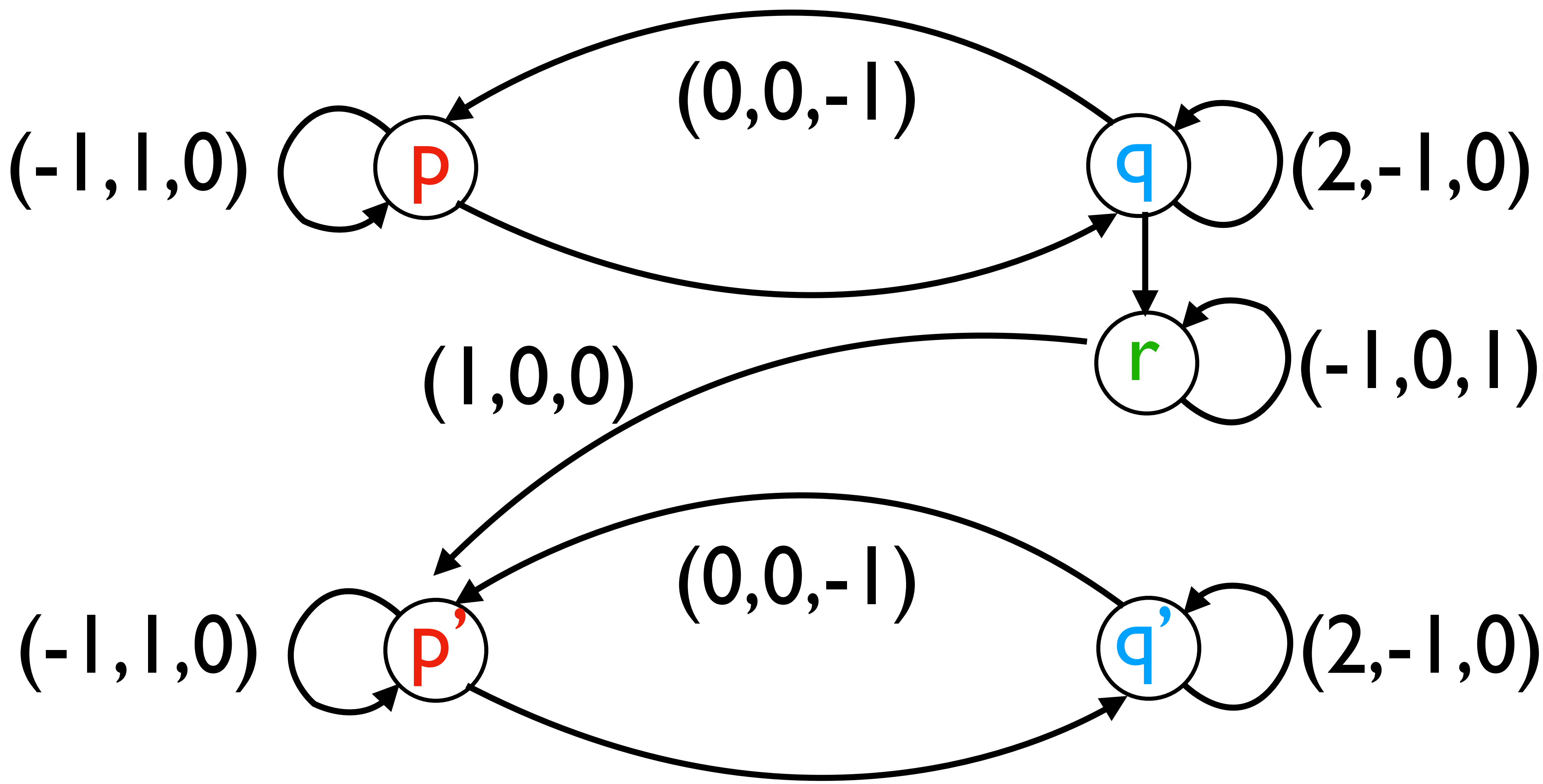
p(1,0,n) ⟶ q(2^n,0,0)

Hard examples



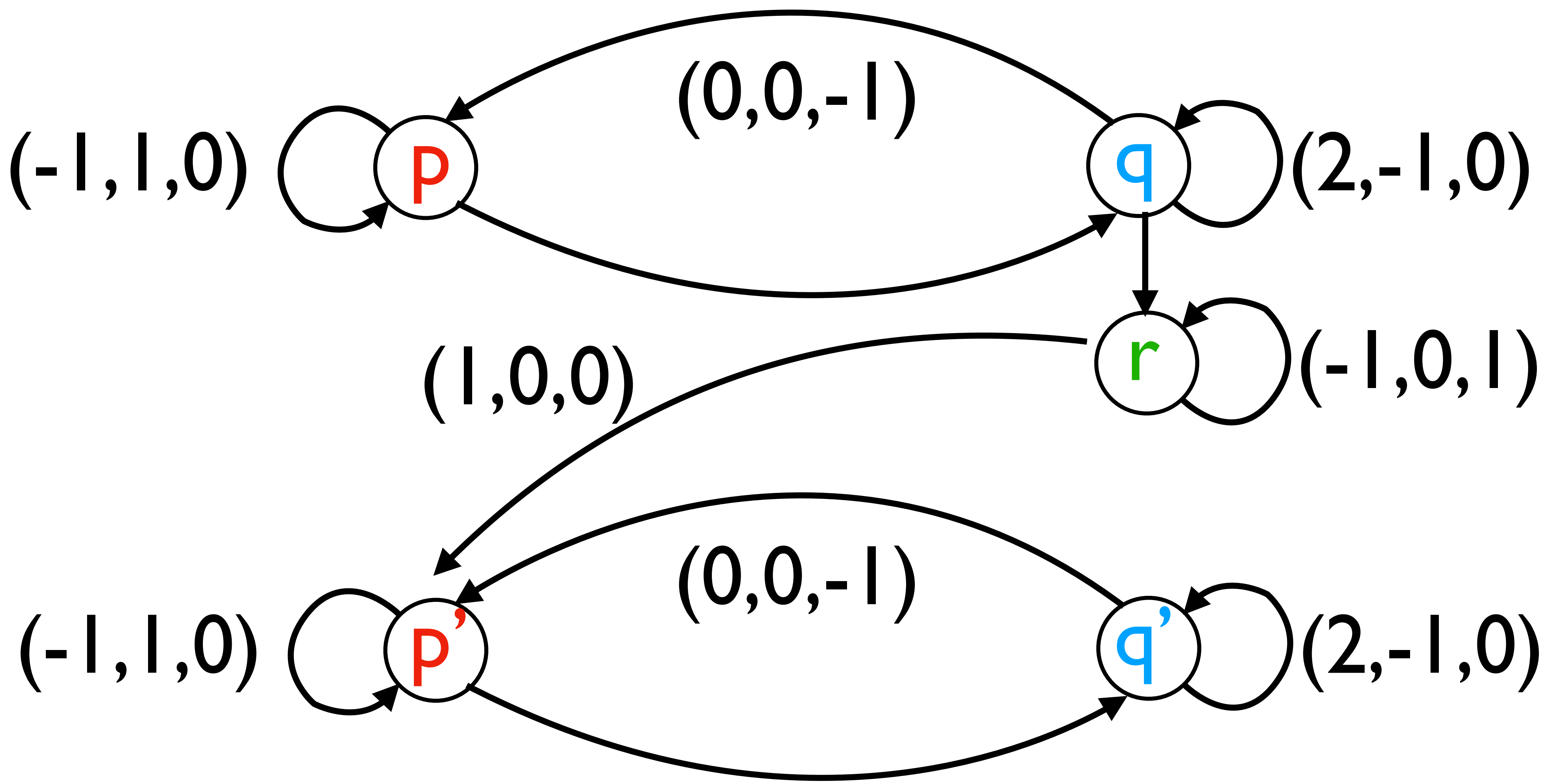
$p(1,0,n) \longrightarrow q(2^n,0,0) \longrightarrow r(2^n,0,0)$

Hard examples



$p(1,0,n) \longrightarrow q(2^n,0,0) \longrightarrow r(2^n,0,0) \longrightarrow r(0,0,2^n)$

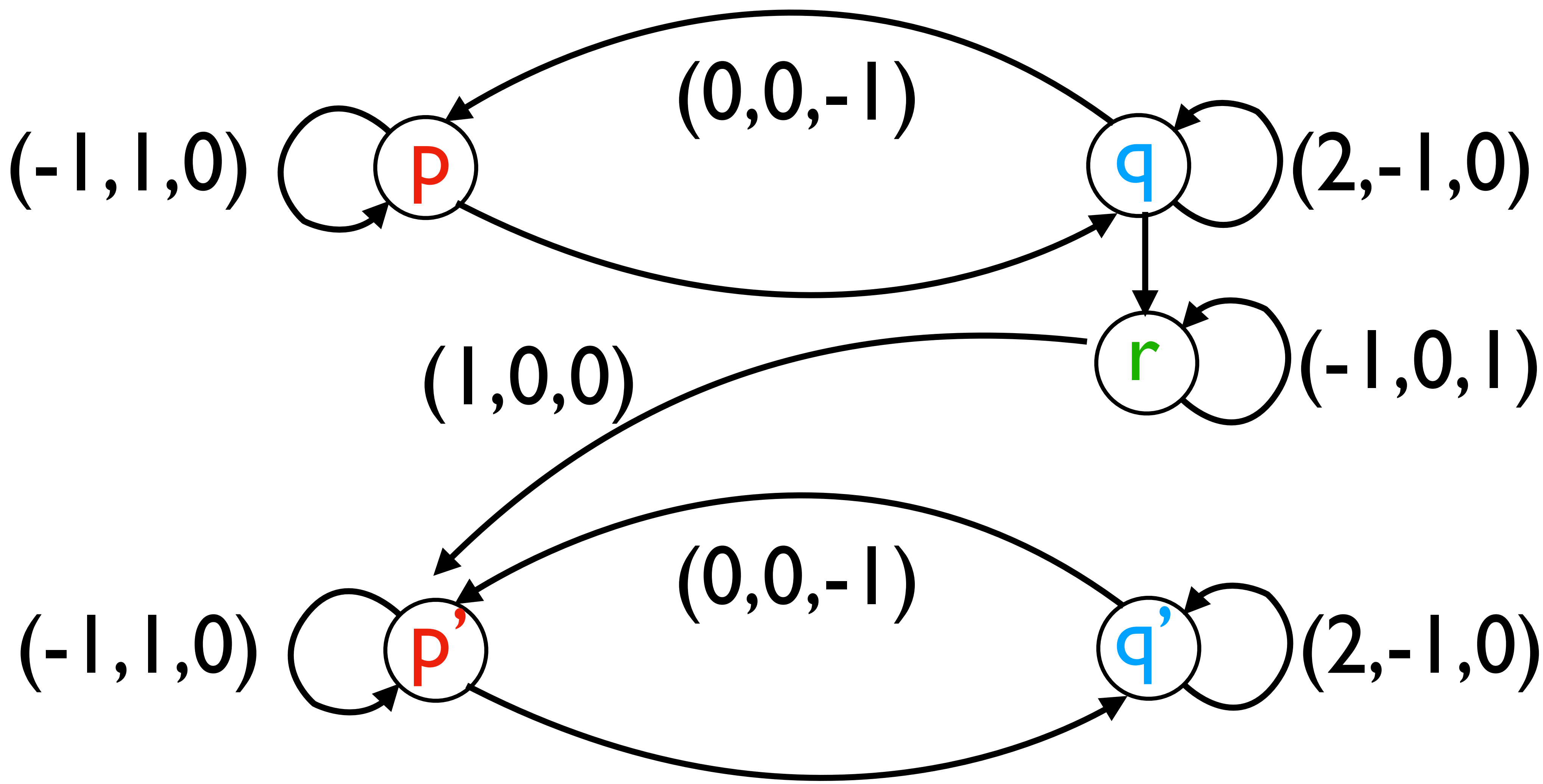
Hard examples



$p(1,0,n) \longrightarrow q(2^n,0,0) \longrightarrow r(2^n,0,0) \longrightarrow r(0,0,2^n)$

$\longrightarrow p'(1,0,2^n)$

Hard examples



$$p(1,0,n) \longrightarrow q(2^n,0,0) \longrightarrow r(2^n,0,0) \longrightarrow r(0,0,2^n)$$

$$\longrightarrow p'(1,0,2^n) \longrightarrow q'(2^{2^n},0,0)$$

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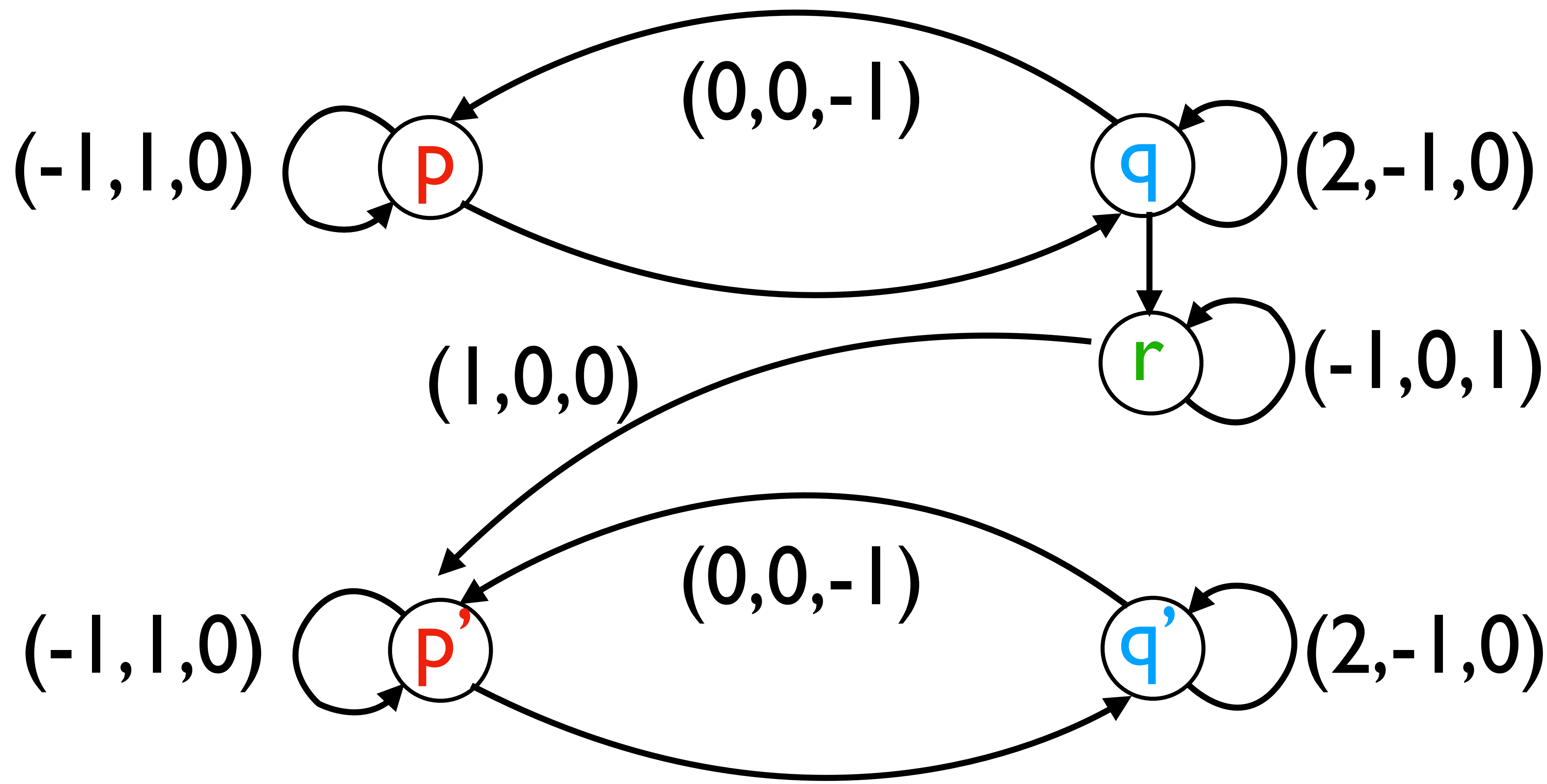

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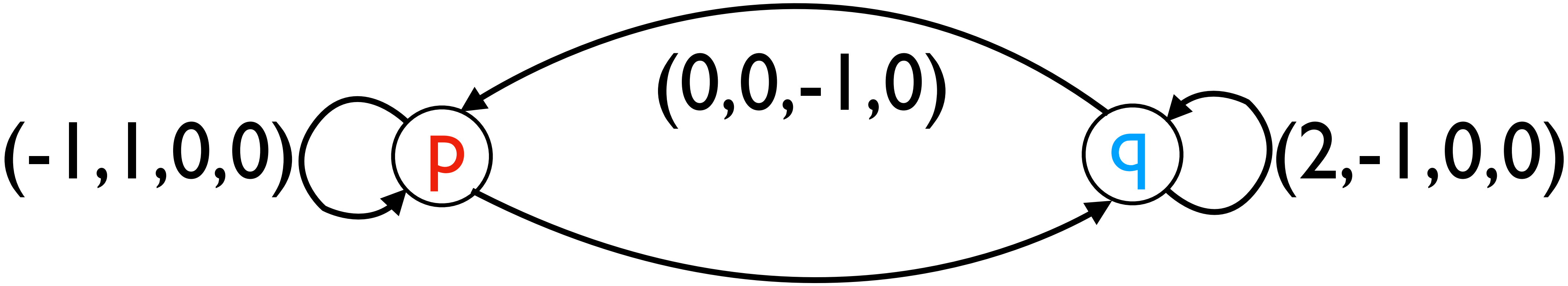
$$\longrightarrow p'(1,0,2^n) \longrightarrow q'(2^{2^n},0,0)$$

finite doubly-exponential reachability set

Hard examples

Hard examples



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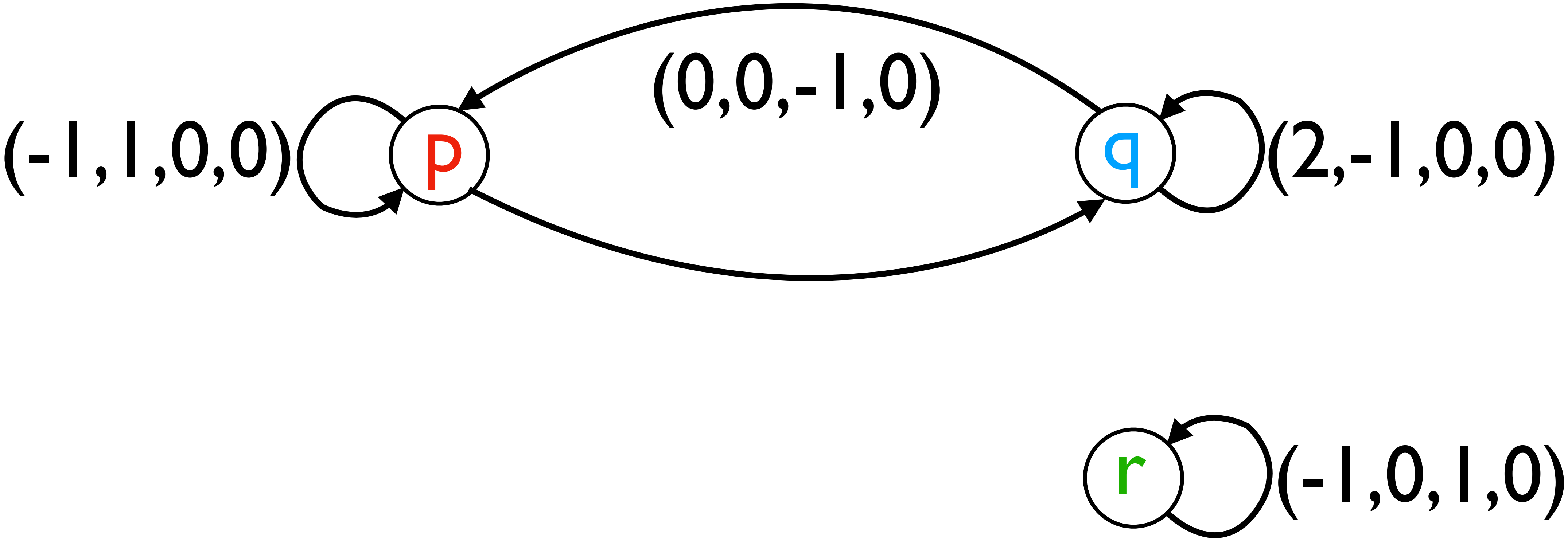

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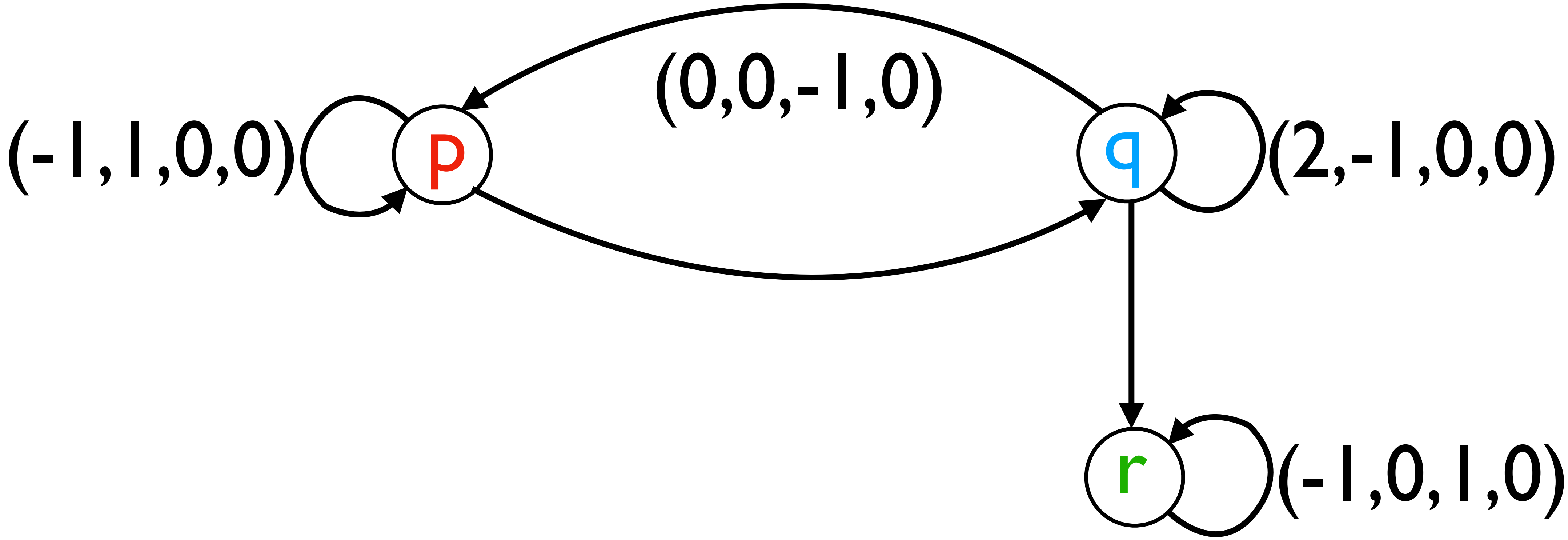

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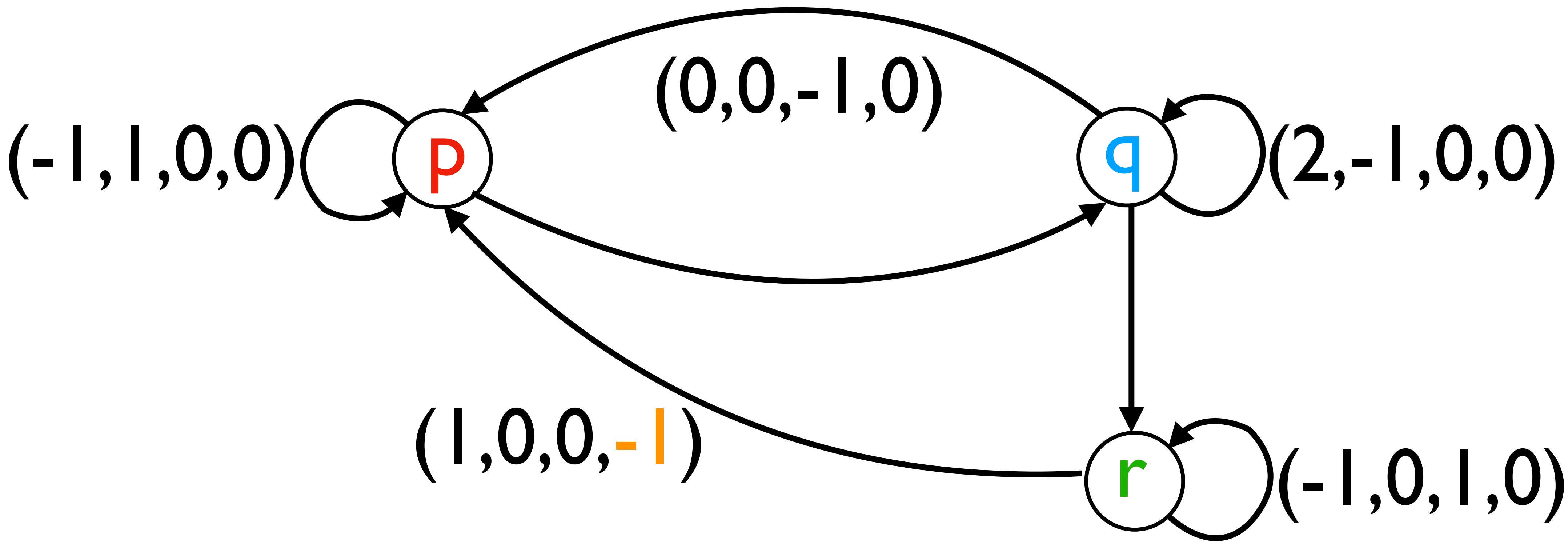

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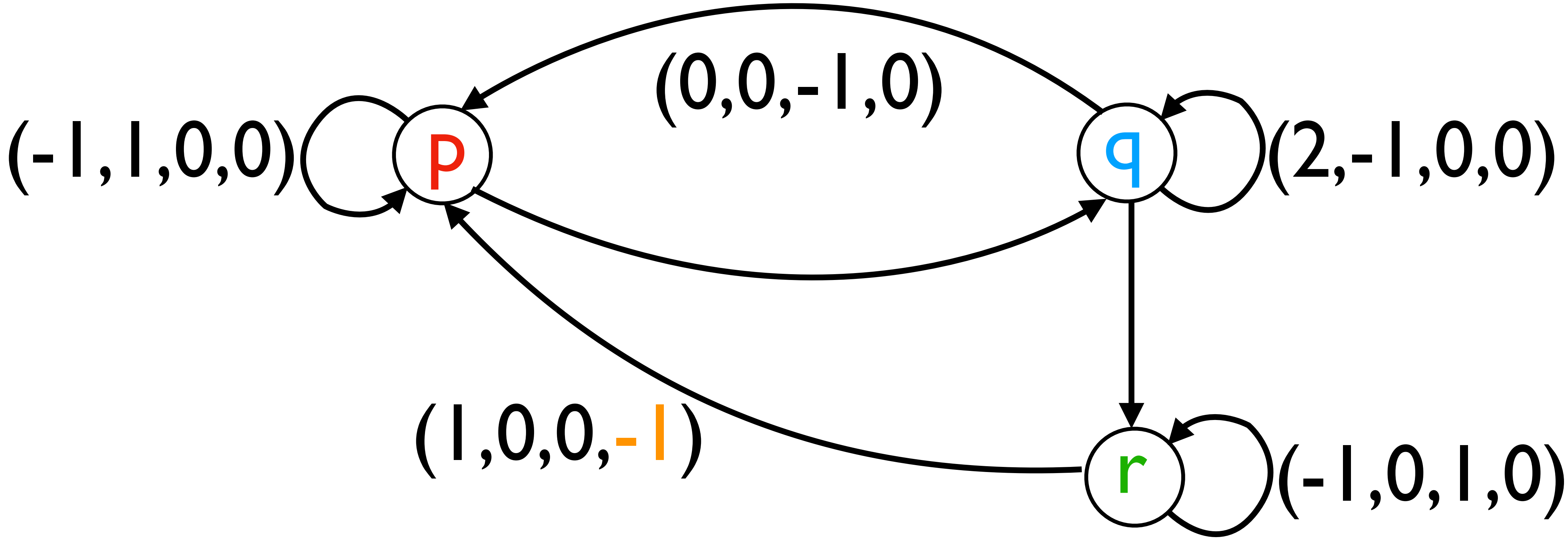
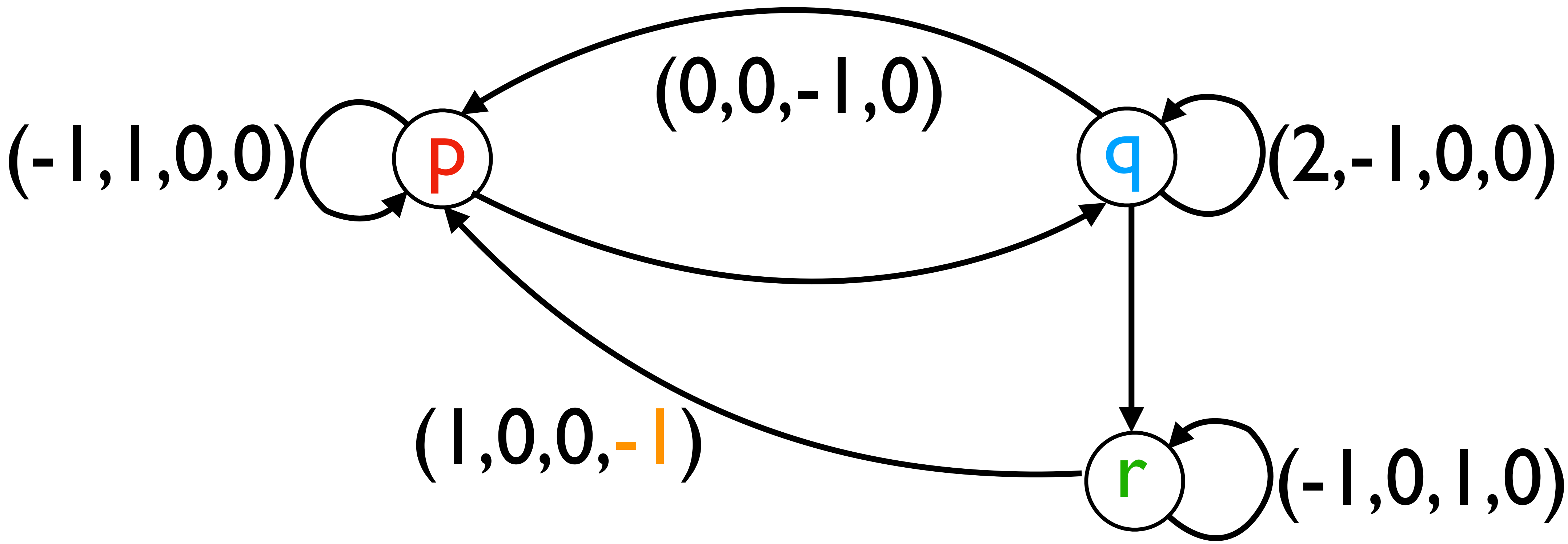


p(1,0,1,n)

Hard examples



p(1,0,1,n) ⟶ p(2^{1},0,1,n-1)

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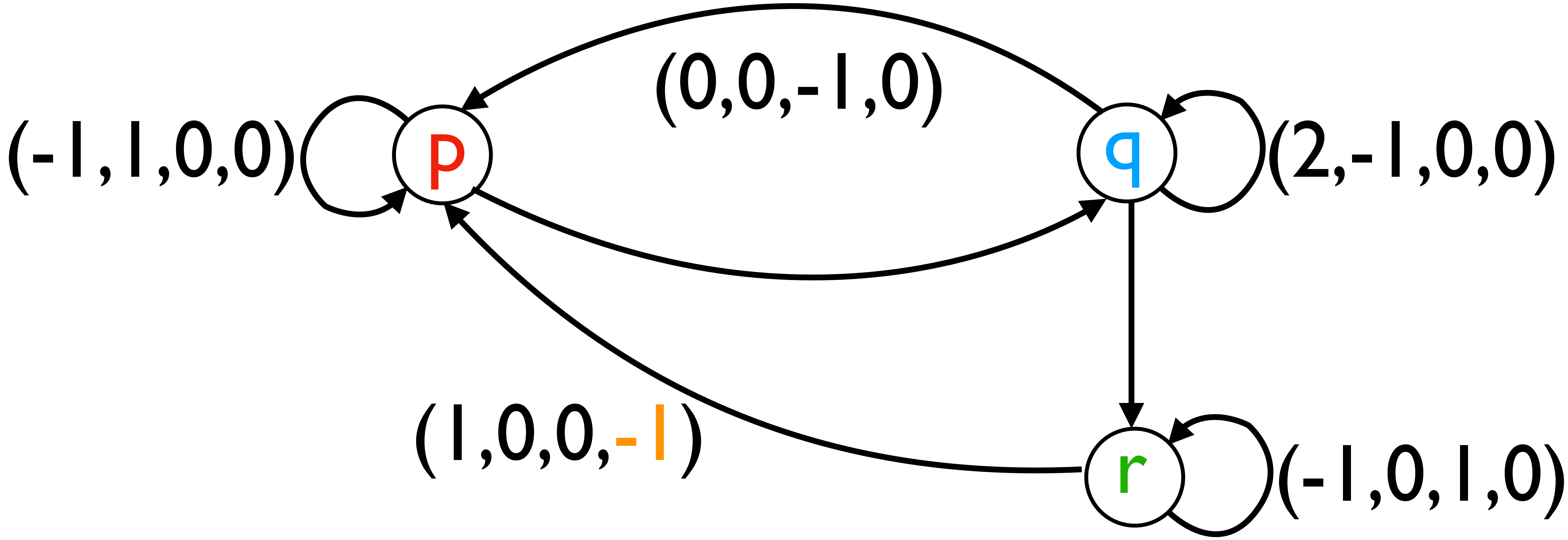

$p(1,0,1,n) \longrightarrow p(2^1,0,1,n-1) \ldots$

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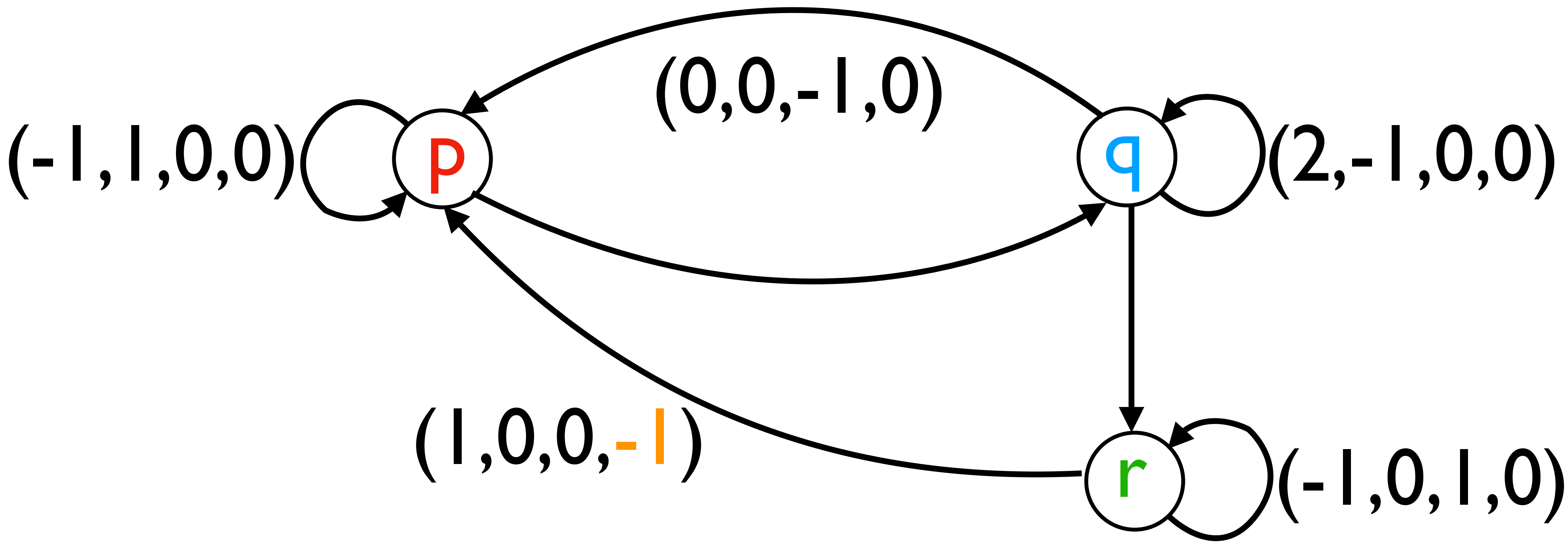
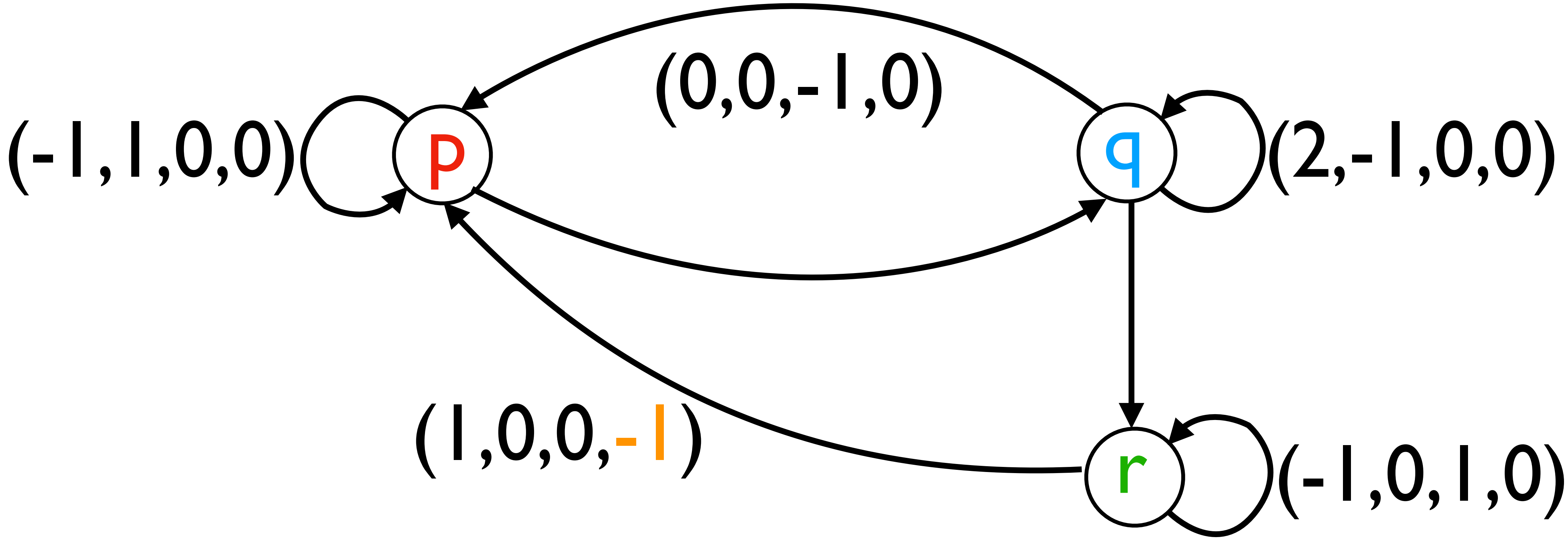


$p(1,0,1,n) \longrightarrow p(2^1,0,1,n-1) \ldots \longrightarrow p(\text{Tower}(n),0,1,0)$

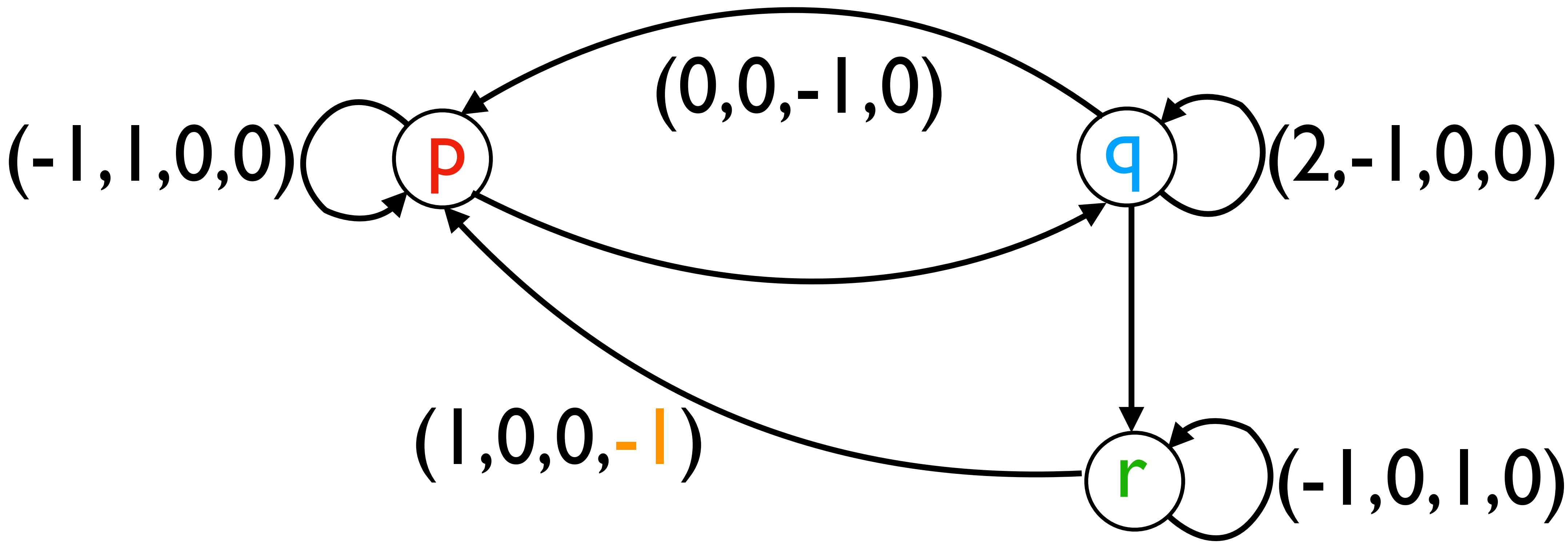
Hard examples



p(1,0,1,n) $\longrightarrow$ p(2^1,0,1,n-1) … $\longrightarrow$ p(Tower(n),0,1,0)

finite tower-size reachability set

Hard examples



p(1,0,1,n) ⟶ p(2^1,0,1,n-1) … ⟶ p(Tower(n),0,1,0)

finite tower-size reachability set

finite F_d-size reachability set

Decidability idea

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Question: does $p(s) \longrightarrow q(t)$?

Decidability idea

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Idea: reachability in $\mathbb{Z}$ is easy!

Decidability idea

Question: does p(s) $\longrightarrow$ q(t)?

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Linear equations (in NP)

Decidability idea

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Assume:

Decidability idea

Question: does $p(s) \longrightarrow q(t)$?

Idea: reachability in $\mathbb{Z}$ is easy!

Assume: $p(s) \longrightarrow p(s+\Delta)$

Decidability idea

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Assume: $p(s) \longrightarrow p(s+\Delta)$ $\quad q(t+\Delta) \longrightarrow q(t)$

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Then: $p(s) \xrightarrow{\mathbb{Z}} q(t)$

Decidability idea

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Idea: reachability in $\mathbb{Z}$ is easy!

Assume: $p(s) \longrightarrow p(s+\Delta)$ $\quad q(t+\Delta) \longrightarrow q(t)$

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Then: $p(s) \xrightarrow{\mathbb{Z}} q(t)$ implies

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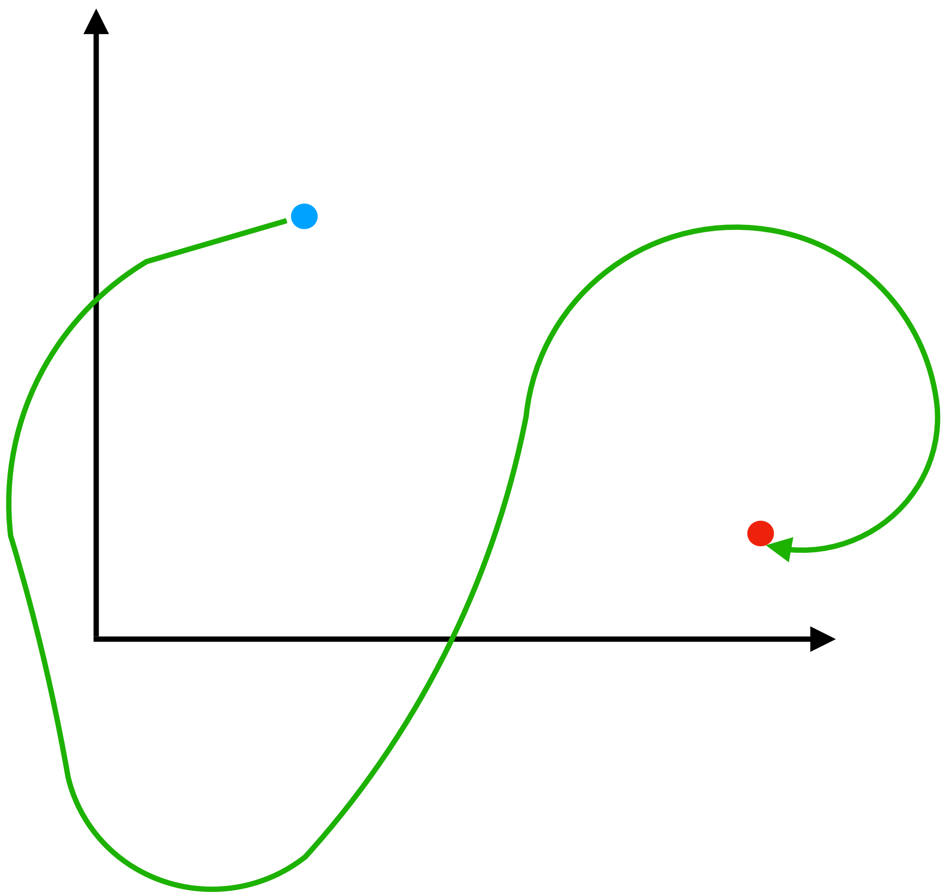

Decidability idea

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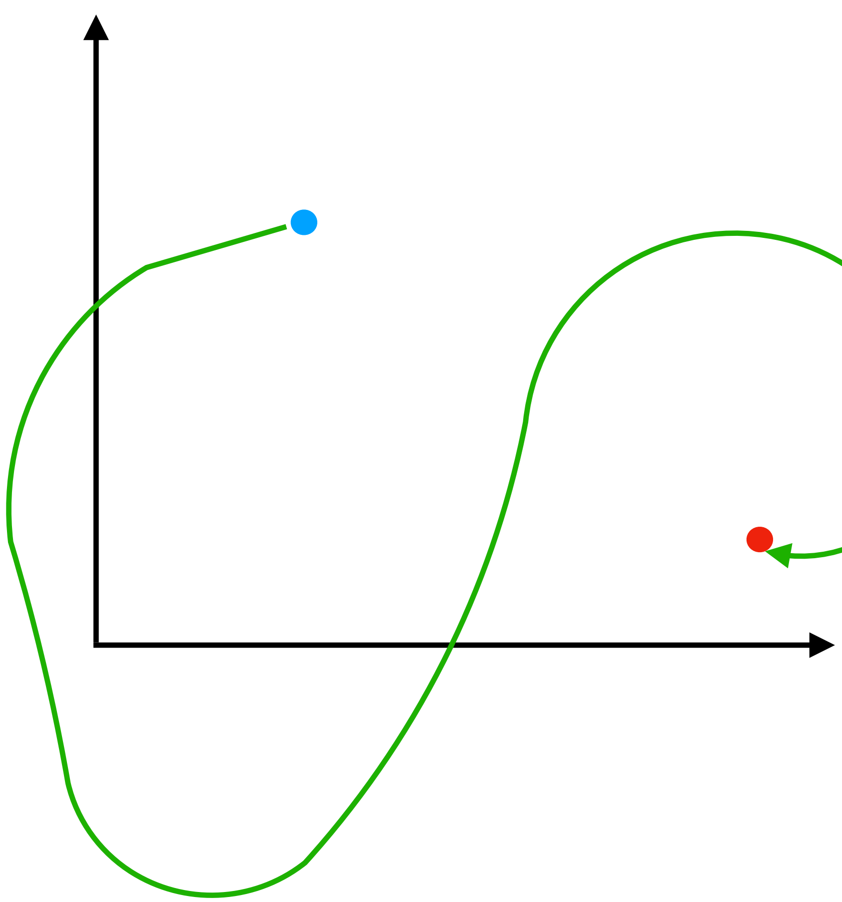

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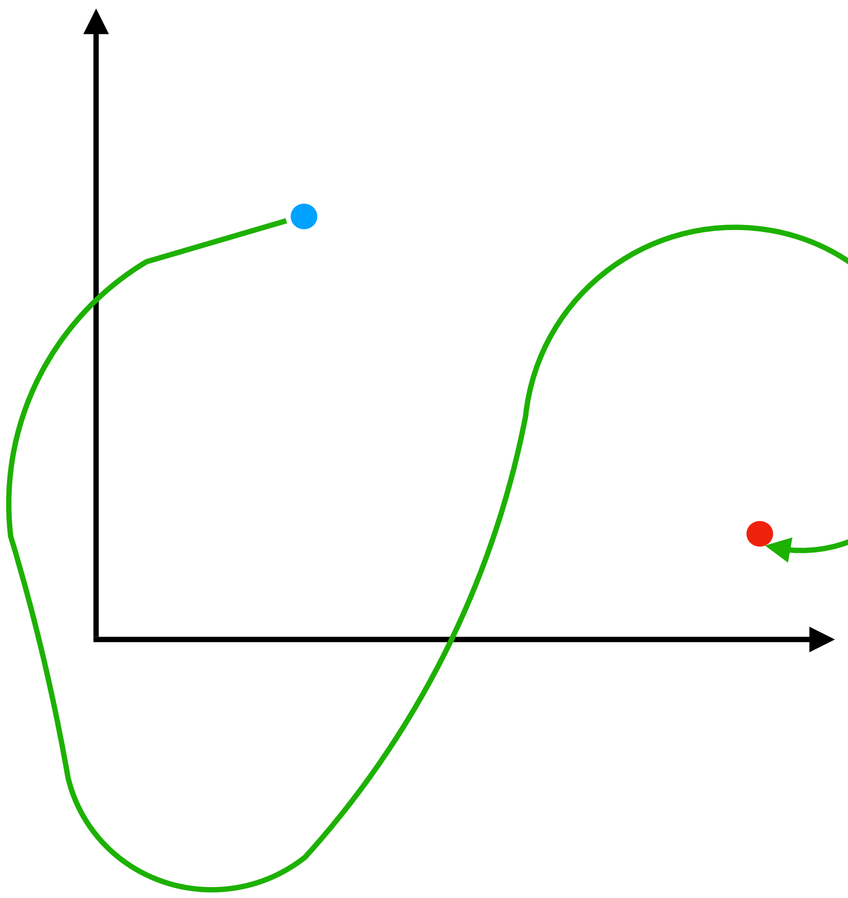

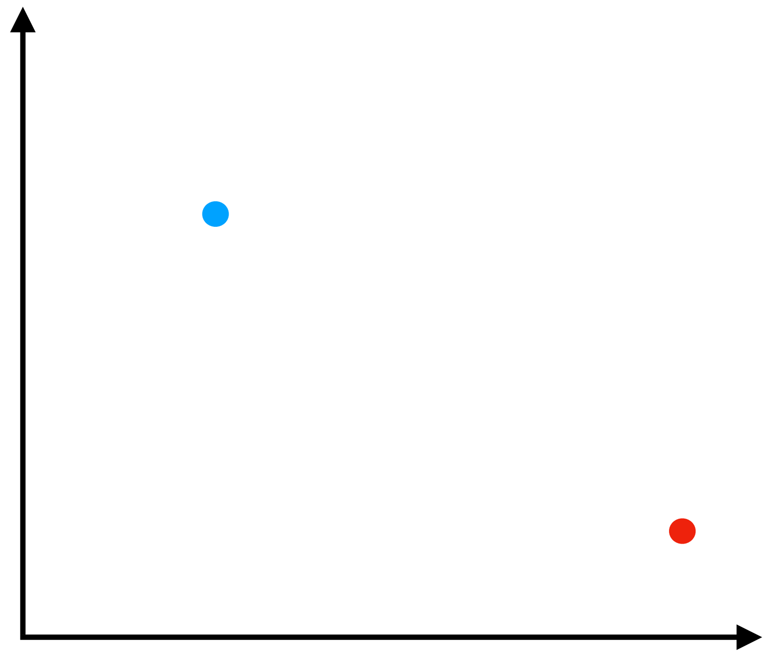

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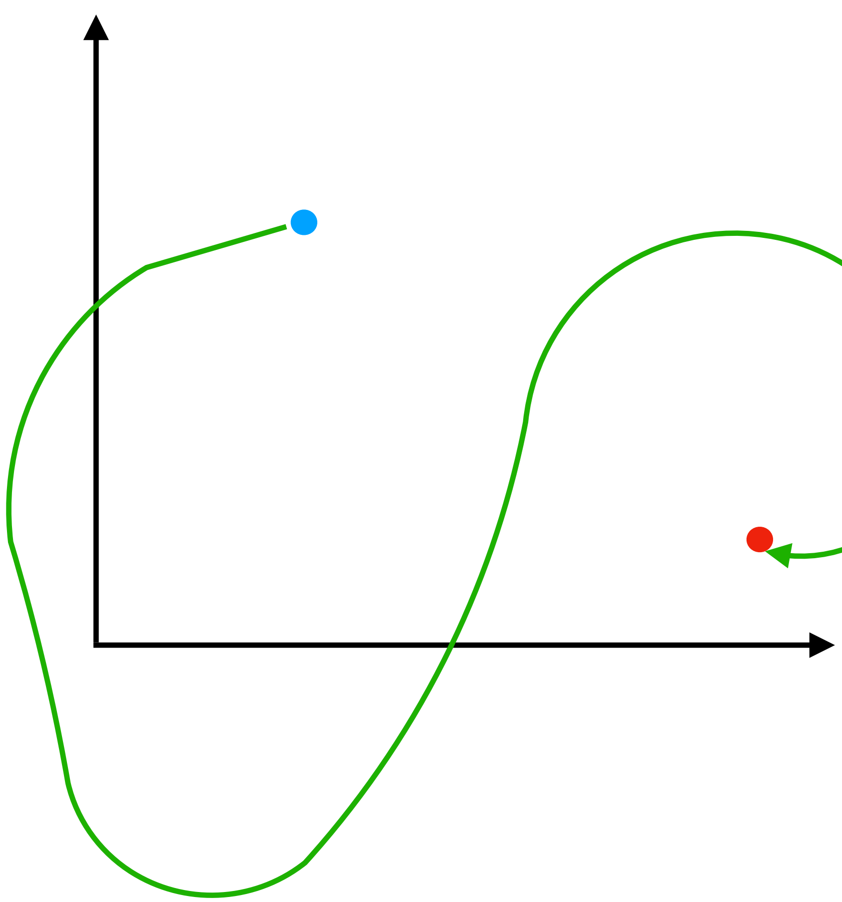

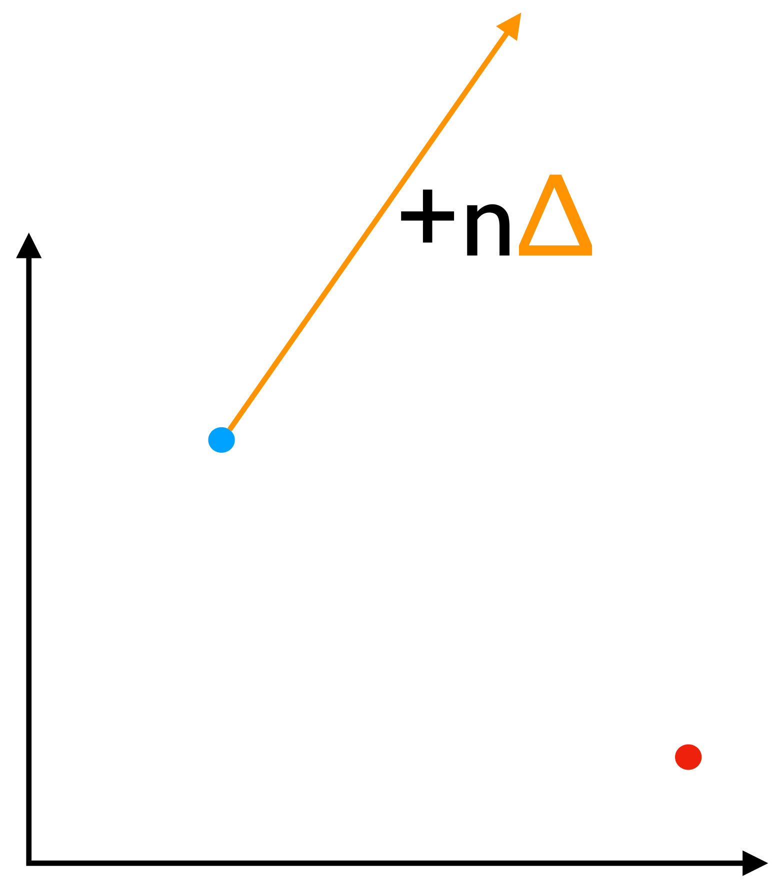

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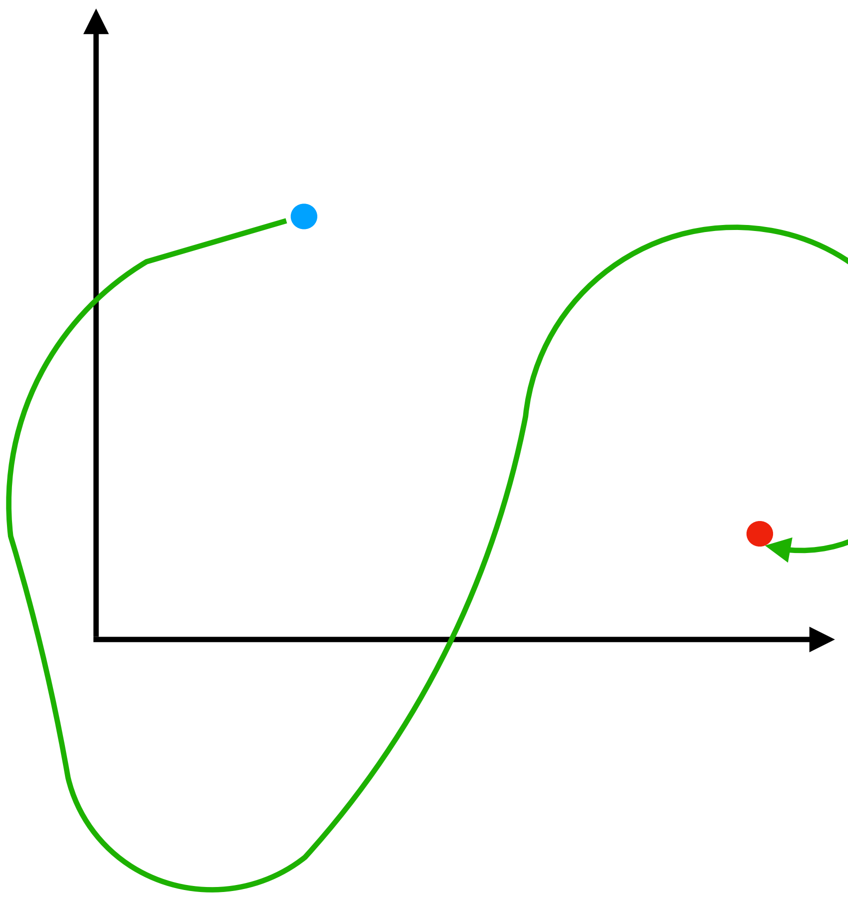

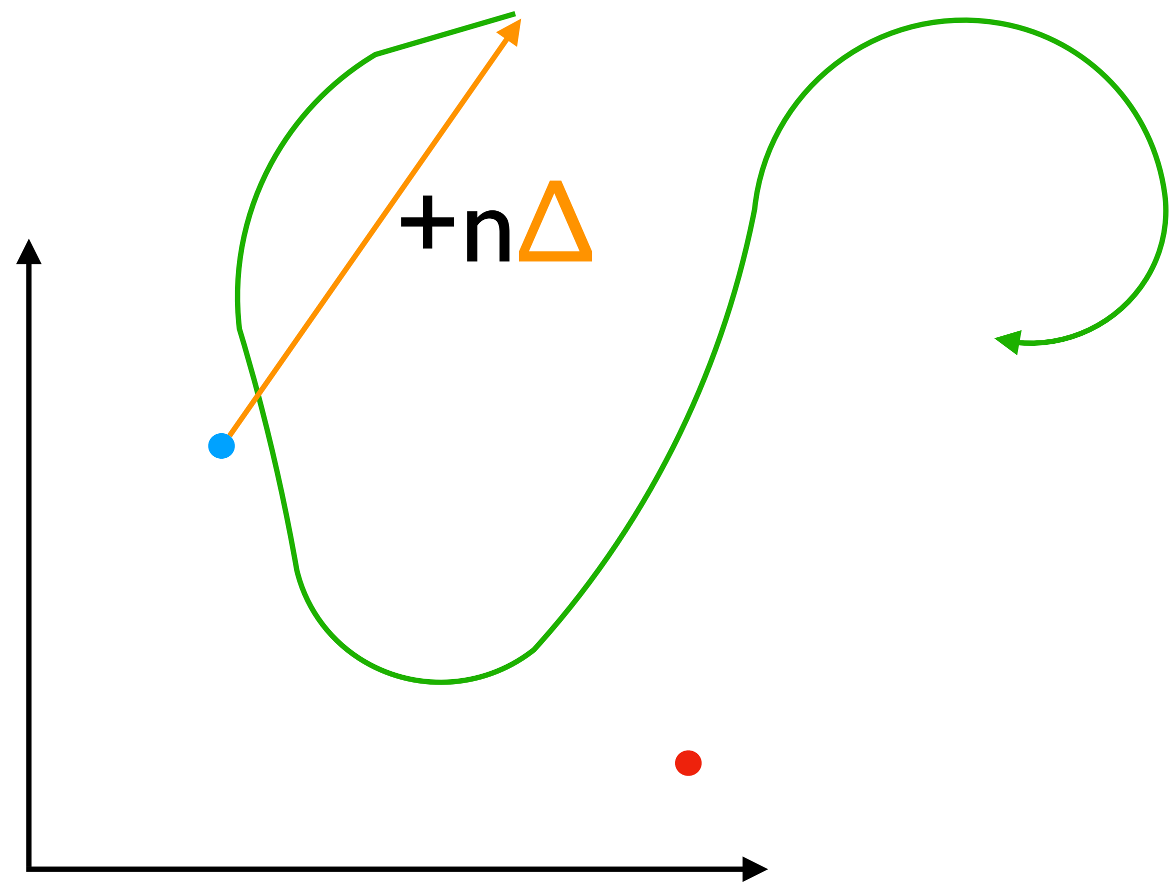

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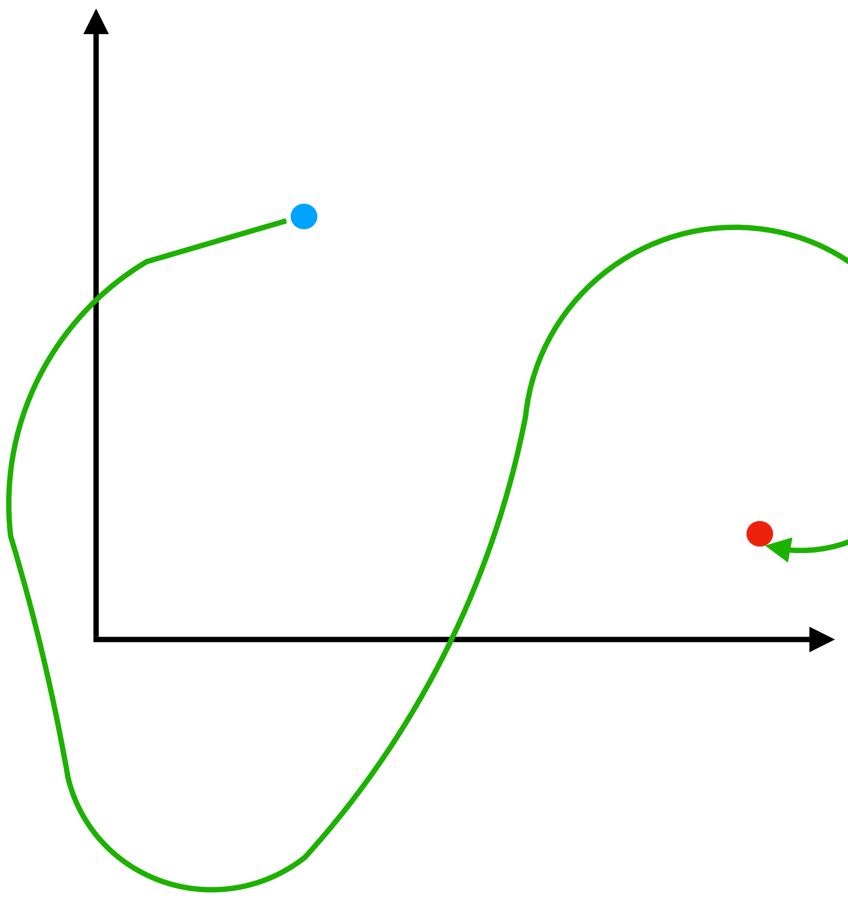

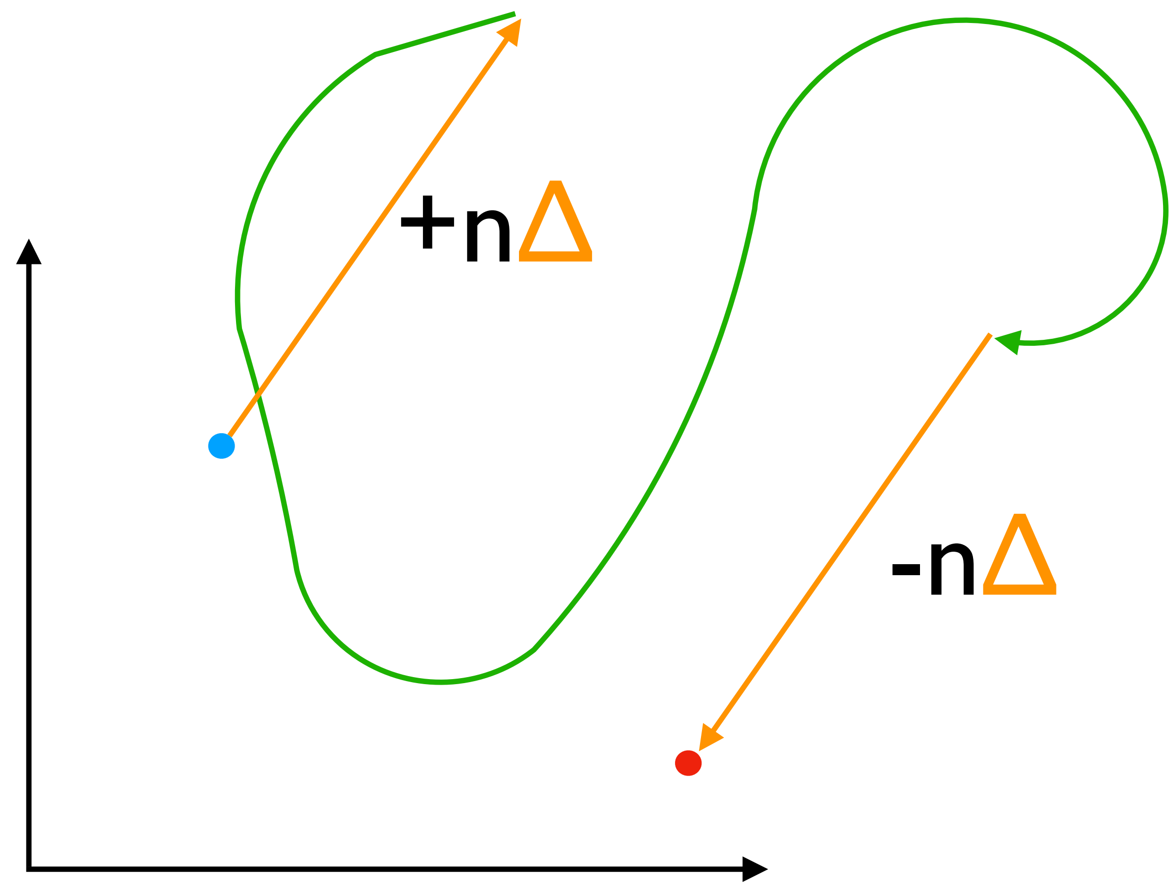

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$p(s) \xrightarrow{\mathbb{Z}} q(t)$

Decidability idea

Assume: $p(s) \longrightarrow p(s+\Delta_1)$ $\quad q(t+\Delta_2) \longrightarrow q(t)$

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$p(s) \xrightarrow{\mathbb{Z}} q(t)$ by runs using each transition many times

Decidability idea

Assume: $p(s) \longrightarrow p(s+\Delta_1) \quad q(t+\Delta_2) \longrightarrow q(t)$

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implies

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implies $p(s) \longrightarrow q(t)$

Why?

Decidability idea

Assume: $p(s) \longrightarrow p(s+\Delta_1)$ $\quad q(t+\Delta_2) \longrightarrow q(t)$

Then:

$p(s) \xrightarrow{\mathbb{Z}} q(t)$ by runs using each transition many times

implies $\quad p(s) \longrightarrow q(t)$

Why? $\quad p(s) \xrightarrow{\mathbb{Z}} p(s+\Delta_2-\Delta_1)$

Decidability idea

Decidability idea

$p(s) \longrightarrow p(s+\Delta_1)$

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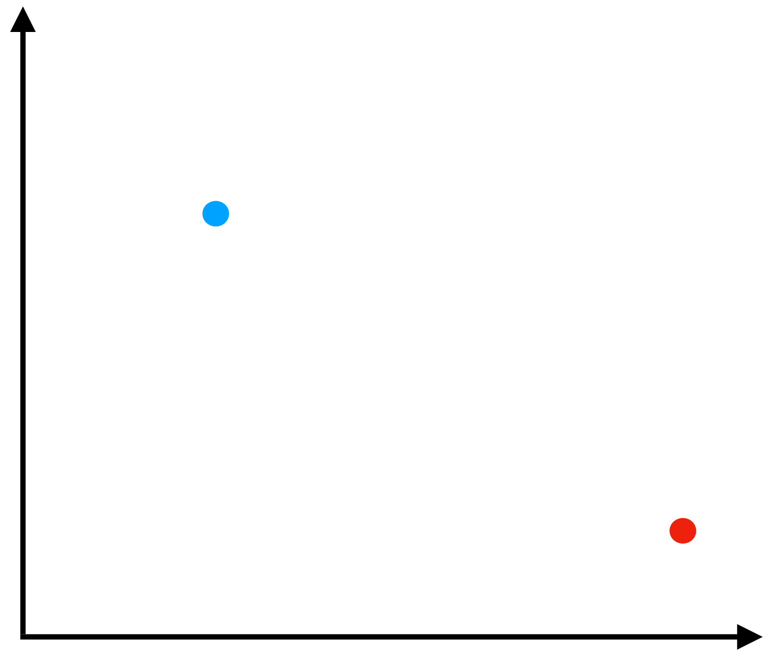

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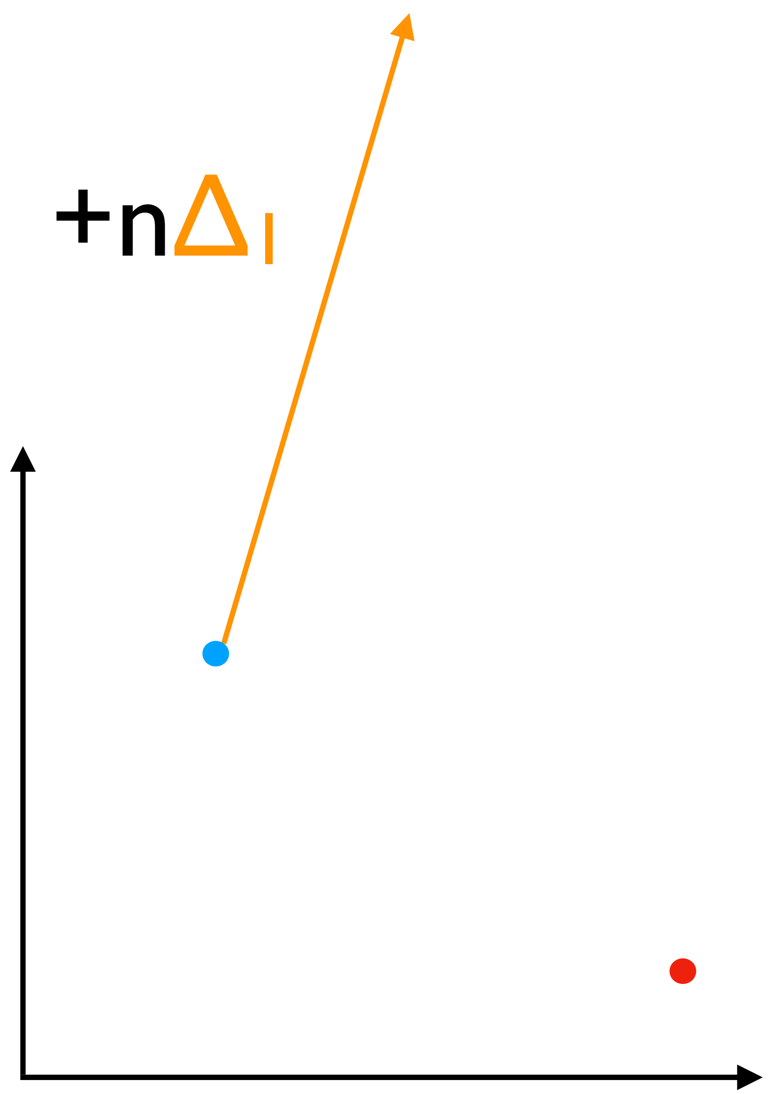

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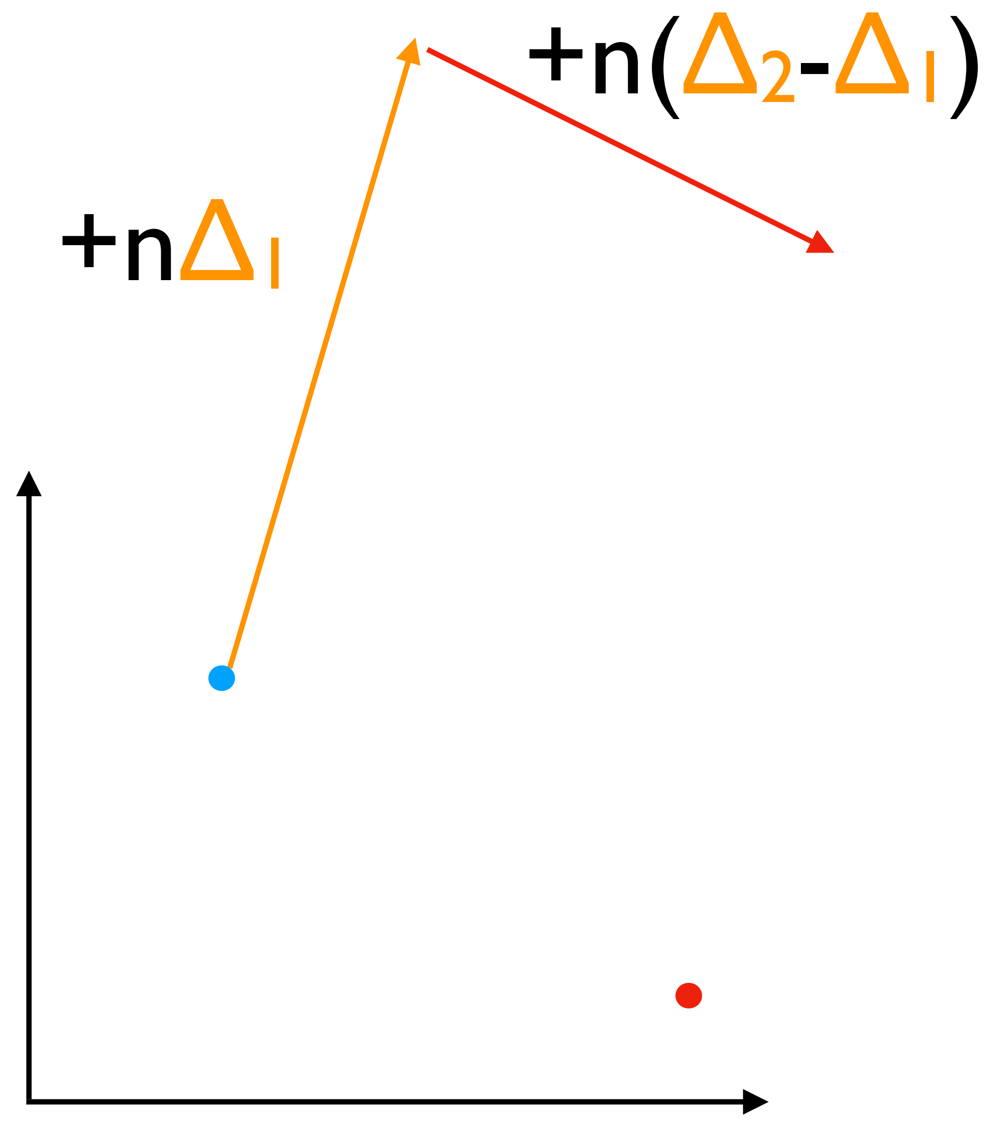

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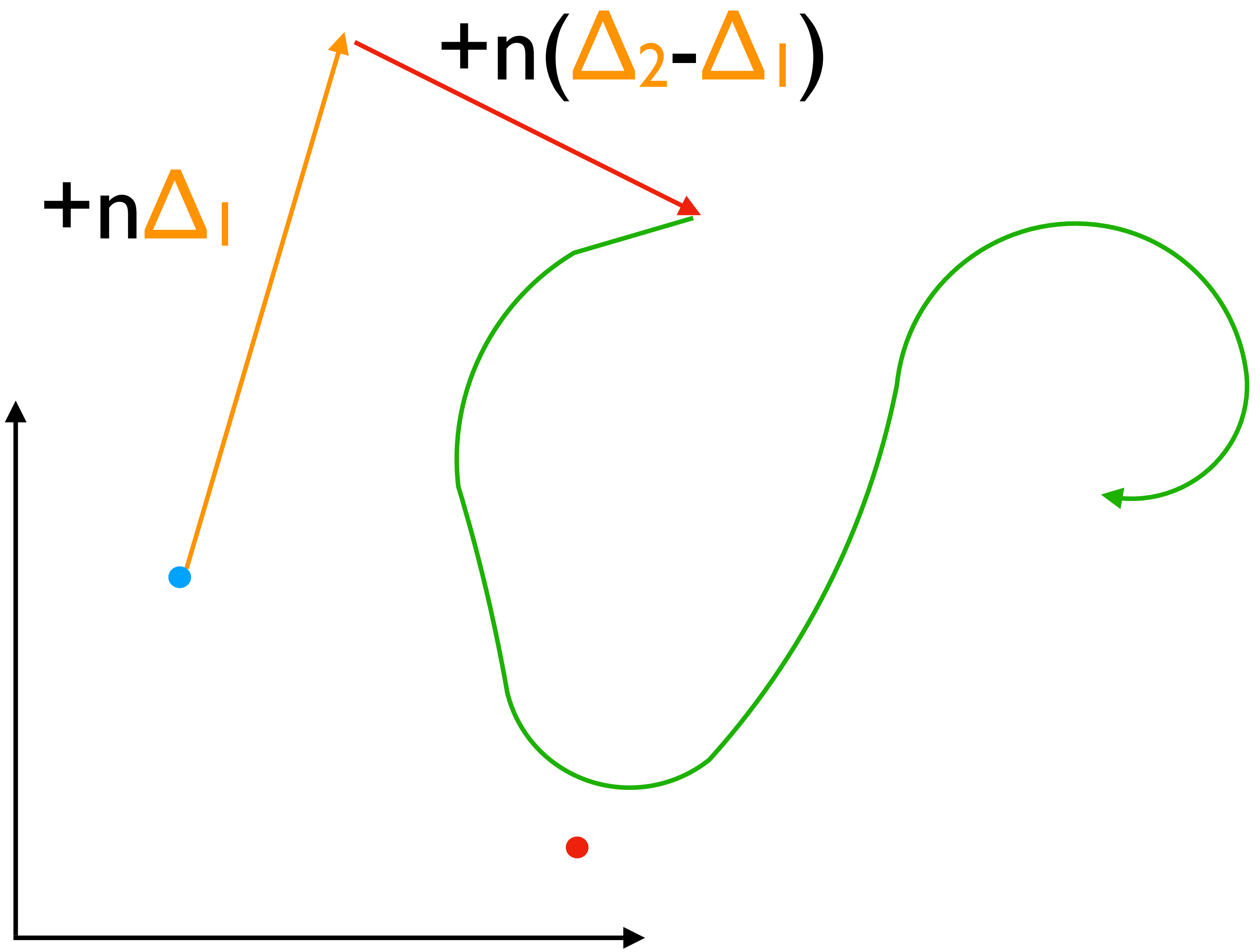

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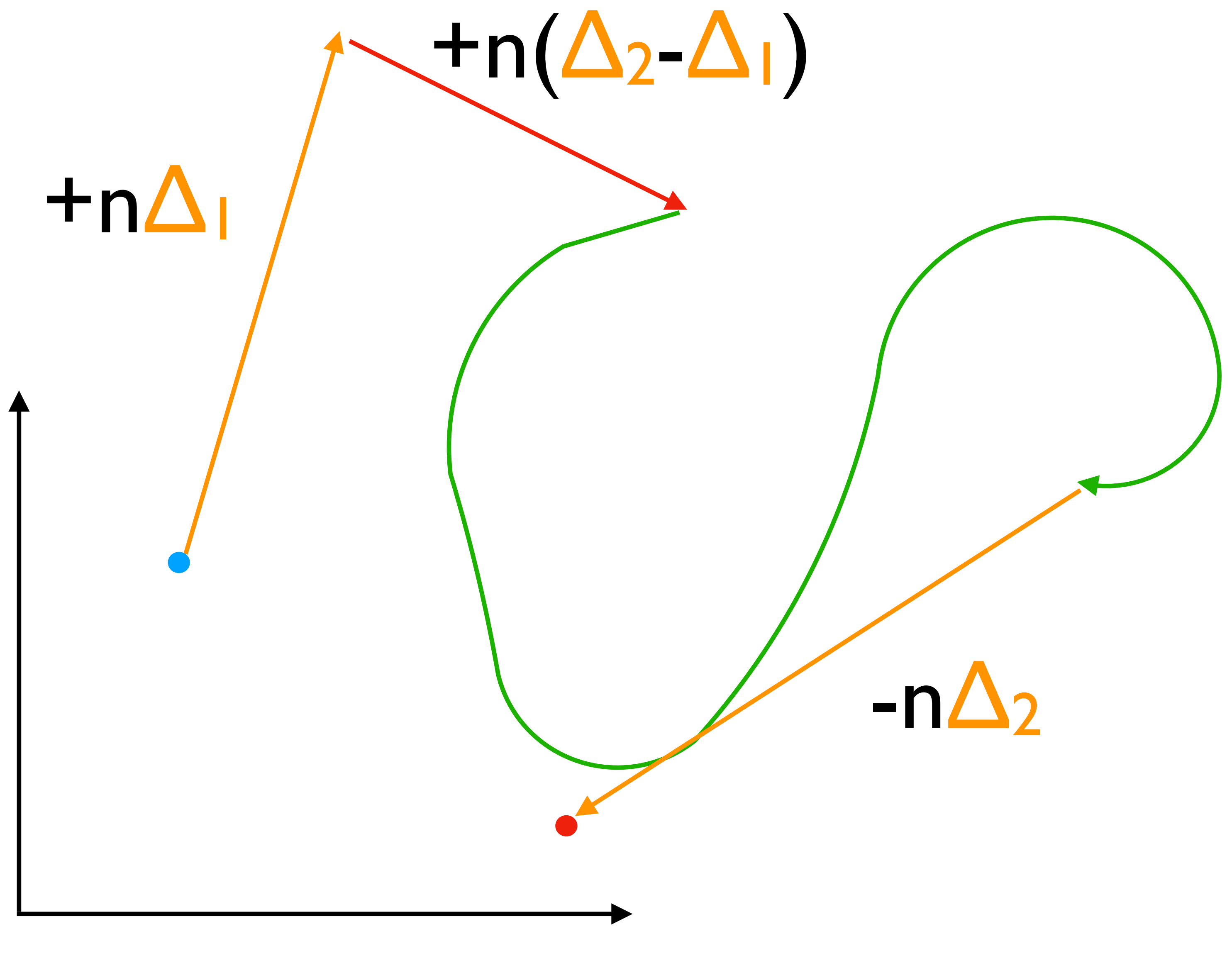

Algorithm

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Check whether:

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$p(s) \xrightarrow{\mathbb{Z}} q(t)$ by runs using each transition many times

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Check whether:

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If yes then return YES

Algorithm

Check whether:

$$p(s) \longrightarrow p(s+\Delta_1) \qquad q(t+\Delta_2) \longrightarrow q(t)$$

$p(s) \xrightarrow{\mathbb{Z}} q(t)$ by runs using each transition many times

If yes then return YES

If no then simplify

Algorithm

Check whether:

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$p(s) \xrightarrow{\mathbb{Z}} q(t)$ by runs using each transition many times

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Involved!

Ackermann-hardness

Ackermann-hardness

Theorem

The Reachability Problem for (3k+2)-VASSes is $\mathbb{F}_k$-hard.

Ackermann-hardness

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↑

Sławomir Lasota

Ackermann-hardness

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Sławomir Lasota

Cz., Łukasz Orlikowski: 6k

Ackermann-hardness

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The Reachability Problem for (3k+2)-VASSes is $\mathbb{F}_k$-hard.

↑

Sławomir Lasota

Cz., Łukasz Orlikowski: 6k

Jerome Leroux (currently): 2k+4

Big counters

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The following problem is $\mathbb{F}_k$-complete ($k \geq 3$)

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Given:

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Given: a two-counter automaton A with zero-tests, number n

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Question:

Big counters

The following problem is $\mathbb{F}_k$-complete ($k \geq 3$)

Given: a two-counter automaton A with zero-tests, number n

Question: does A have an $F_k(n)$-bounded run?

Multiplication triples

Multiplication triples

Lemma

If for each n there is a d-VASS with transitions of size $\leq$ n such that

Multiplication triples

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If for each n there is a d-VASS with transitions of size ≤ n such that

arbitrary big m guessed
some counters reach 0

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then reachability for d-VASSes is $\mathbb{F}_k$-hard

Multiplication triples

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If for each n there is a d-VASS with transitions of size $\leq n$ such that

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→ $(F_k(n), m, F_k(n)\, m)$

then reachability for d-VASSes is $\mathbb{F}_k$-hard

Proof: simulate $F_k(n)$-bounded run

Triples

Triples

Triples (B, m, Bm) allow zero-testing

Triples

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for m/2 zero-tests on B-bounded counters

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Triples (B, m, Bm) allow zero-testing

for $m/2$ zero-tests on B-bounded counters

Goal: compute $(F_k(n), m, F_k(n)\ m)$

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Triples (B, m, Bm) allow zero-testing

for $m/2$ zero-tests on B-bounded counters

Goal: compute $(F_k(n), m, F_k(n)\ m)$

For k = 1 easy: $(2n, 0, 0) + m(0, 1, 2n)$

Triples

Triples

Assume $(x, y, z) = (B, m, Bm)$

Triples

Assume $(x, y, z) = (B, m, Bm)$

Let $x' = 0$

Triples

Assume $(x, y, z) = (B, m, Bm)$

keep $x+x' = B$

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Triples

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Let x’ = 0

zero-test(x’):

Triples

Assume (x, y, z) = (B, m, Bm)

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Let x’ = 0

zero-test(x’):

loop {inc(x’), dec(x), dec(z)}

Triples

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Triples

Assume (x, y, z) = (B, m, Bm)

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Let x’ = 0

zero-test(x’):

loop {inc(x’), dec(x), dec(z)}

loop {dec(x’), inc(x), dec(z)}

y := y-2

Triples

Assume (x, y, z) = (B, m, Bm)

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zero-test(x’):

loop {inc(x’), dec(x), dec(z)}

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y dec by 2

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Triples

Assume (x, y, z) = (B, m, Bm)

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Let x' = 0

zero-test(x'):

loop {inc(x'), dec(x), dec(z)}

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z dec by ≤2B

Triples

Assume (x, y, z) = (B, m, Bm)

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zero-test(x'):

loop {inc(x'), dec(x), dec(z)}

y dec by 2

loop {dec(x'), inc(x), dec(z)}

z dec by ≤2B

y := y-2

At the end check if z = 0

Testing more counters

Testing more counters

Goal: z dec by $\leq 2B$

Testing more counters

Goal: z dec by $\leq 2B$

Let $x_1 + \ldots + x_k = B$

Testing more counters

Goal: z dec by $\leq 2B$

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zero-test(x_1):

Testing more counters

Goal: z dec by ≤ 2B

Let $x_1 + \ldots + x_k =$ B

zero-test(x_1):

transfer(x_2, x_1, z)

Testing more counters

Goal: z dec by $\leq 2B$

Let $x_1 + \ldots + x_k = B$

loop {dec(x_2), inc(x_1), dec(z)}

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Testing more counters

Goal: z dec by $\leq 2B$

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transfer(x_3, x_2, z)

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transfer(x_1, x_2, z)

$y := y-2$

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each token moved at most twice

Testing more counters

Goal: z dec by $\leq 2B$

Let $x_1 + \ldots + x_k = B$

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If there is a d-VASS such that

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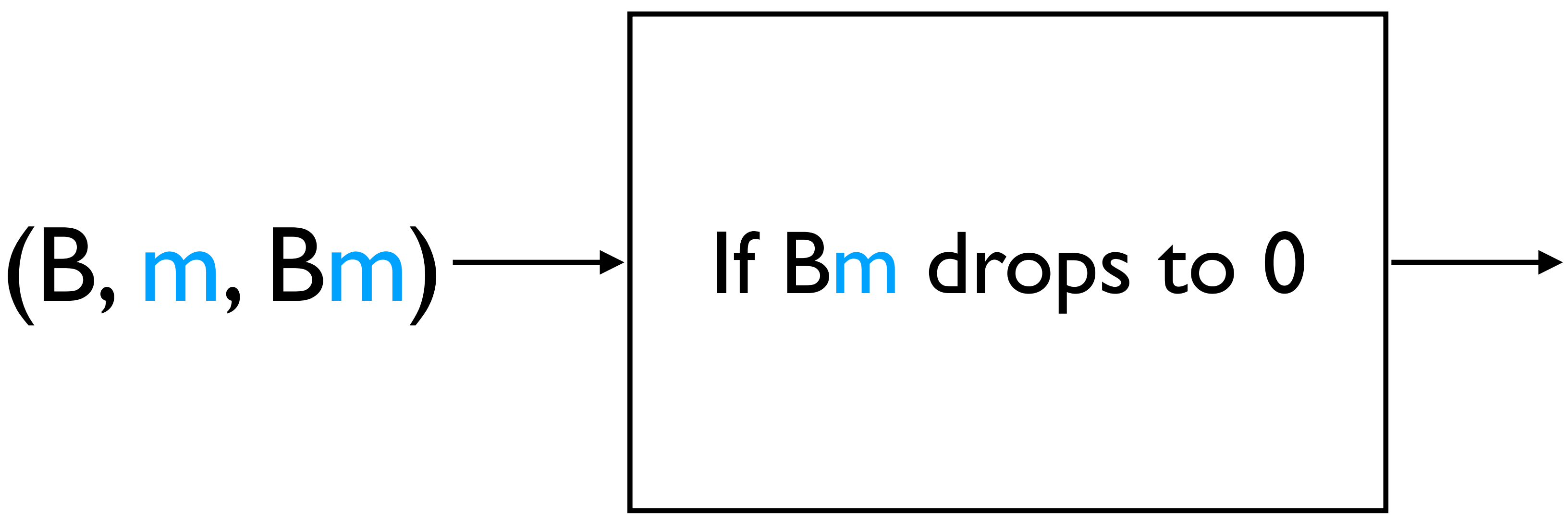

Triples

Lemma

If there is a d-VASS such that

(B, m, Bm) → [If Bm drops to 0] → (F_k(B), m', F_k(B) m')

Triples

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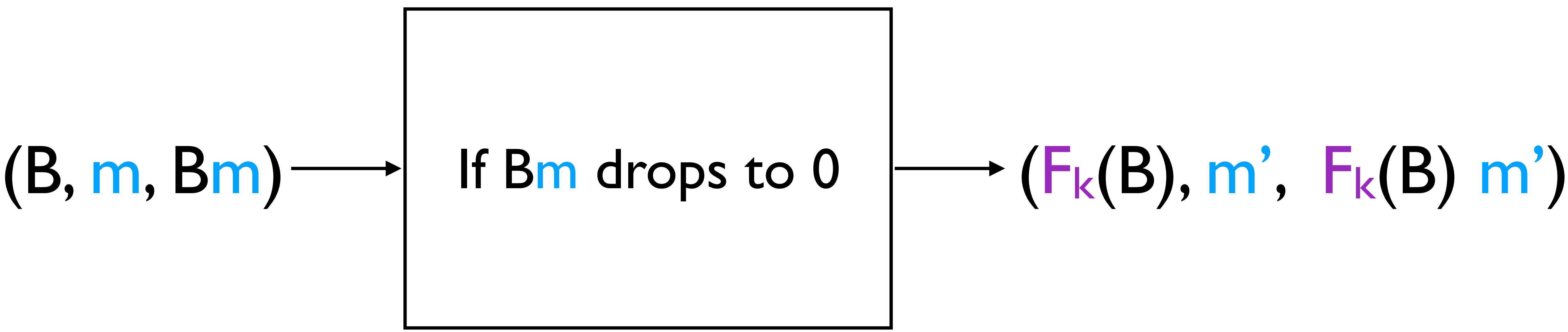

then reachability for d-VASSes is $\mathbb{F}_k$-hard.

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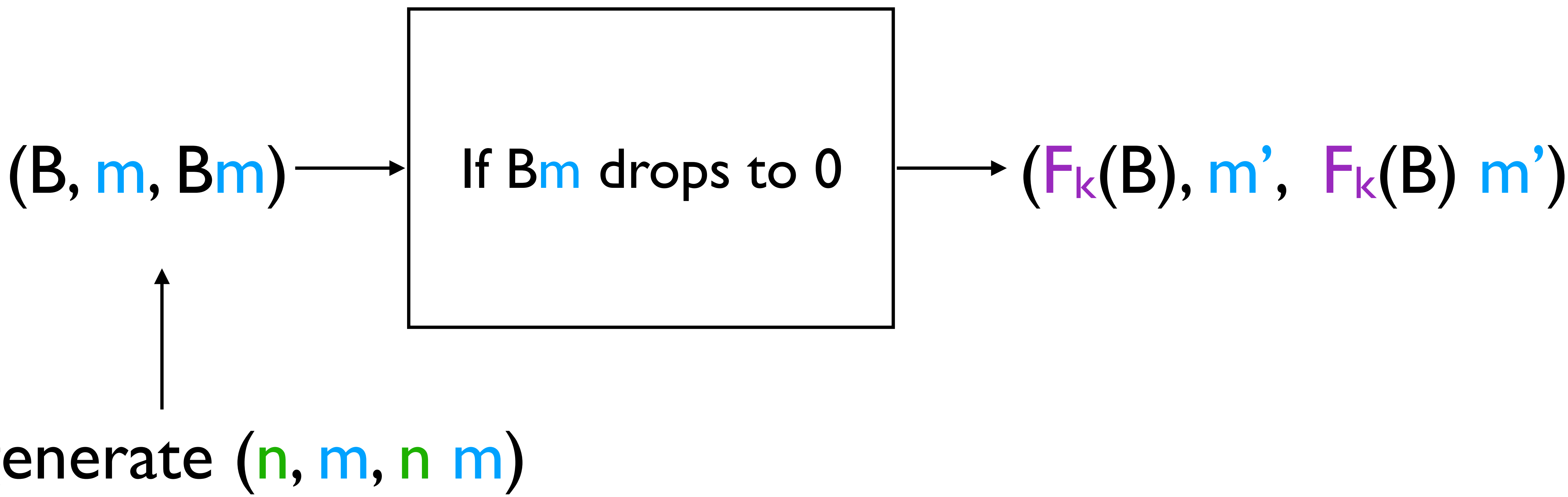

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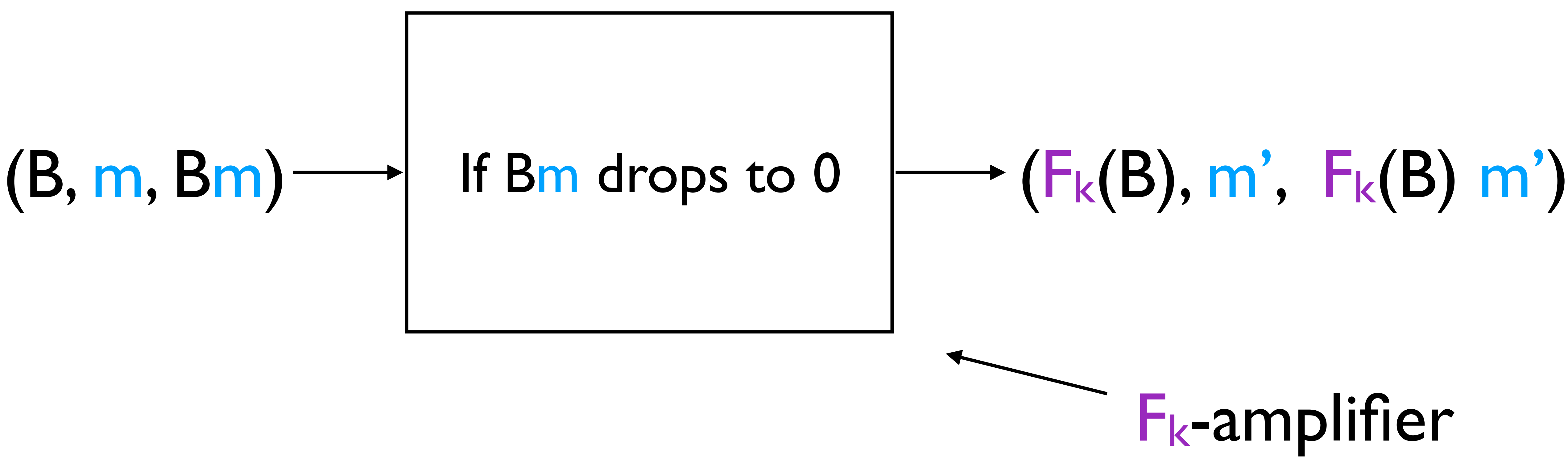

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F_k-amplifiers of dimension 3k+2 exist.

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Proof idea: induction on k

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Lemma

F_k-amplifiers of dimension $3k+2$ exist.

Proof idea: induction on k

To get F_k-amplifier apply n times F_{k-1}-amplifier

General strategy

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from algorithm computing $(F_{k-1}(n), m, F_{k-1}(n)\ m)$

$$F_k(n) = F_{k-1} \circ \ldots \circ F_{k-1}(1)$$

Idea: start from triple $(1, m, m)$

n times apply F_{k-1} to 1

F_k-amplifiers

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(B, m, Bm) gives B/2 zero-tests on m-bounded counters

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repeat B/12 times:

apply (x, y, z) $\xrightarrow{F_{k-1}\text{-amplifier}}$ (x', y', z')

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repeat B/12 times:

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zero-test x, y, z

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transfer back x' to x, y' to y, z' to z

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we have (x, y, z) = (F_k(B), m', F_k(B) m')

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F'_k-amplifiers

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F_k-amplifiers

(B, m, Bm) gives B/2 zero-tests on m-bounded counters

goal: (F_k(B), m', F_k(B) m')   guess (1, M, M)

repeat B/12 times:

apply (x, y, z) —F_{k-1}-amplifier→ (x', y', z')

zero-test x, y, z

transfer back x' to x, y' to y, z' to z

zero-test x', y', z'

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Thank you!