The Reachability Problem for Vector Addition Systems is Not Elementary

Wojciech Czerwiński

Sławomir Lasota

Ranko Lazić

Jérôme Leroux

Filip Mazowiecki

Plan

Plan

- basic notions and problem

Plan

- basic notions and problem
- short history

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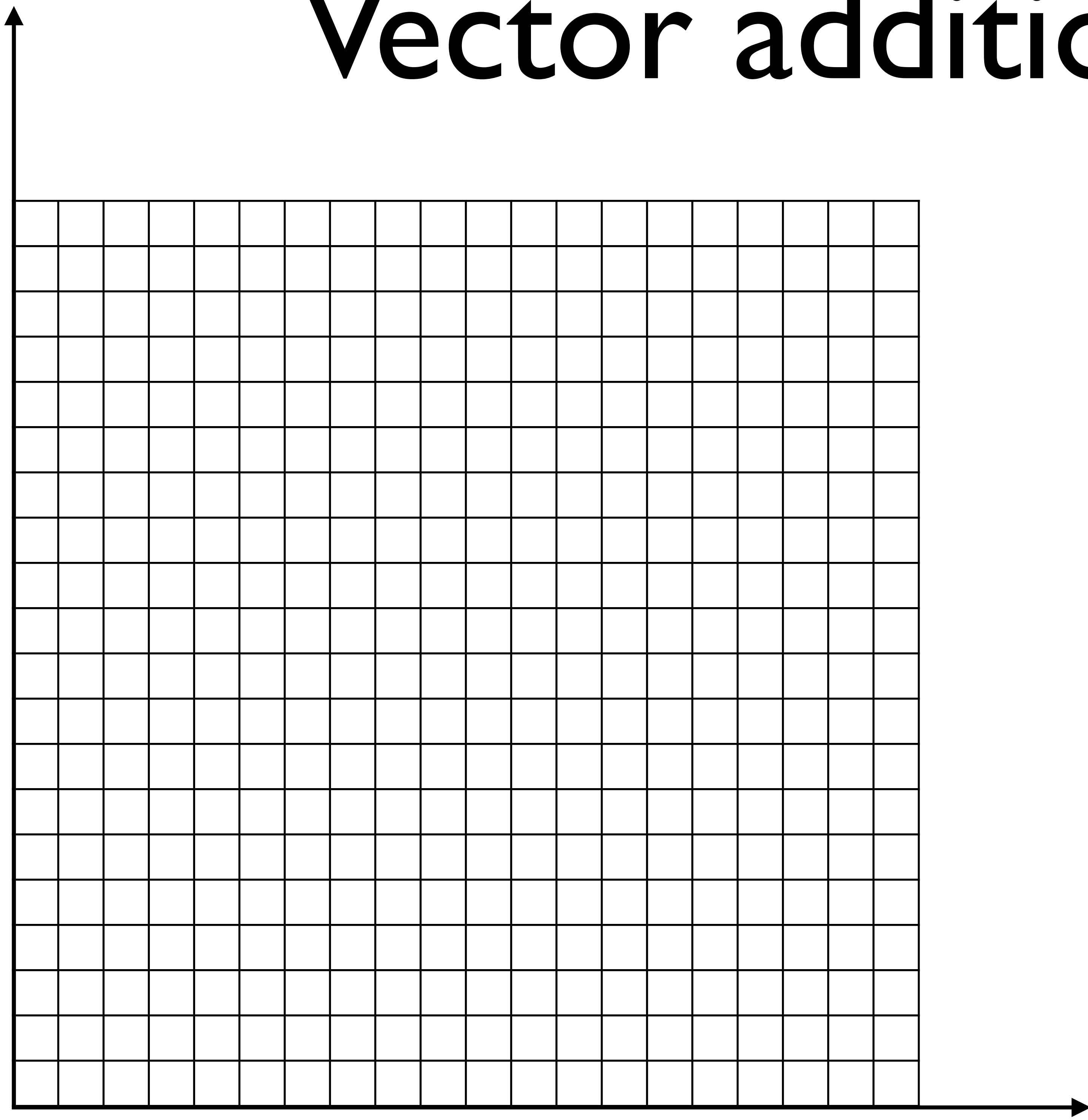
- basic notions and problem
- short history
- our result

Plan

- basic notions and problem
- short history
- our result
- intuitions behind the solution

Vector addition systems

Vector addition systems



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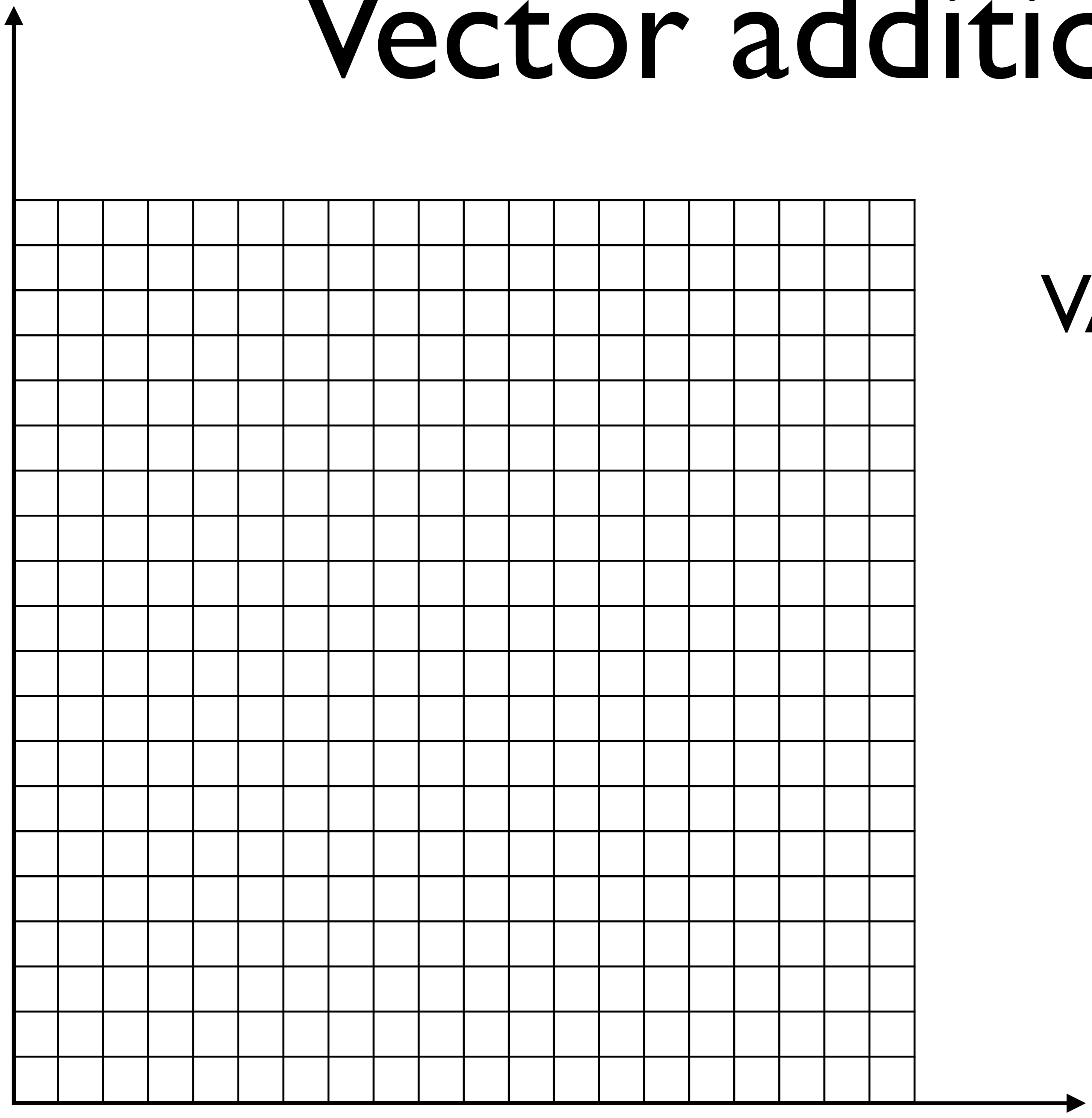
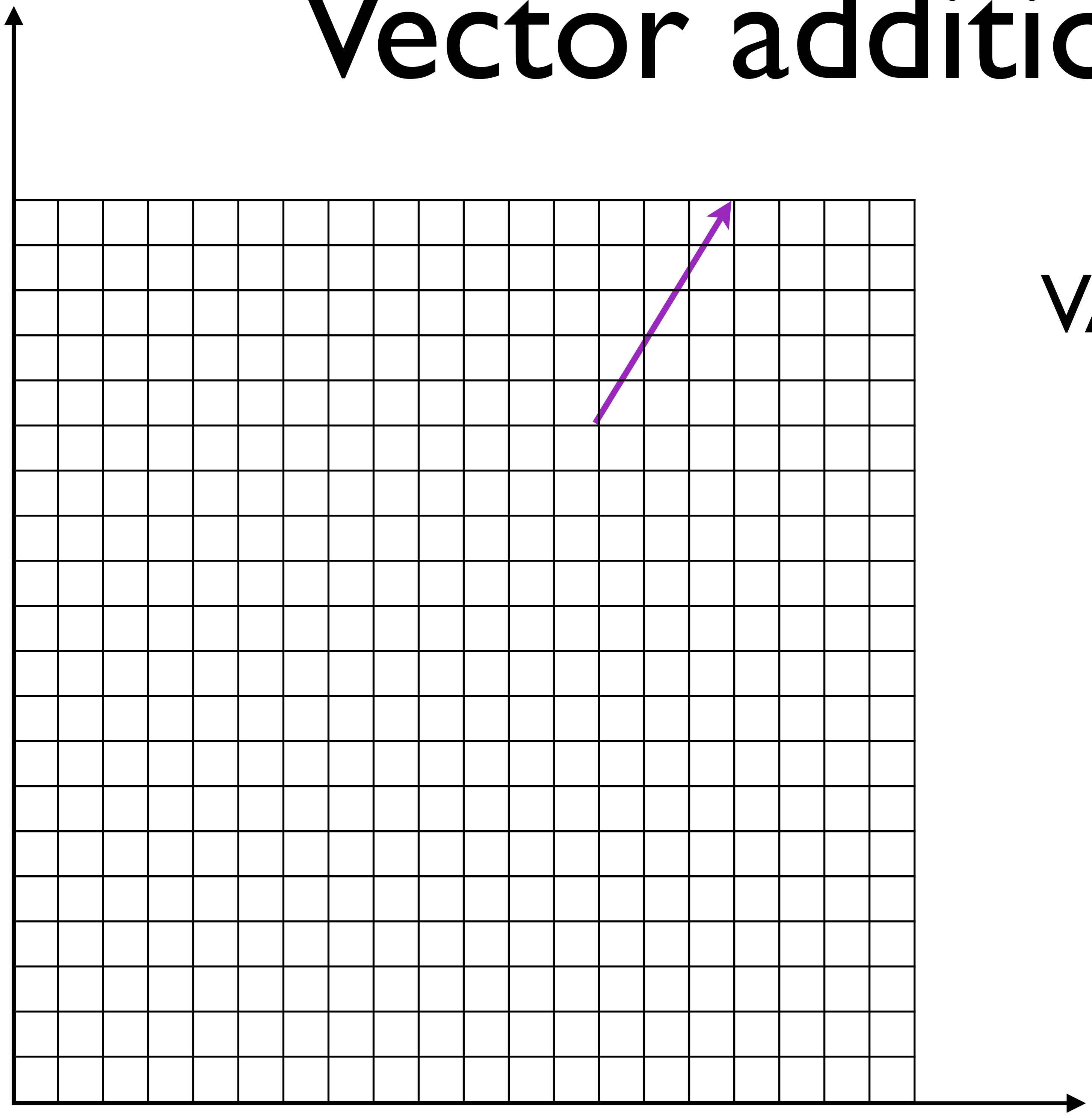


VAS = set of transitions

Vector addition systems



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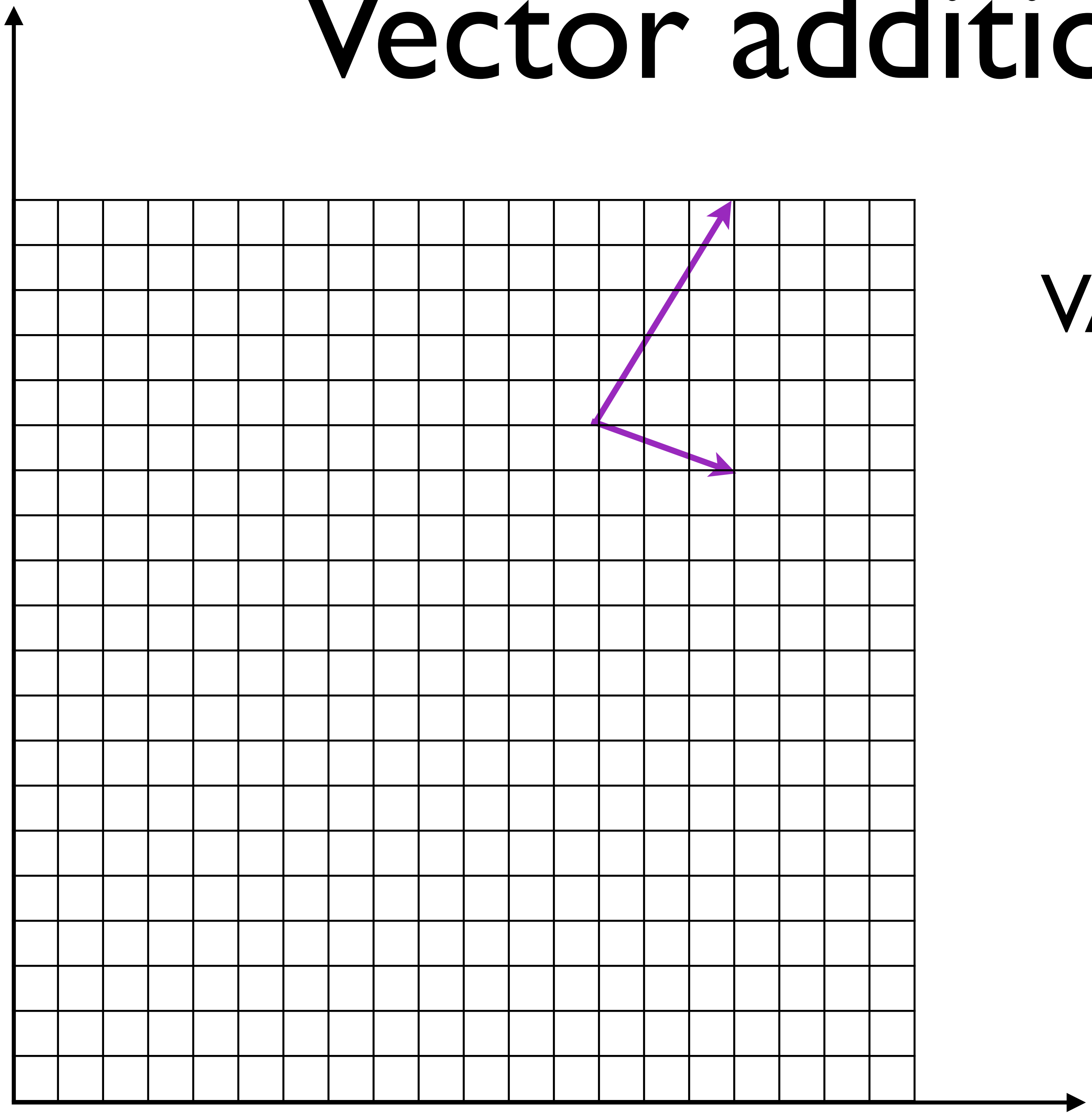

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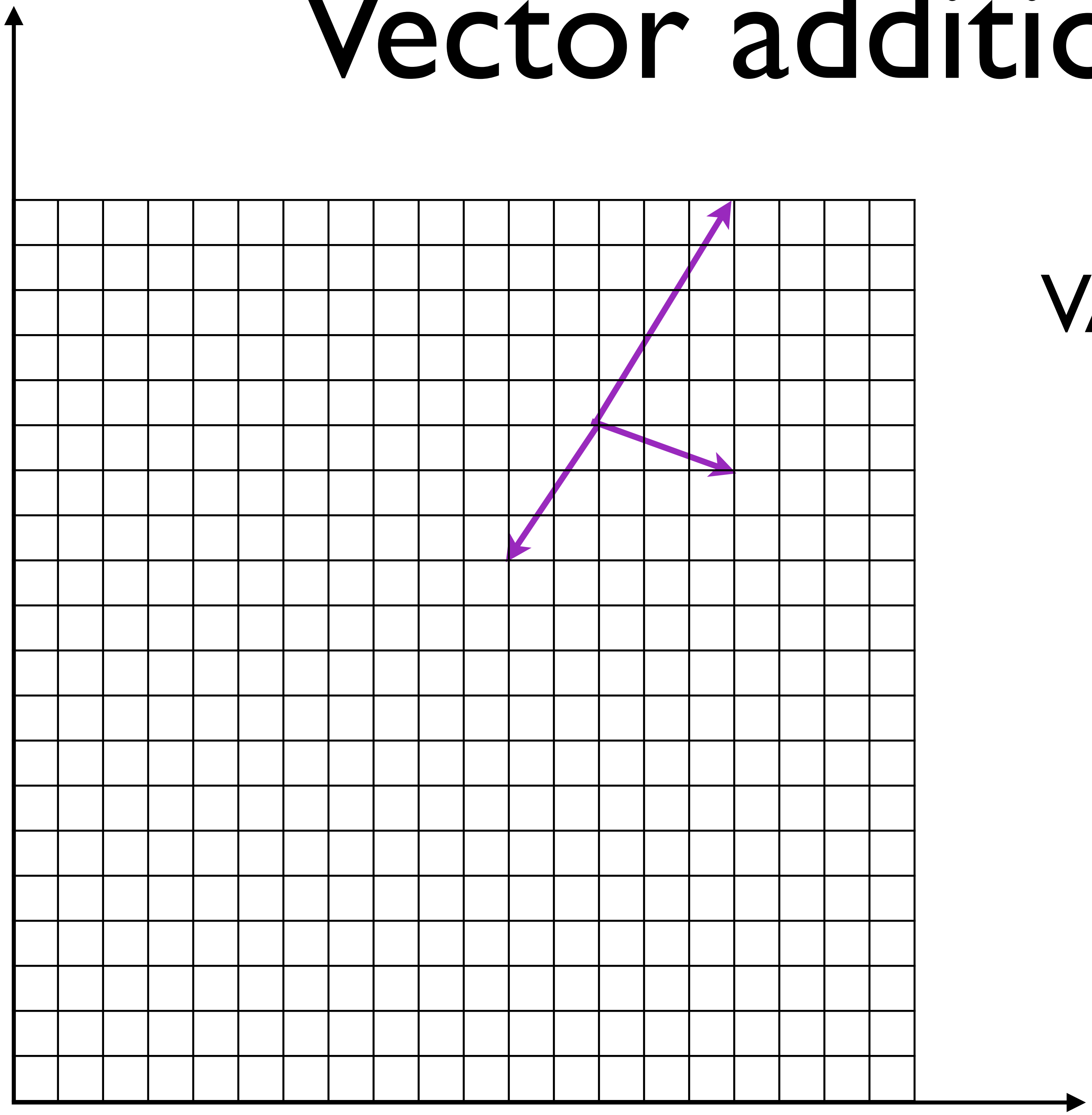

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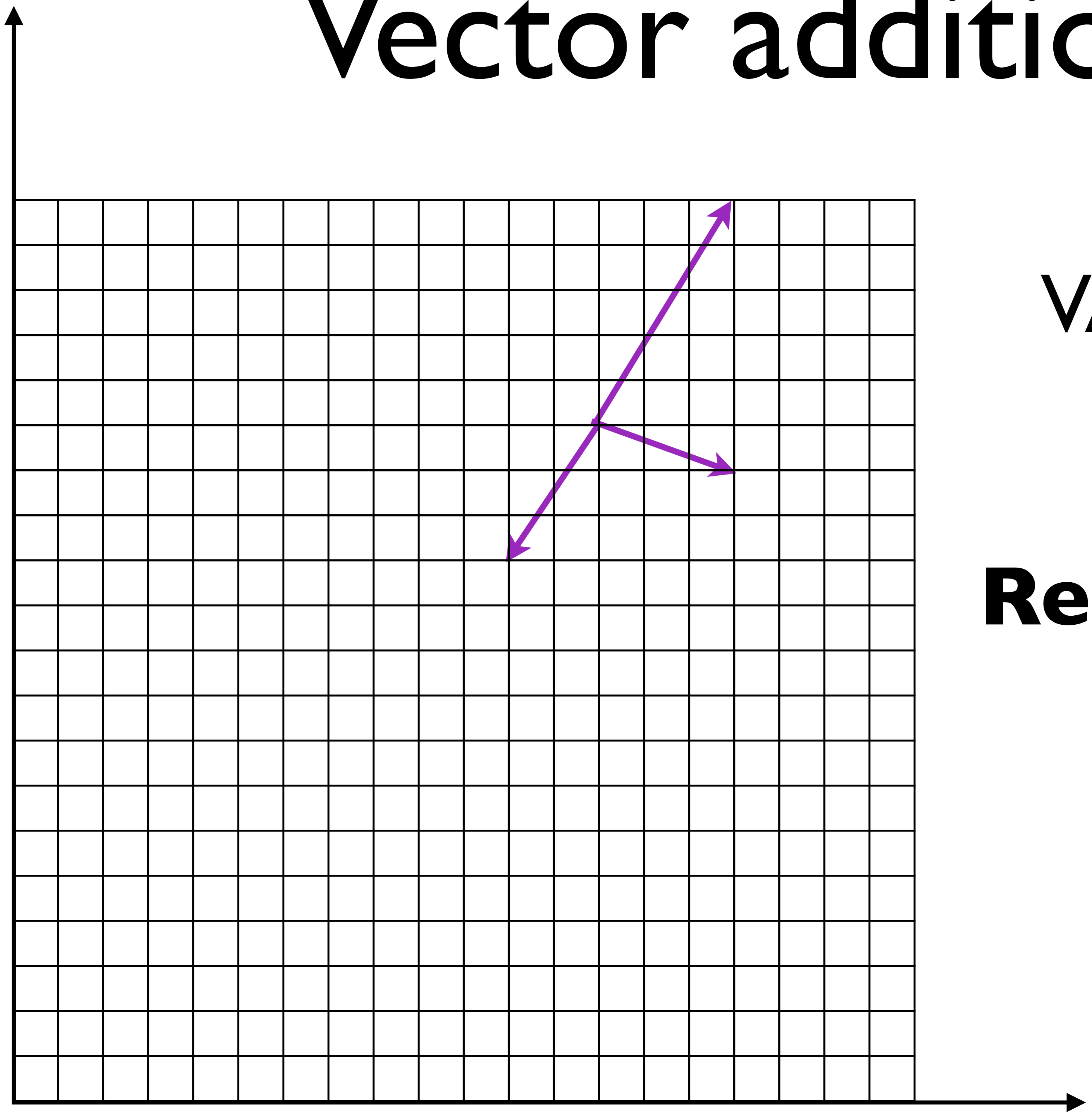

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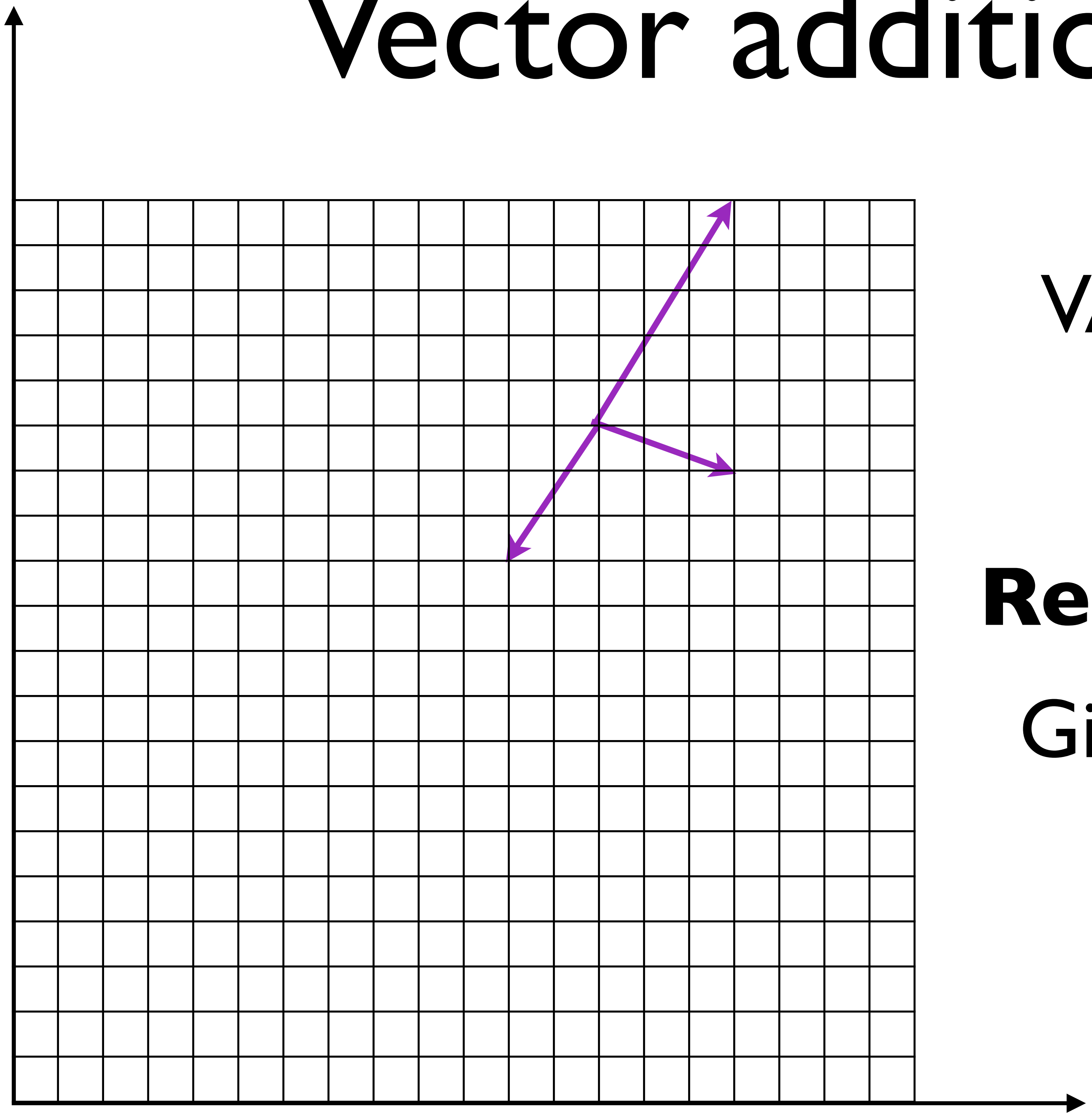

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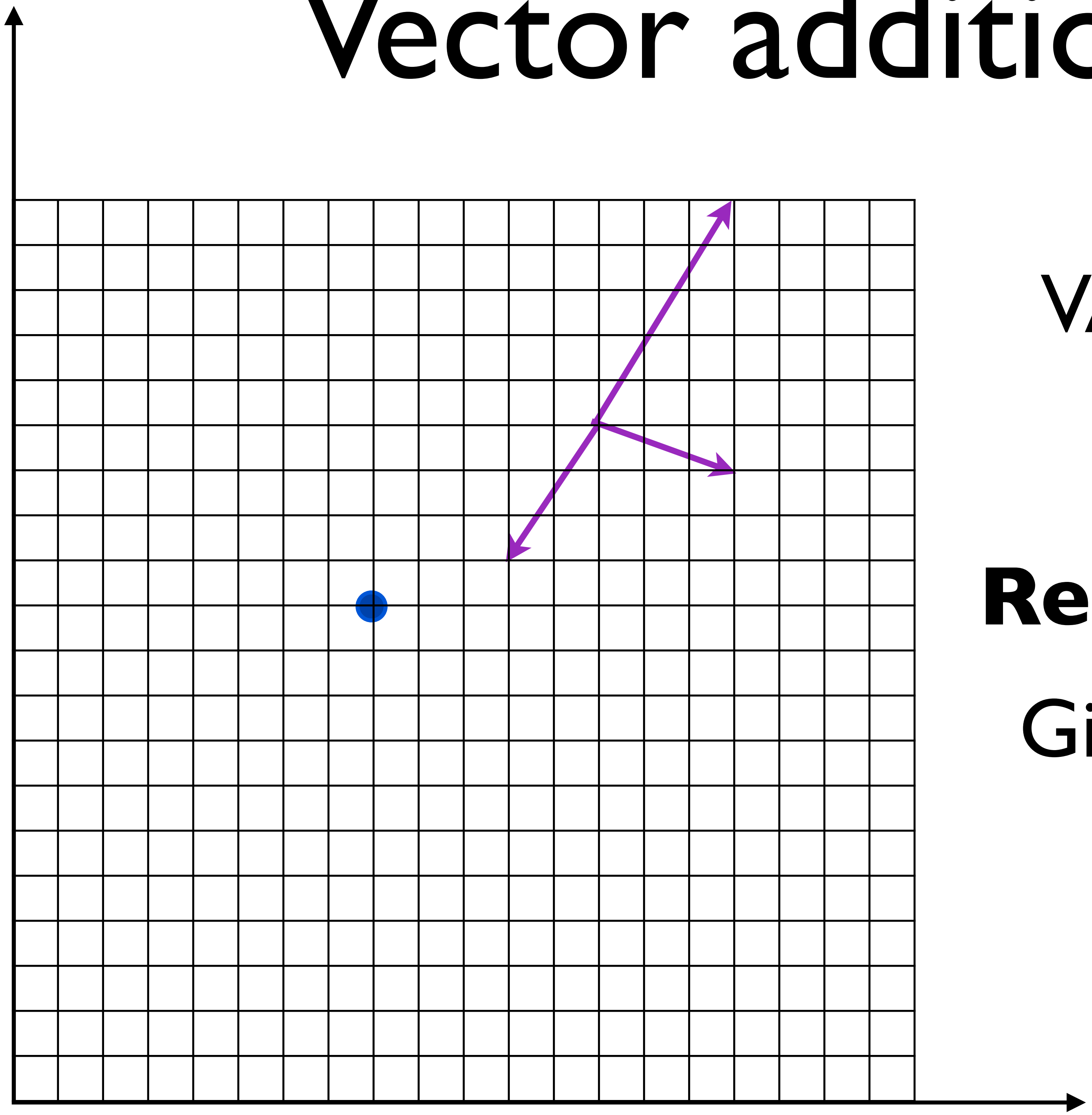

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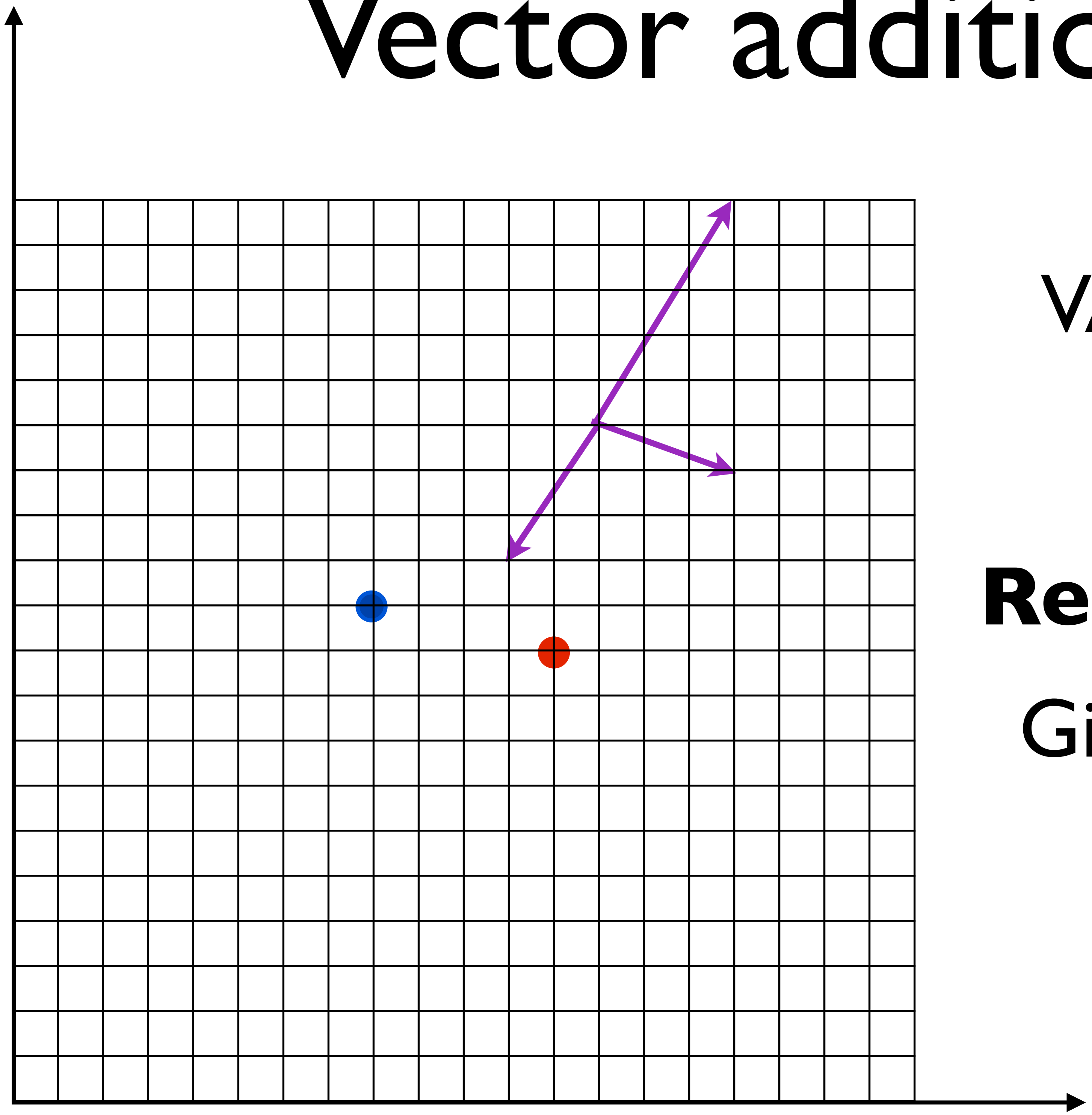

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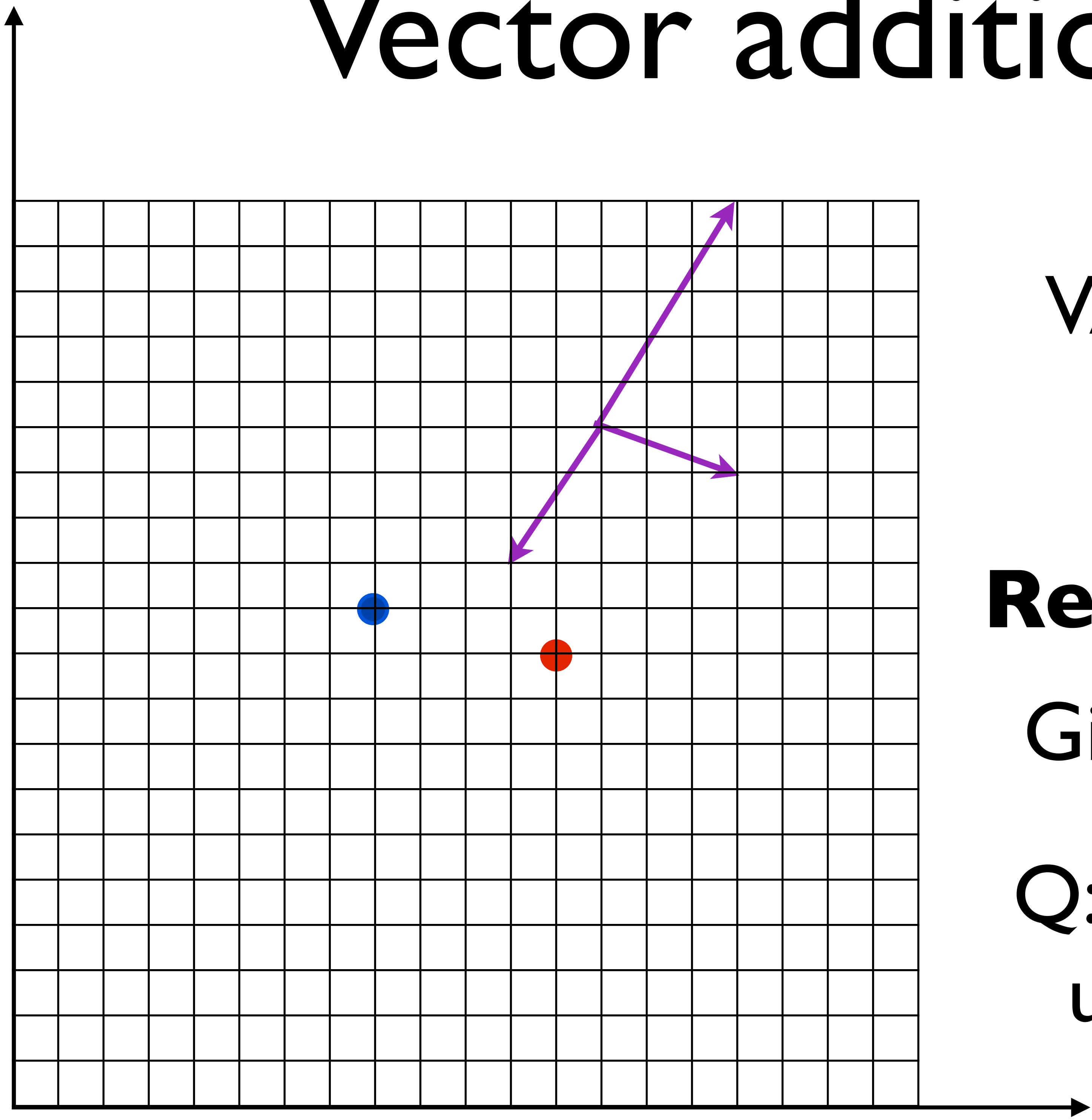

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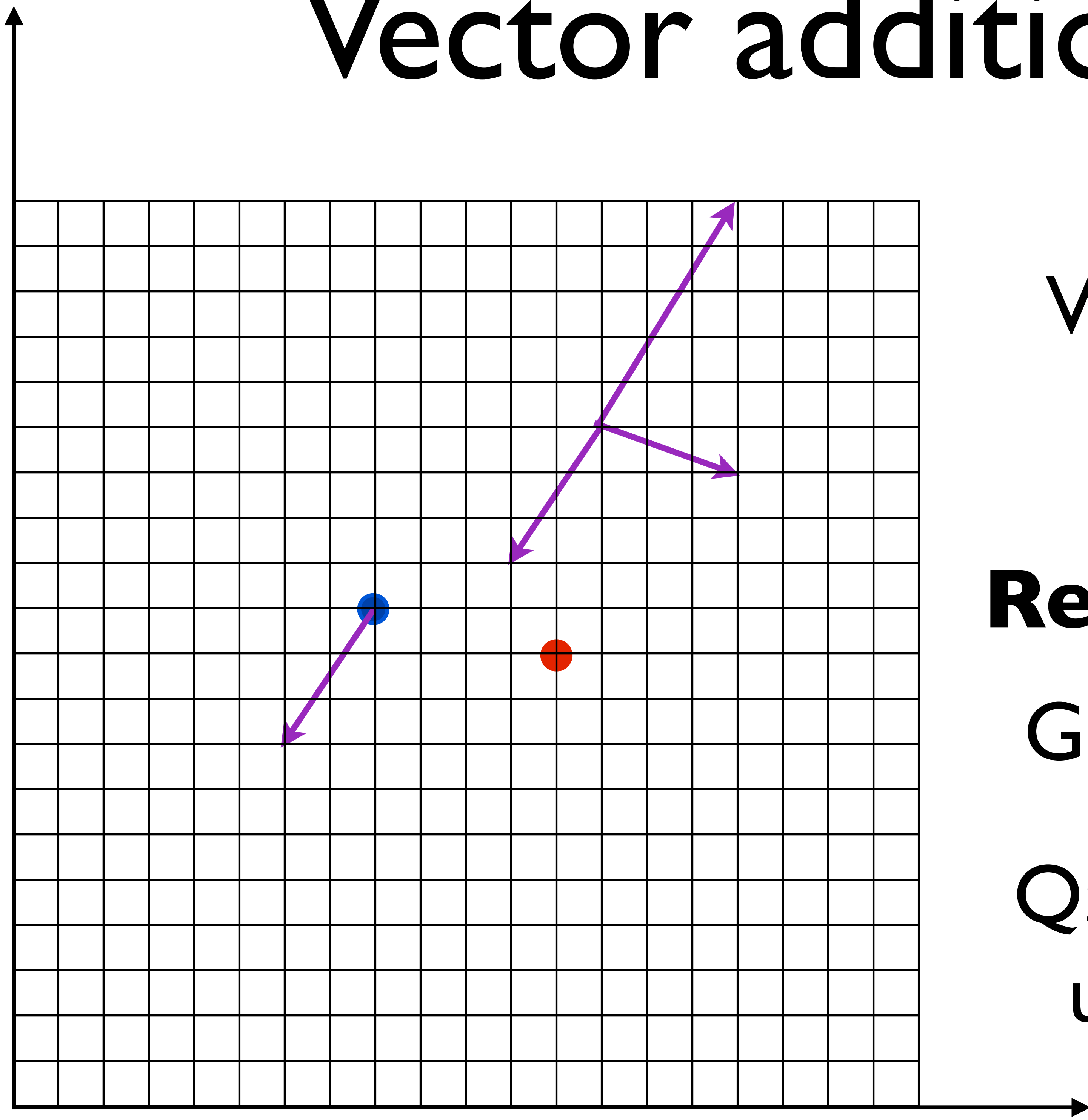

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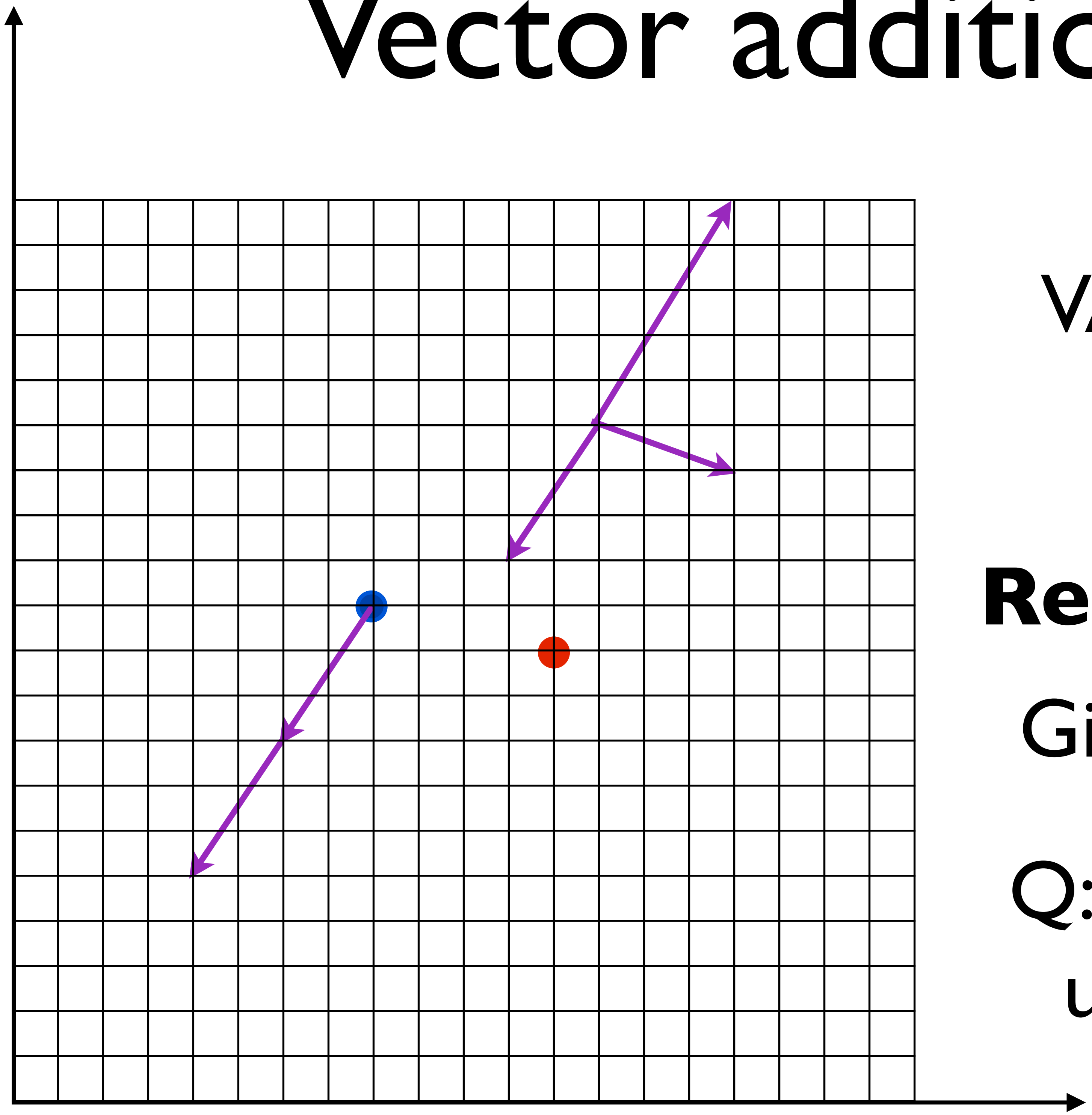

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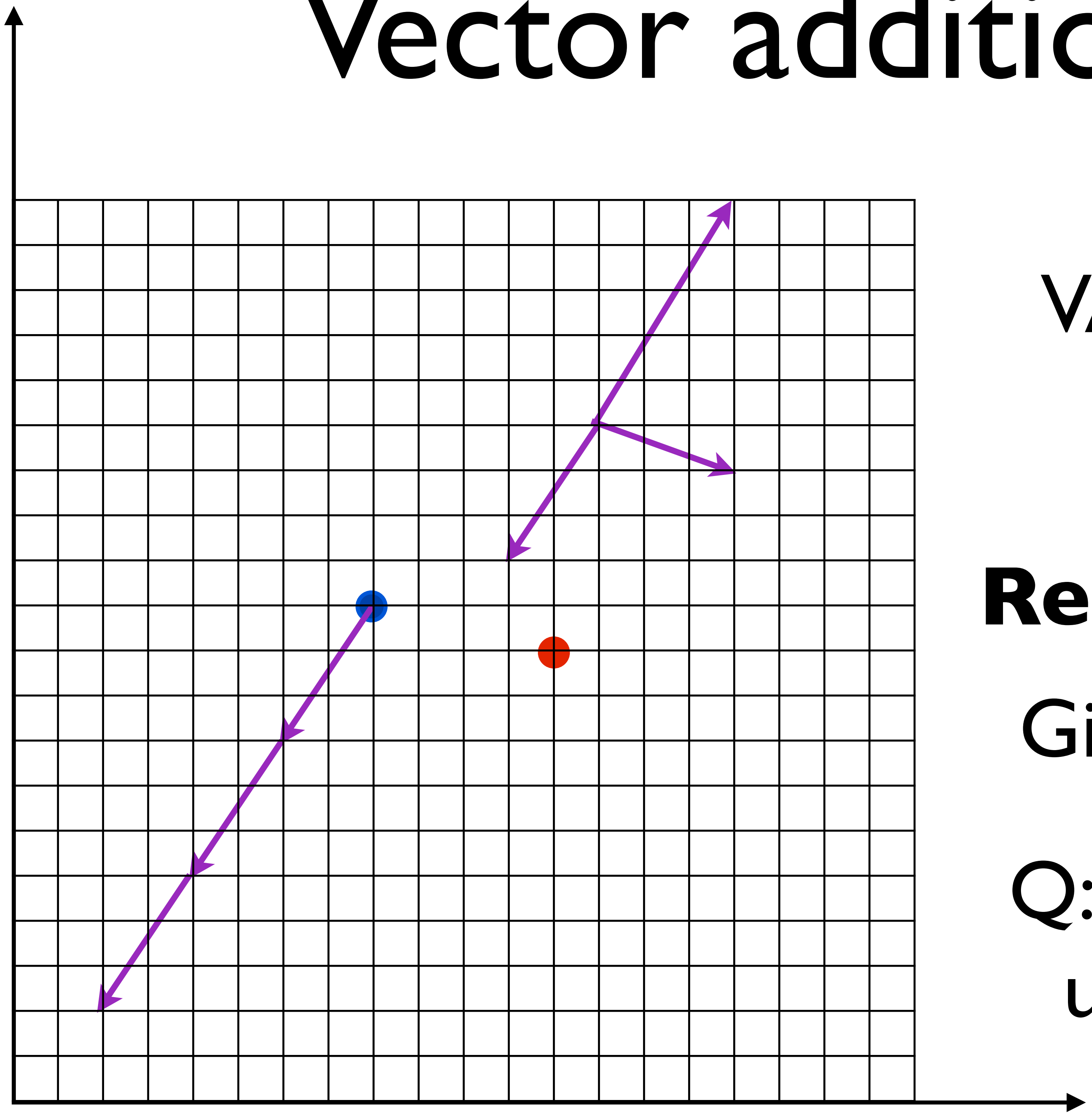

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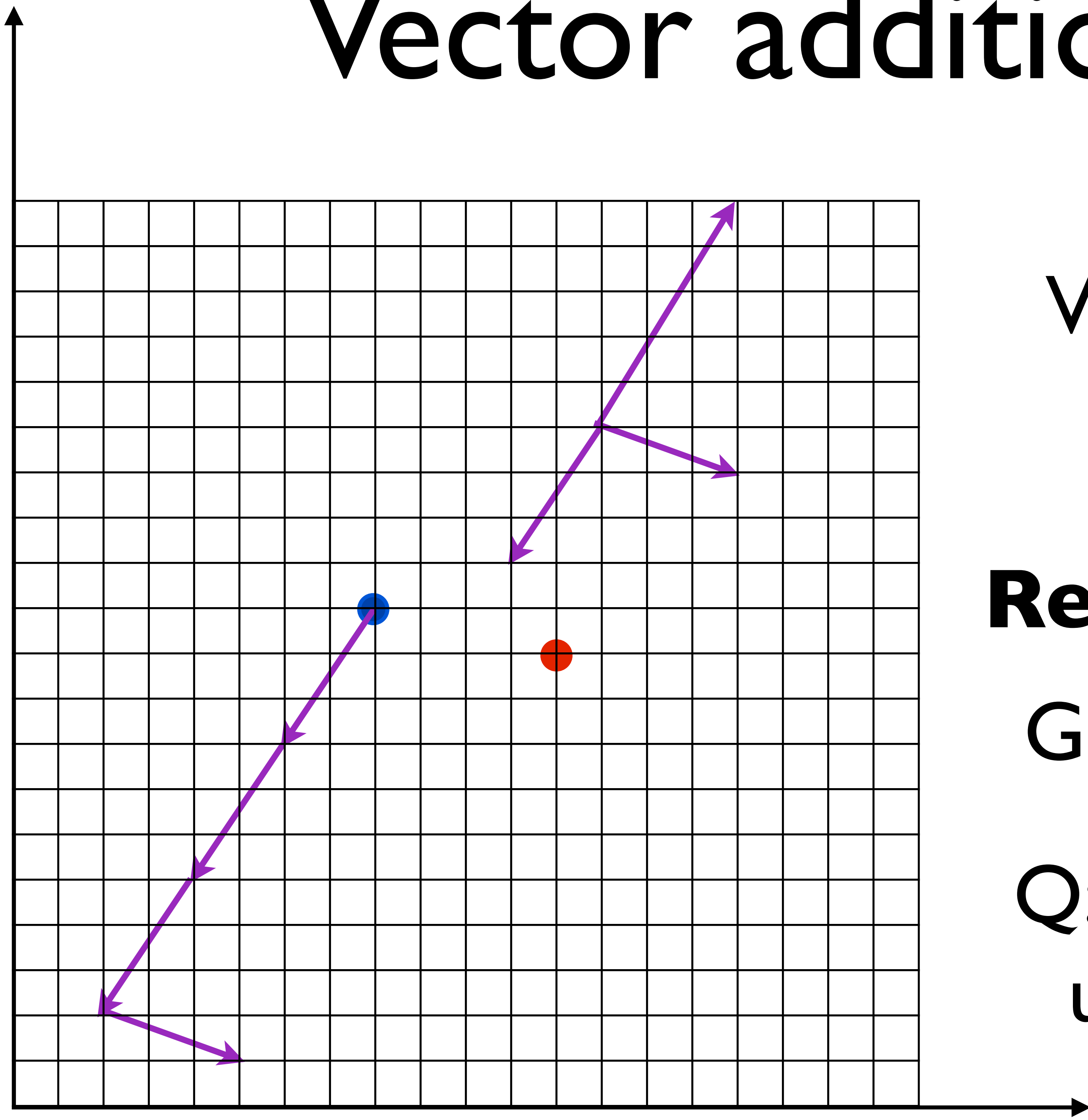

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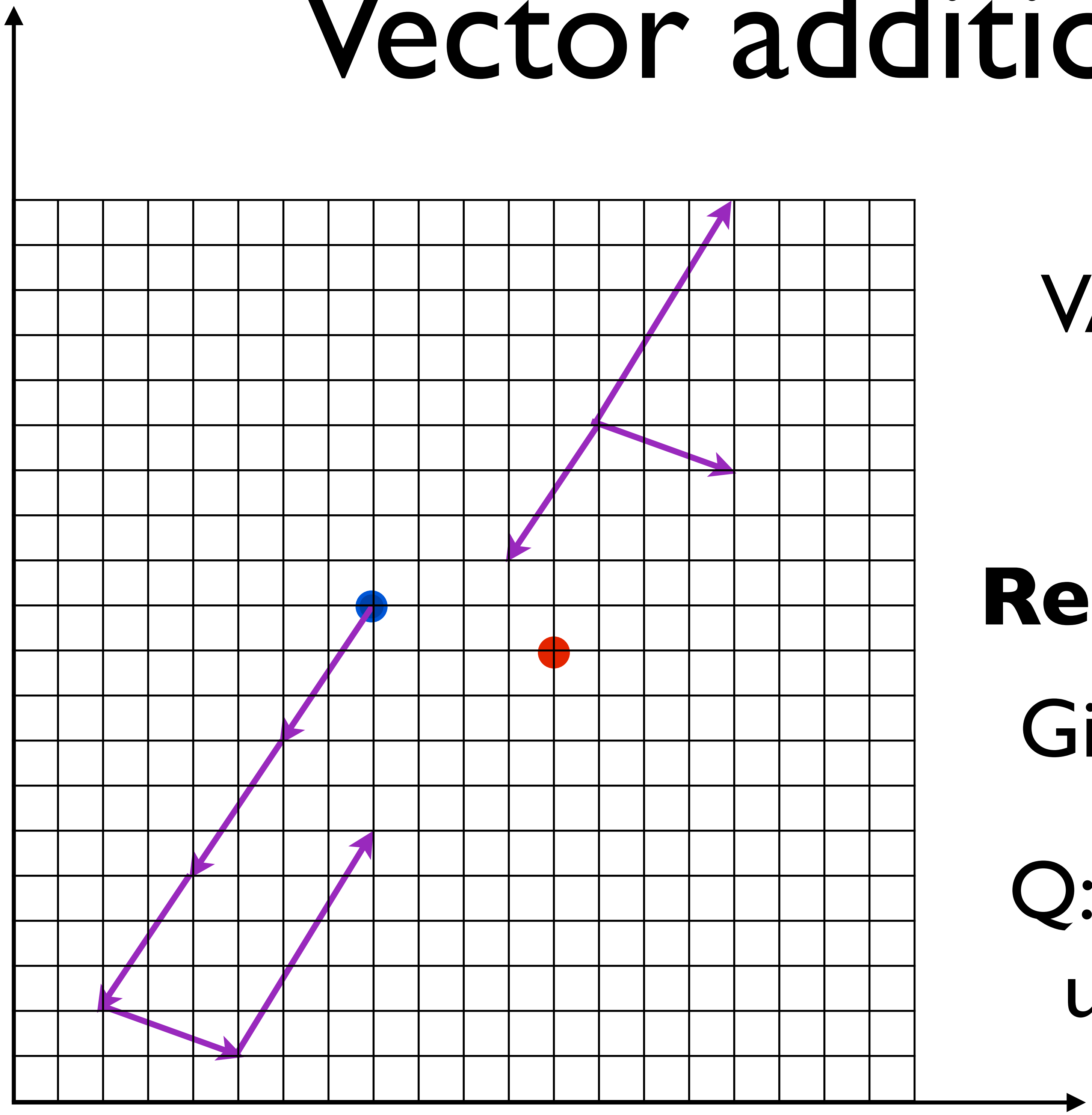

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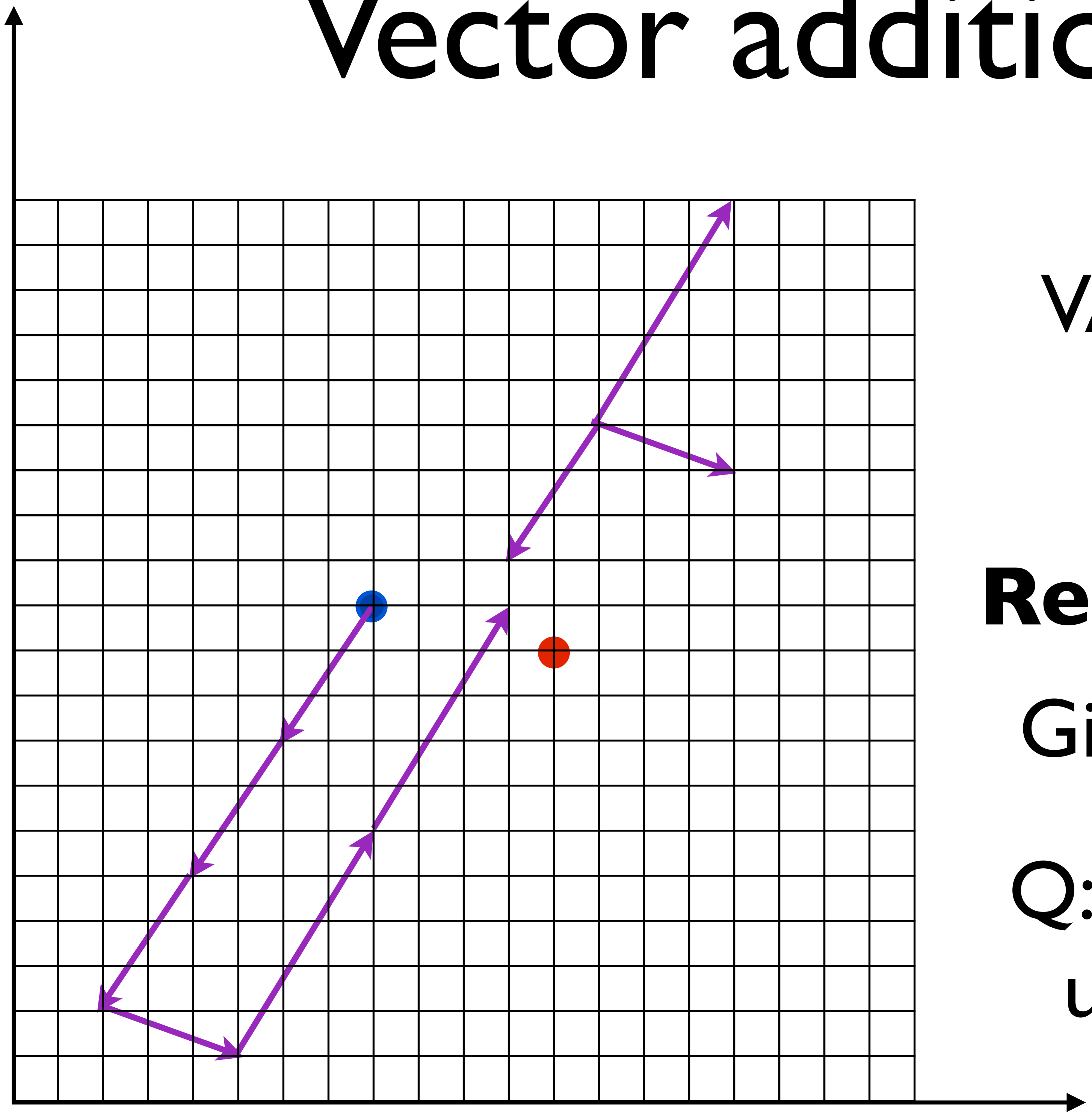

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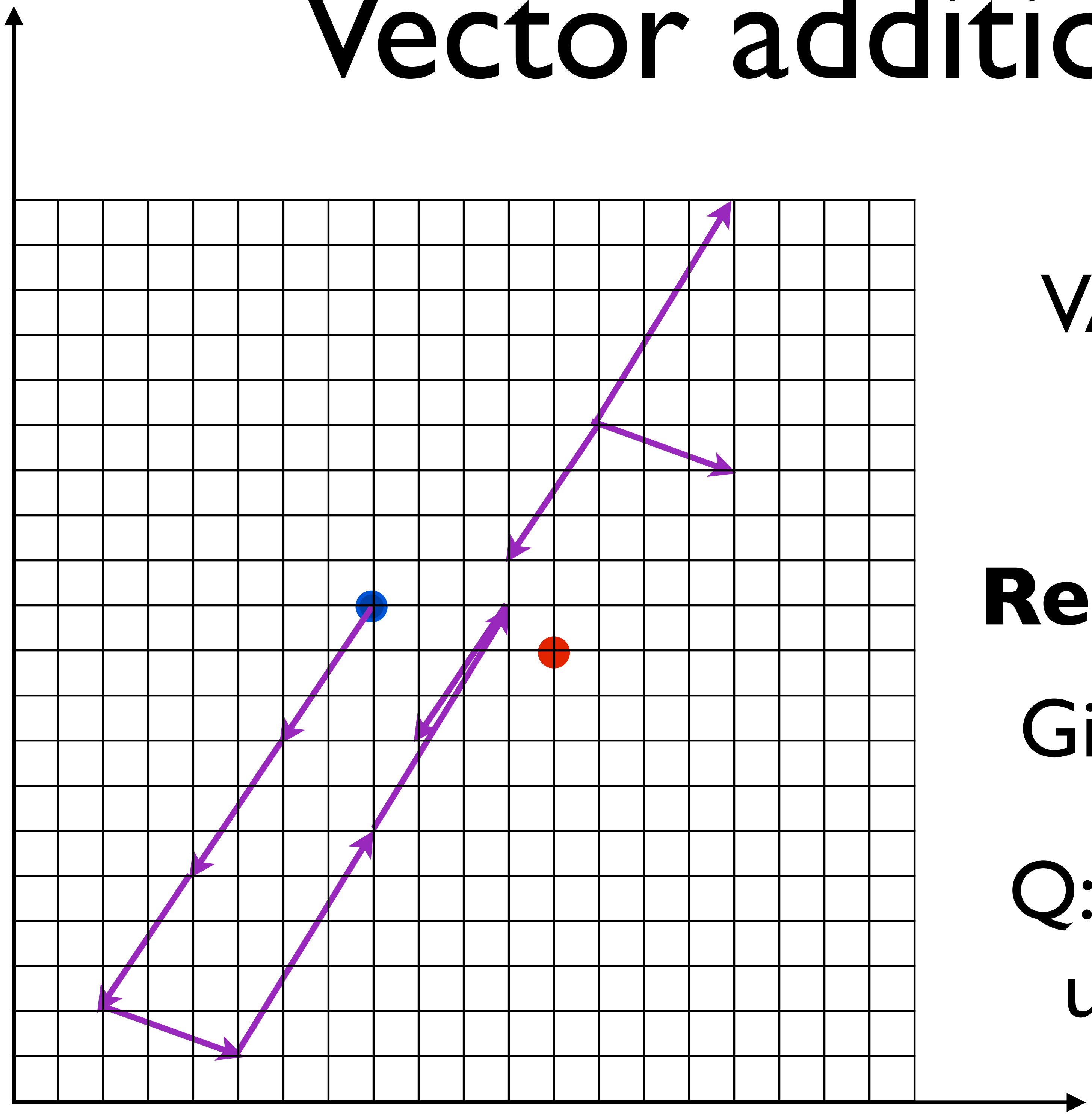

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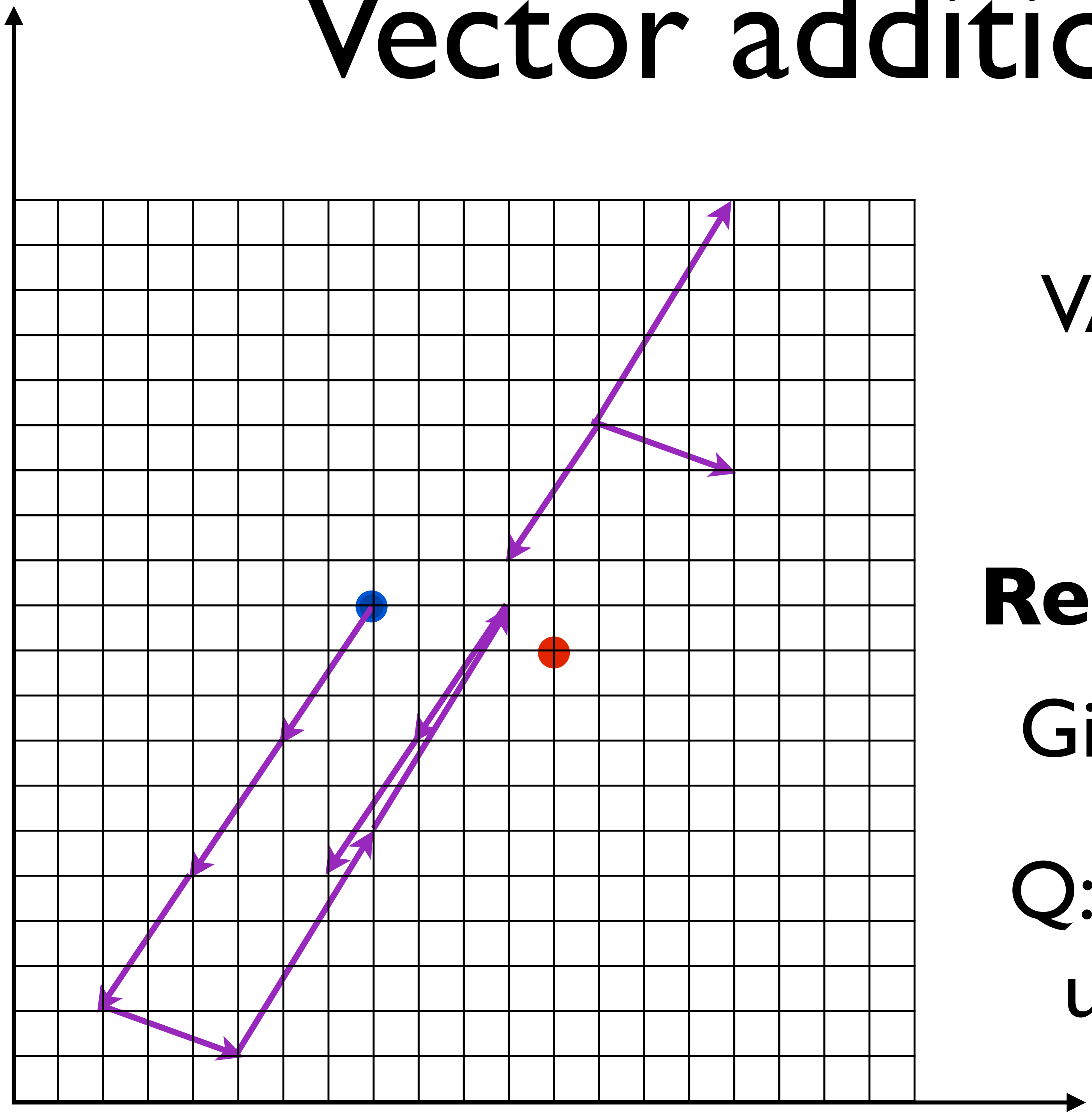

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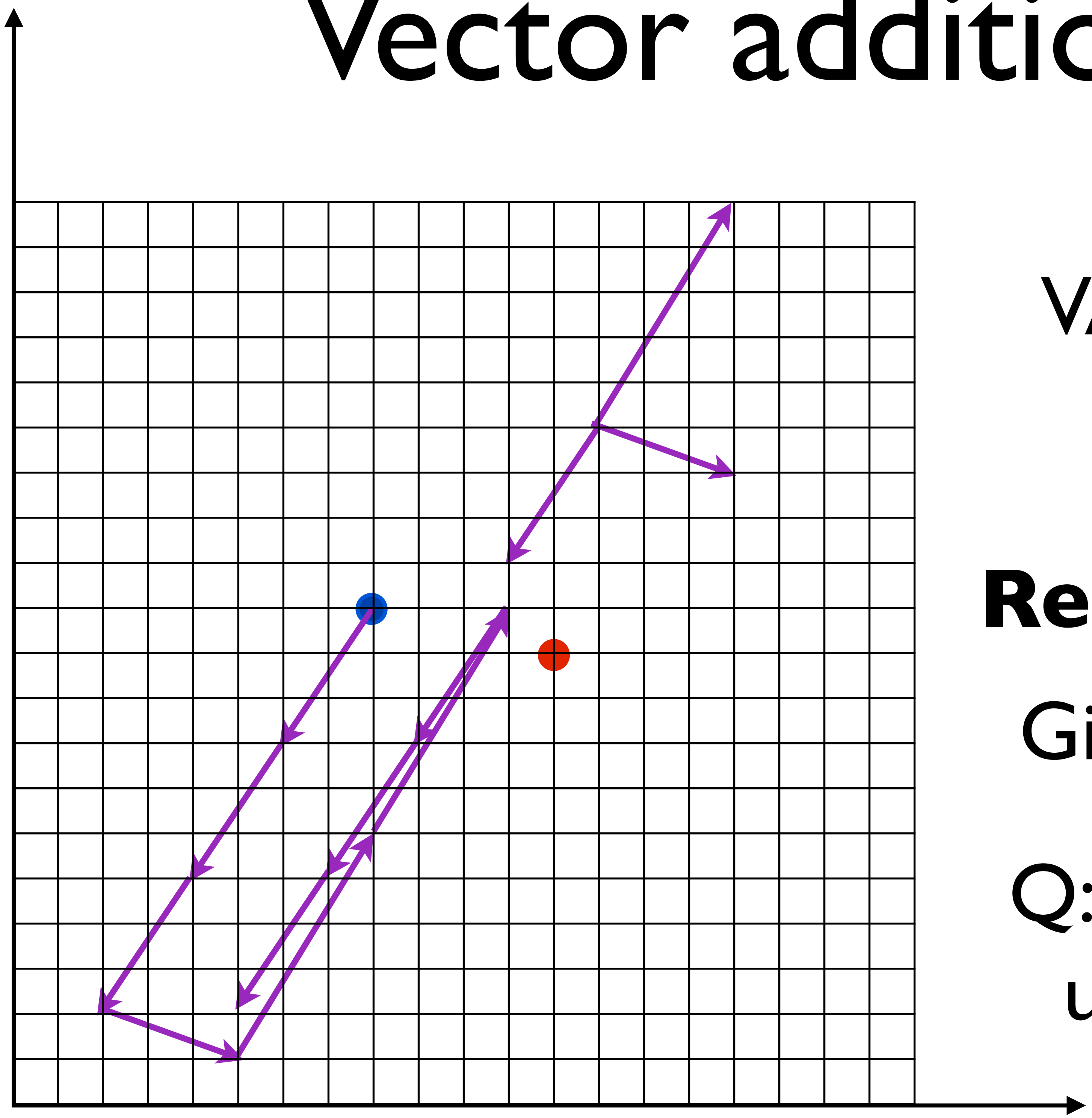

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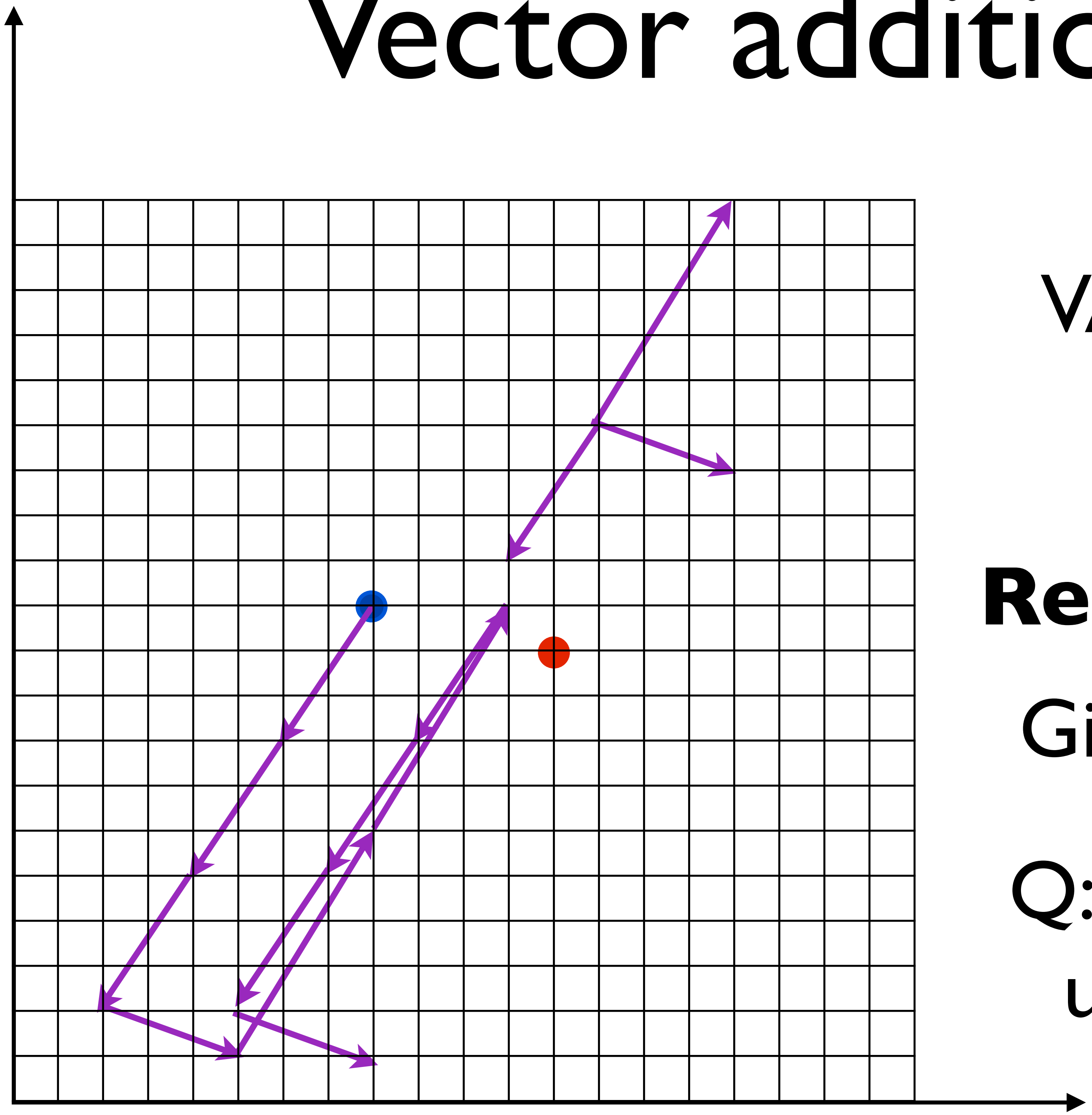

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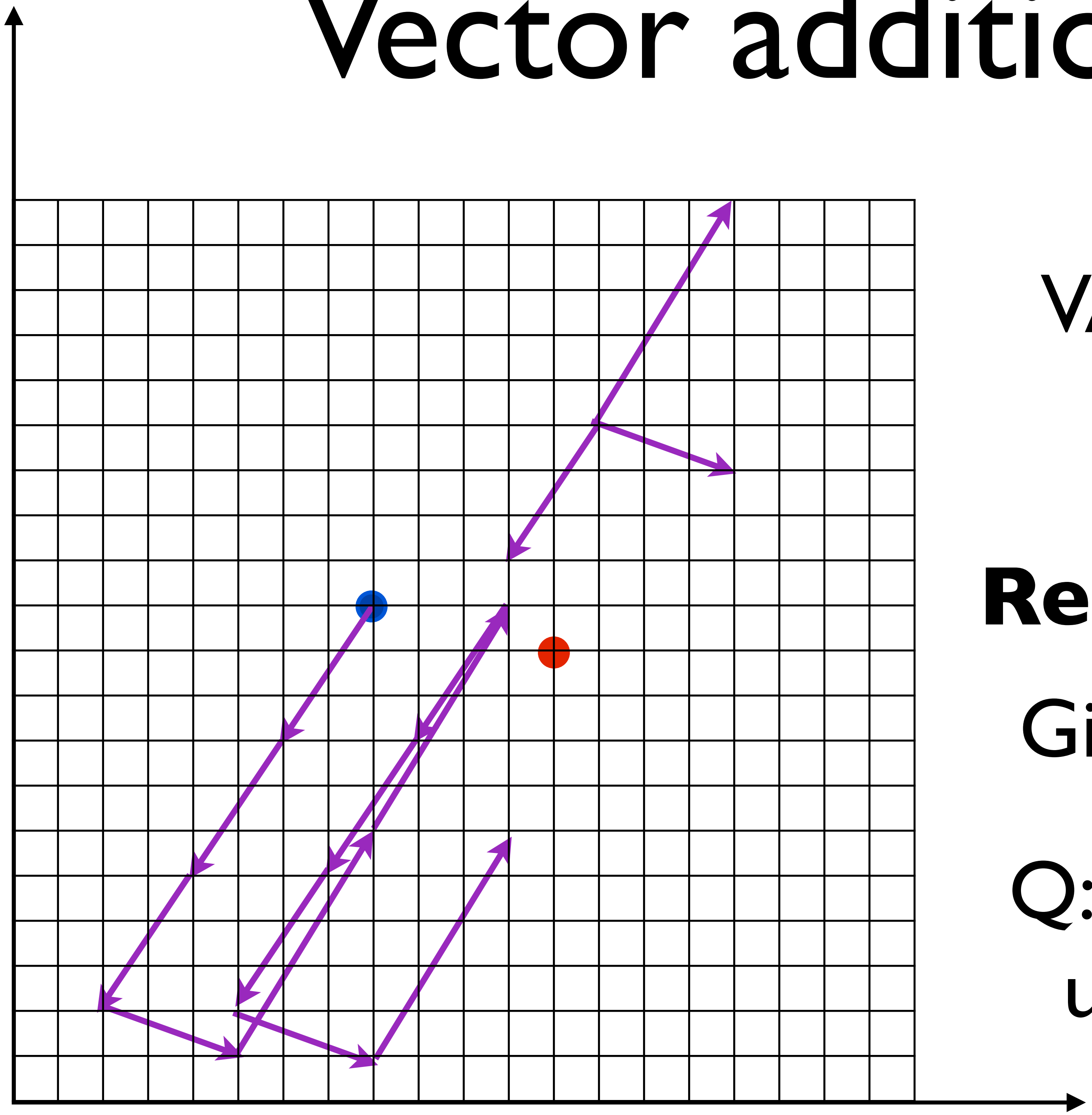

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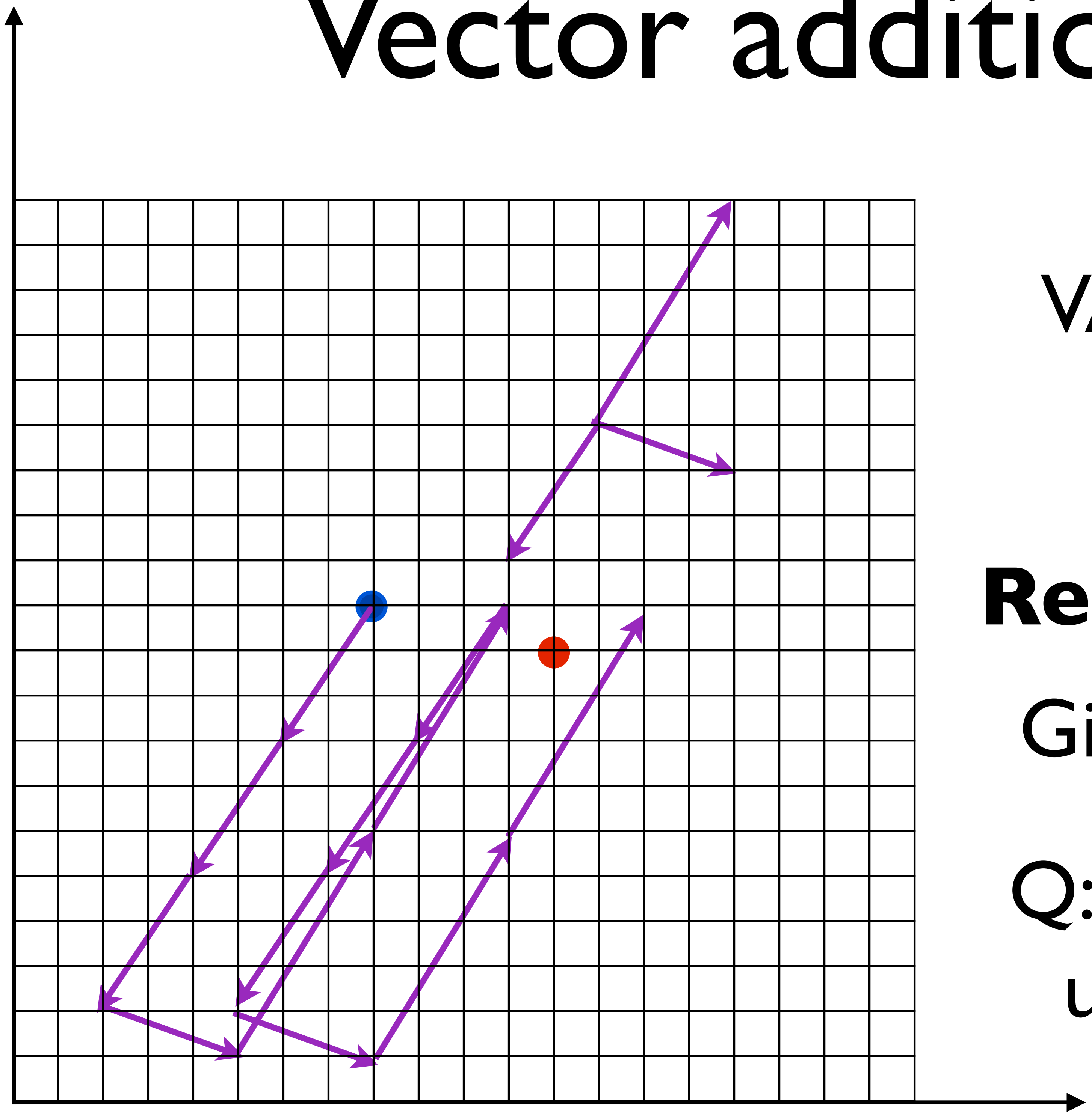

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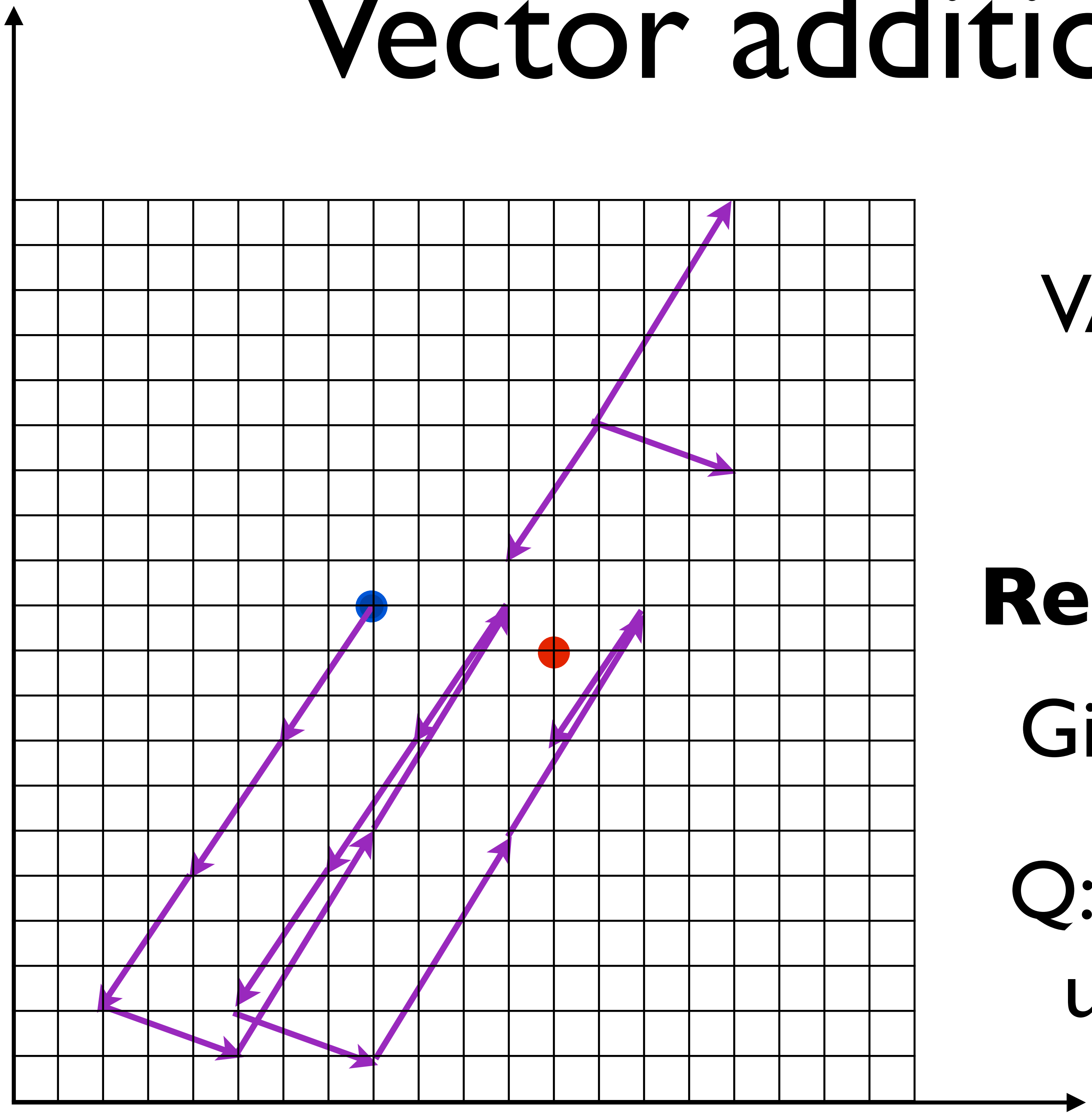

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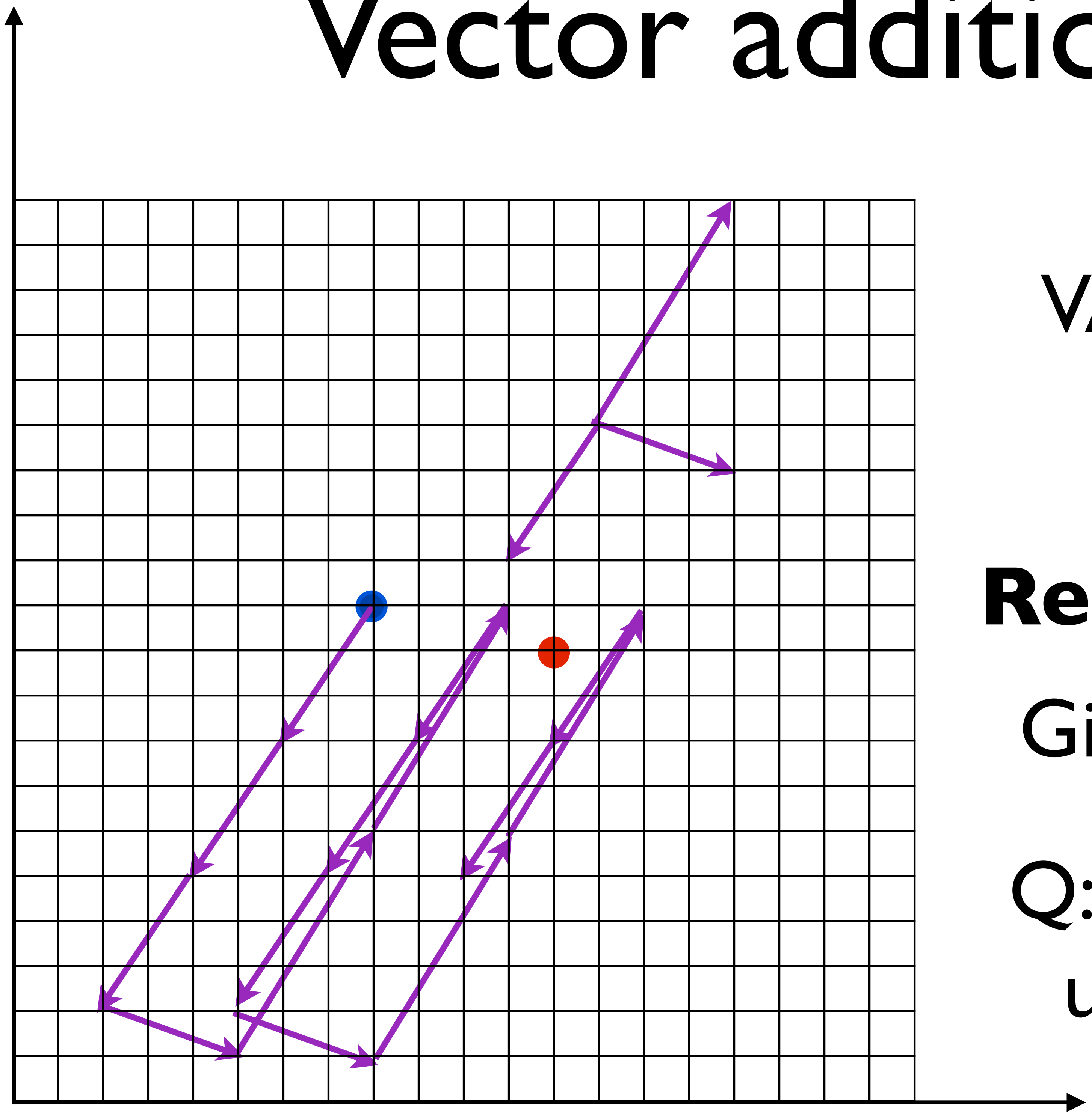

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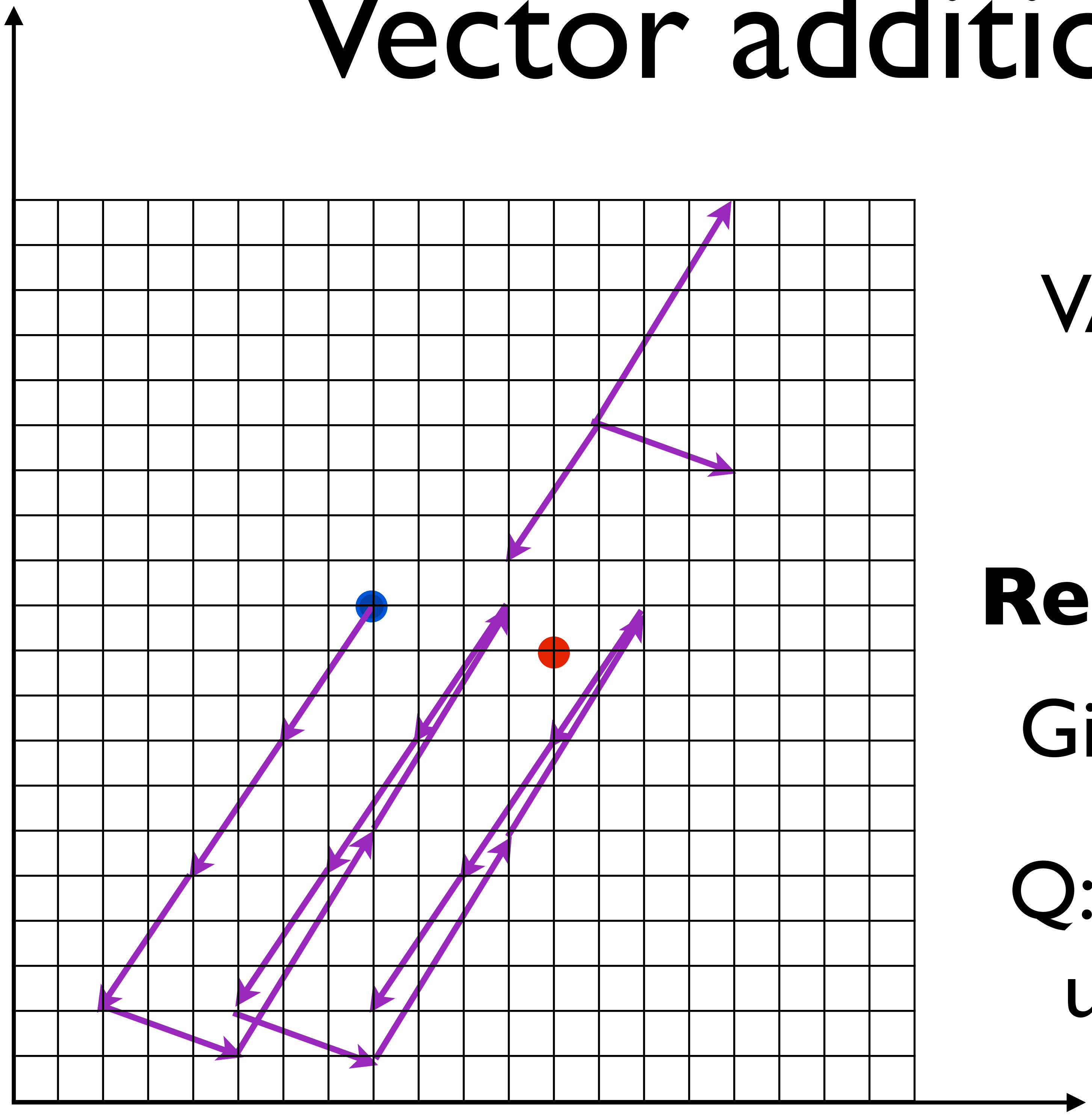

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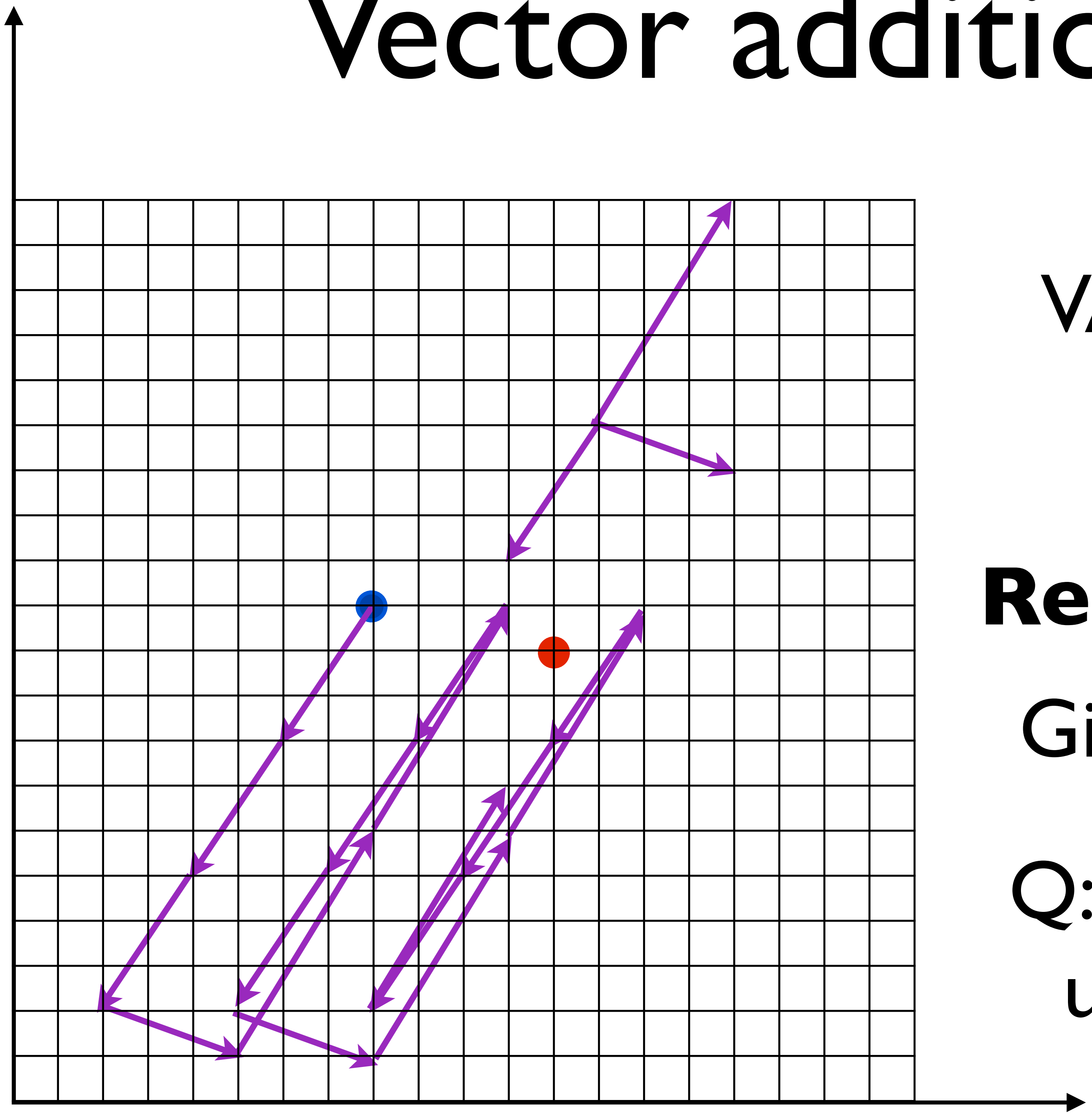

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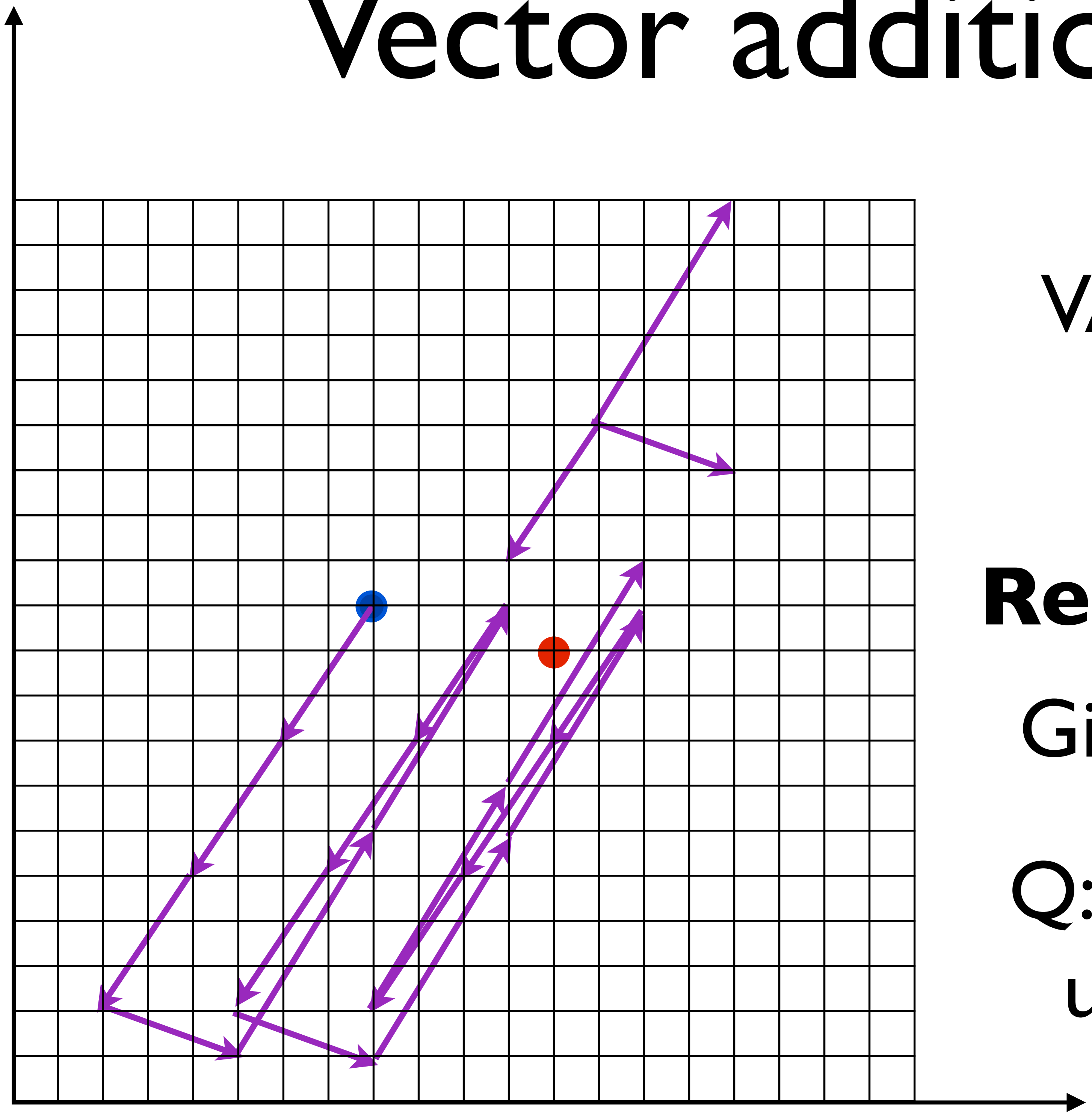

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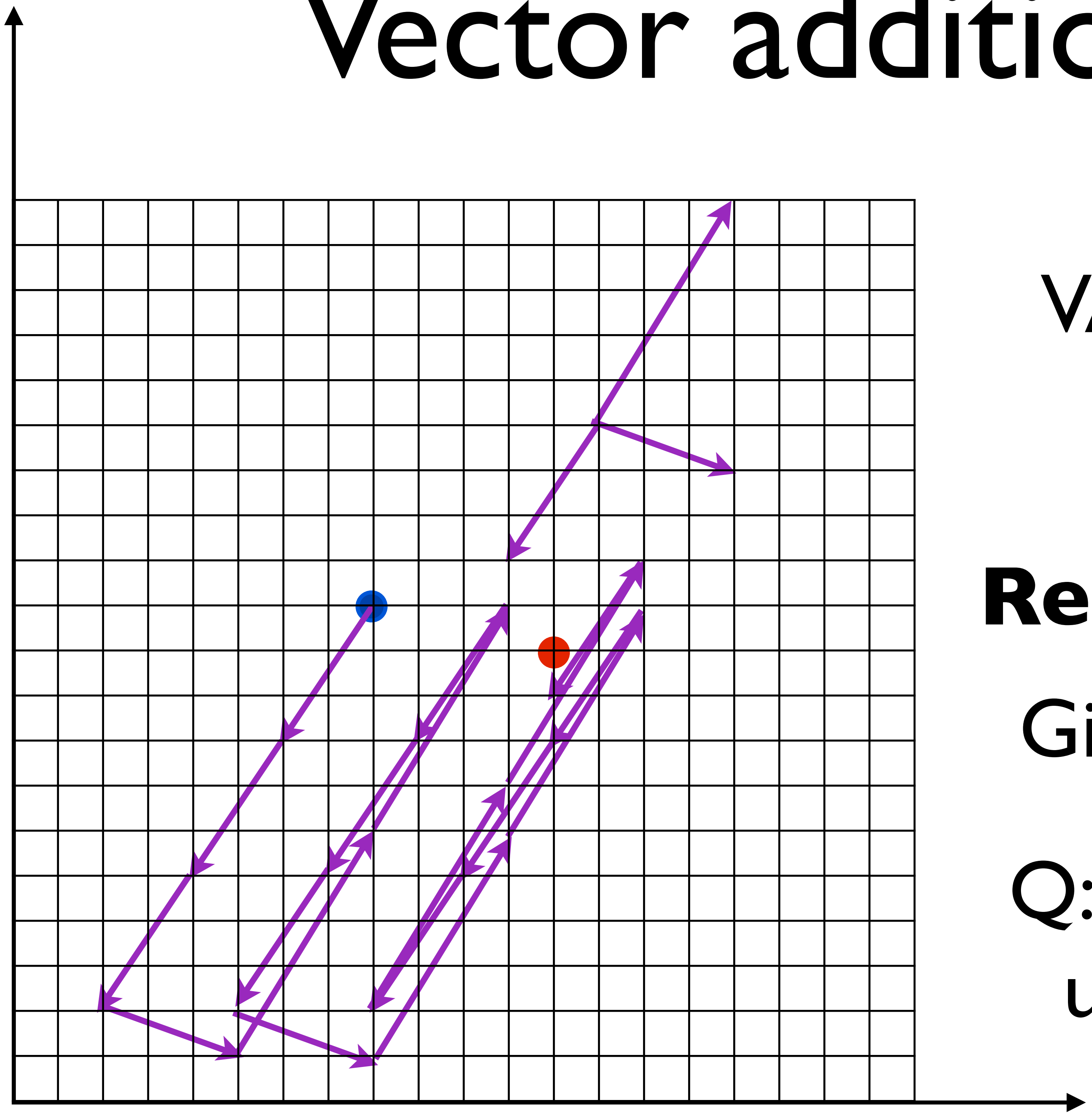

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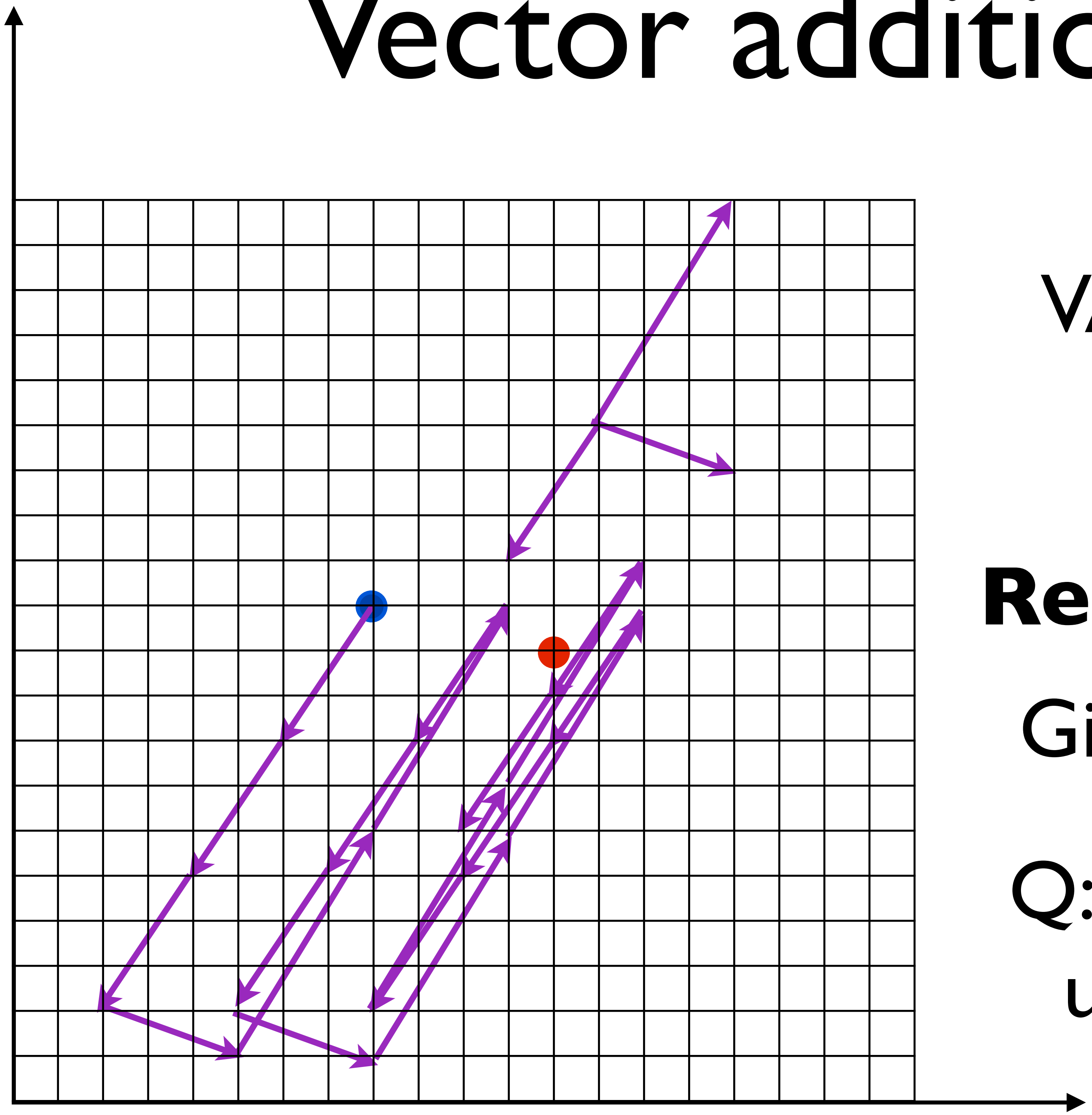

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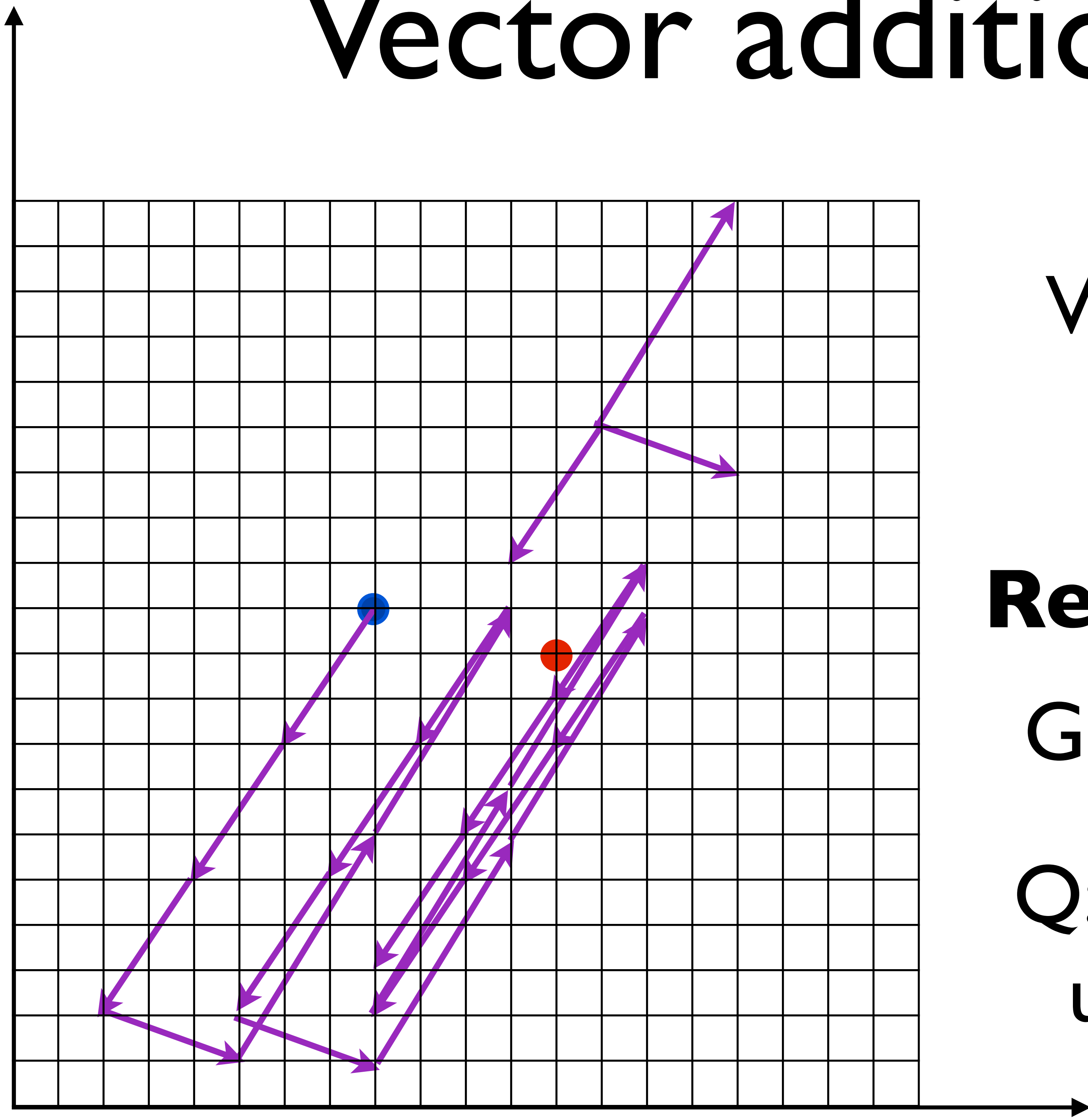

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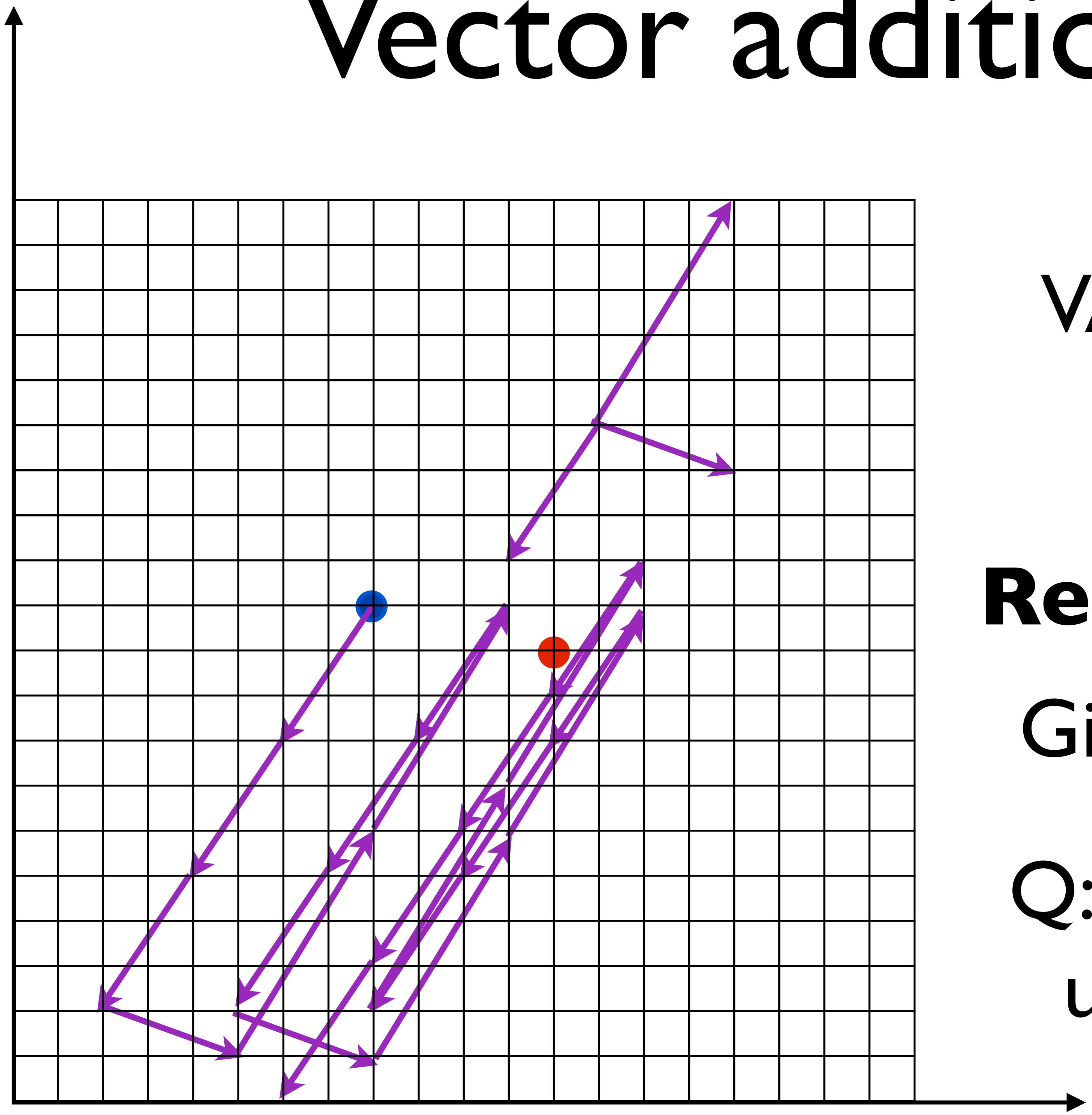

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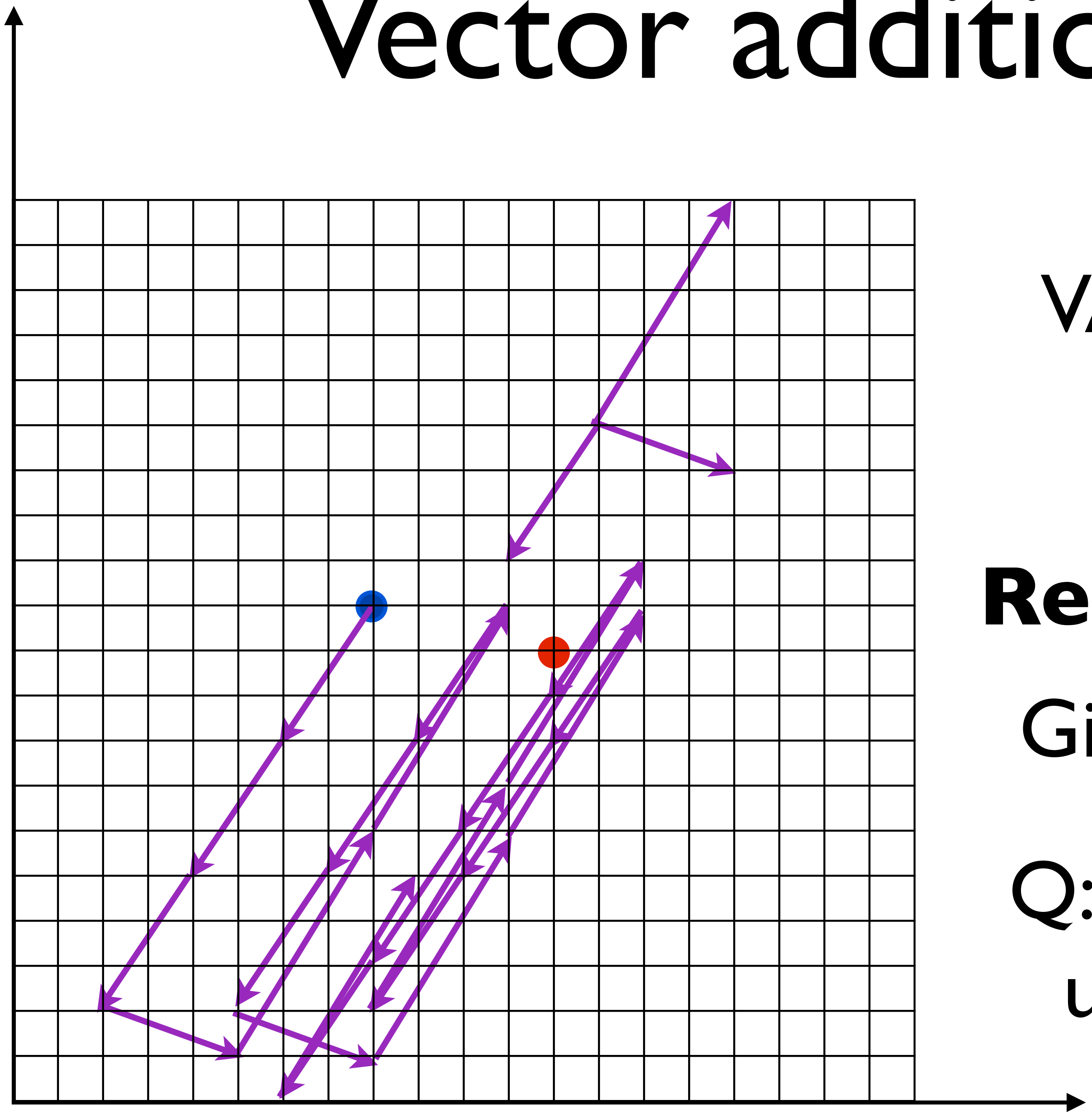

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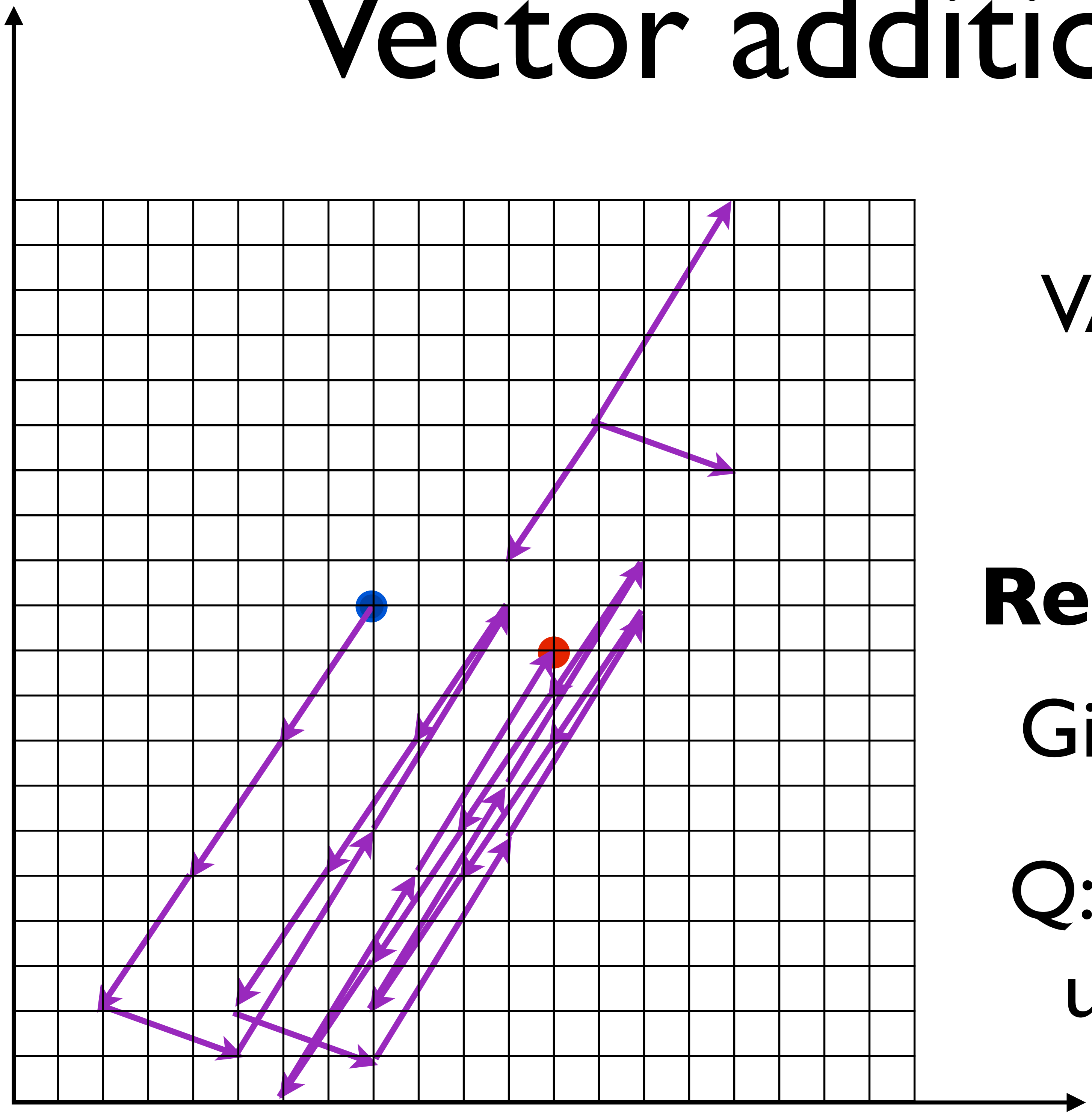

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Vector Addition Systems
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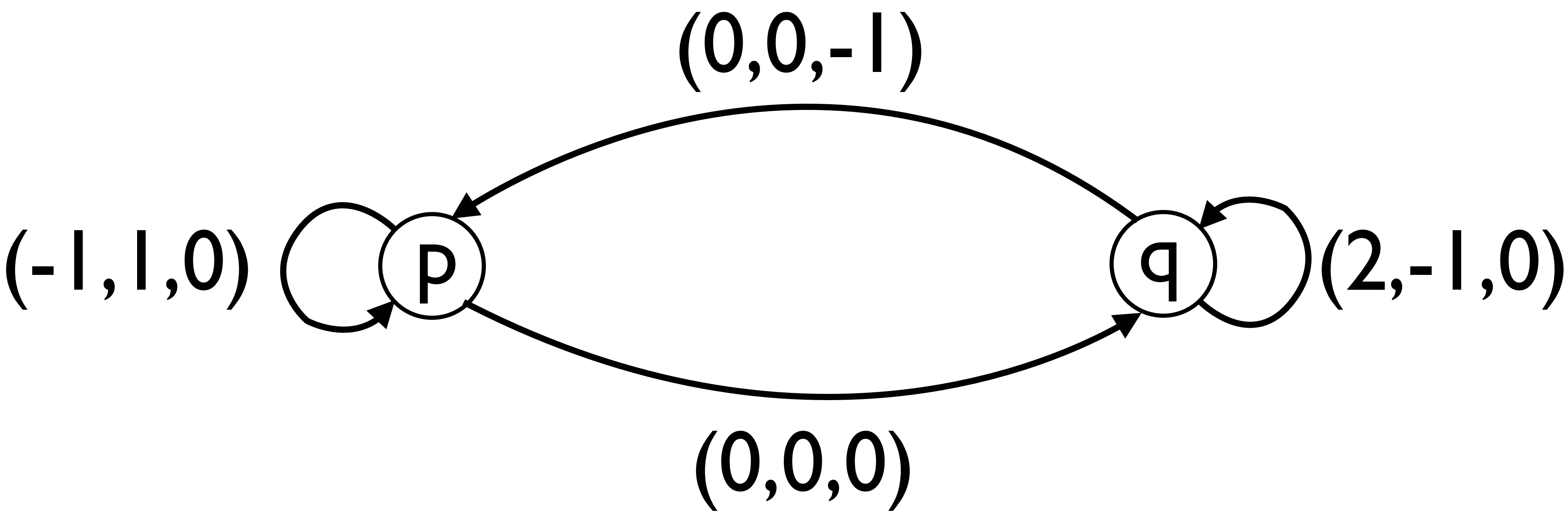

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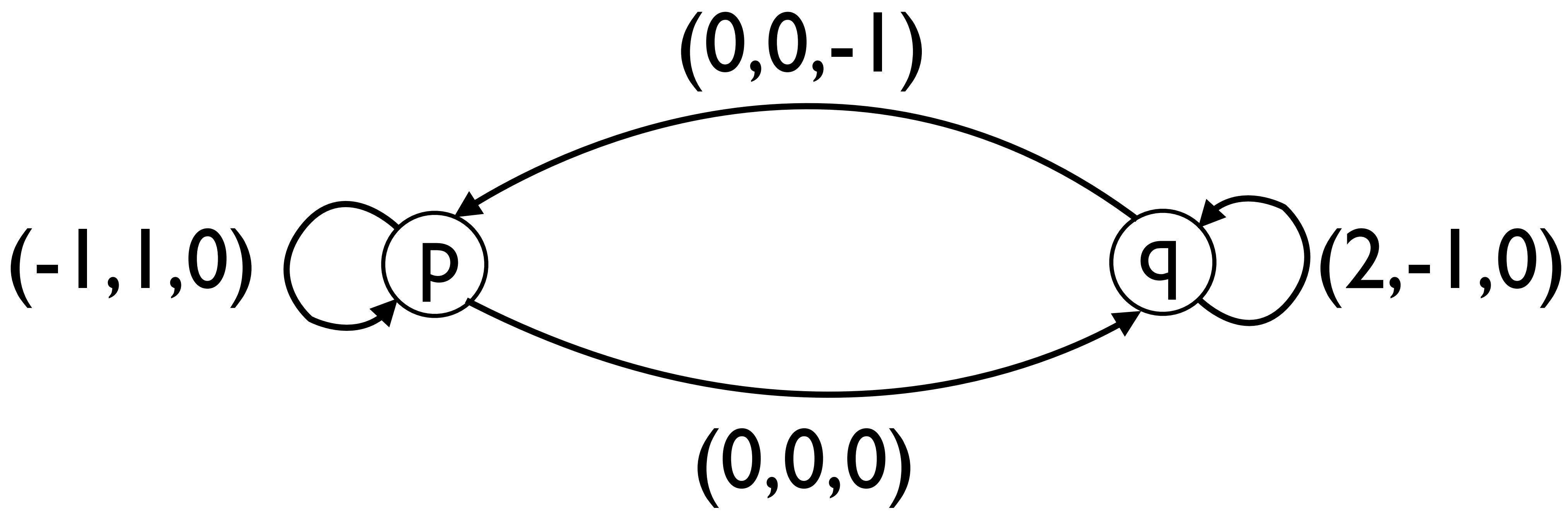
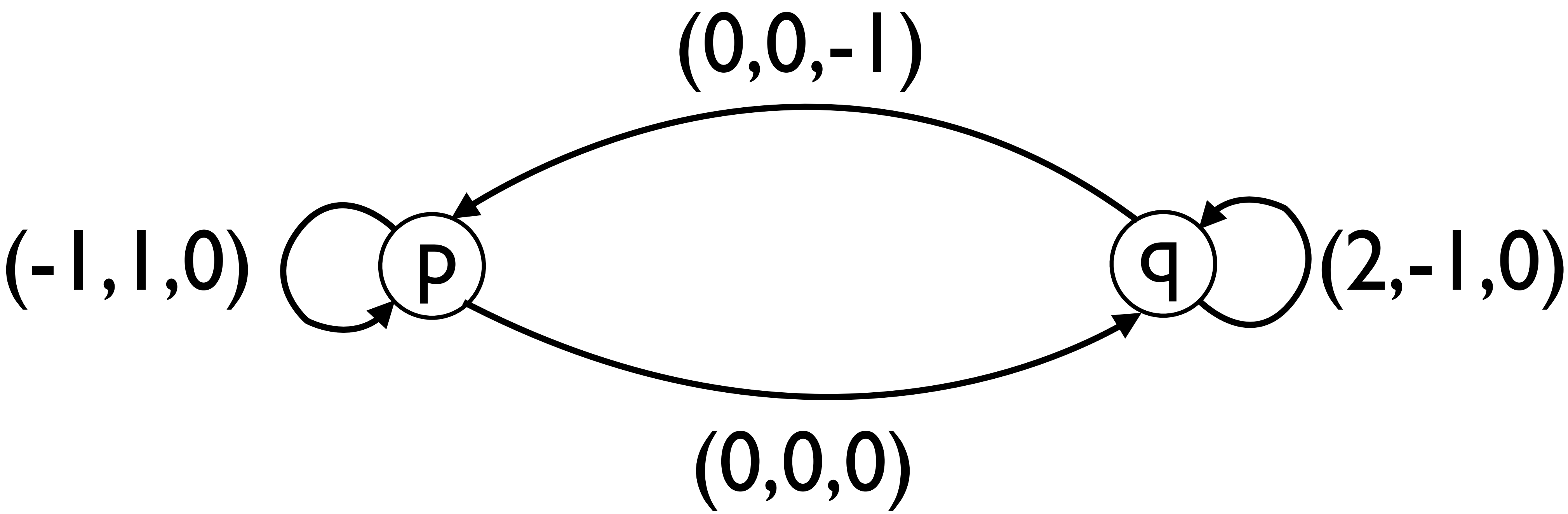


p(1,0,4)

Vector Addition Systems with States



$p(1,0,4) \longrightarrow p(0,1,4)$

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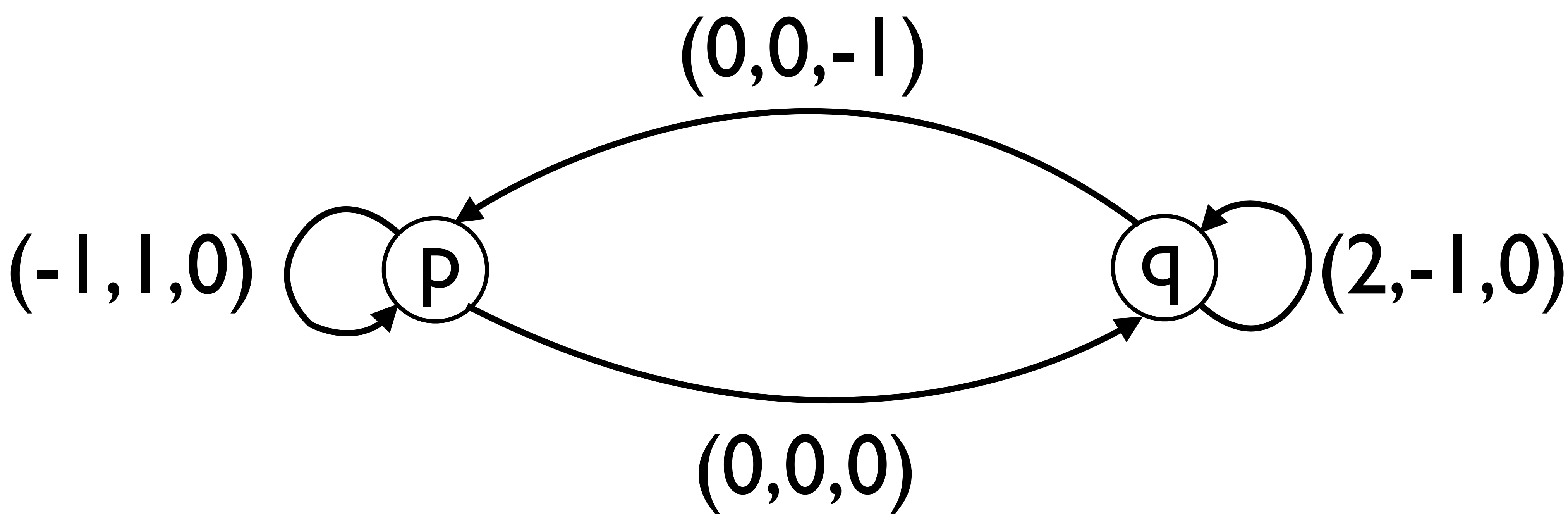

p(1,0,4) ⟶ p(0,1,4) ⟶ q(0,1,4)

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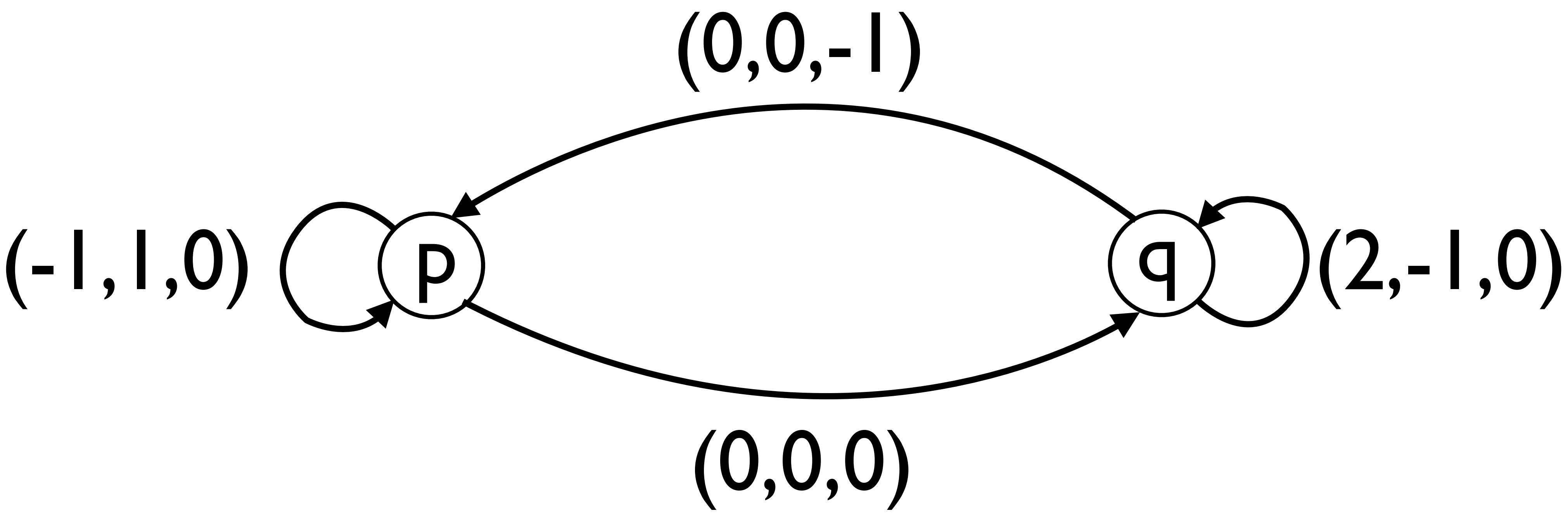

p(1,0,4) ⟶ p(0,1,4) ⟶ q(0,1,4) ⟶ q(2,0,4)

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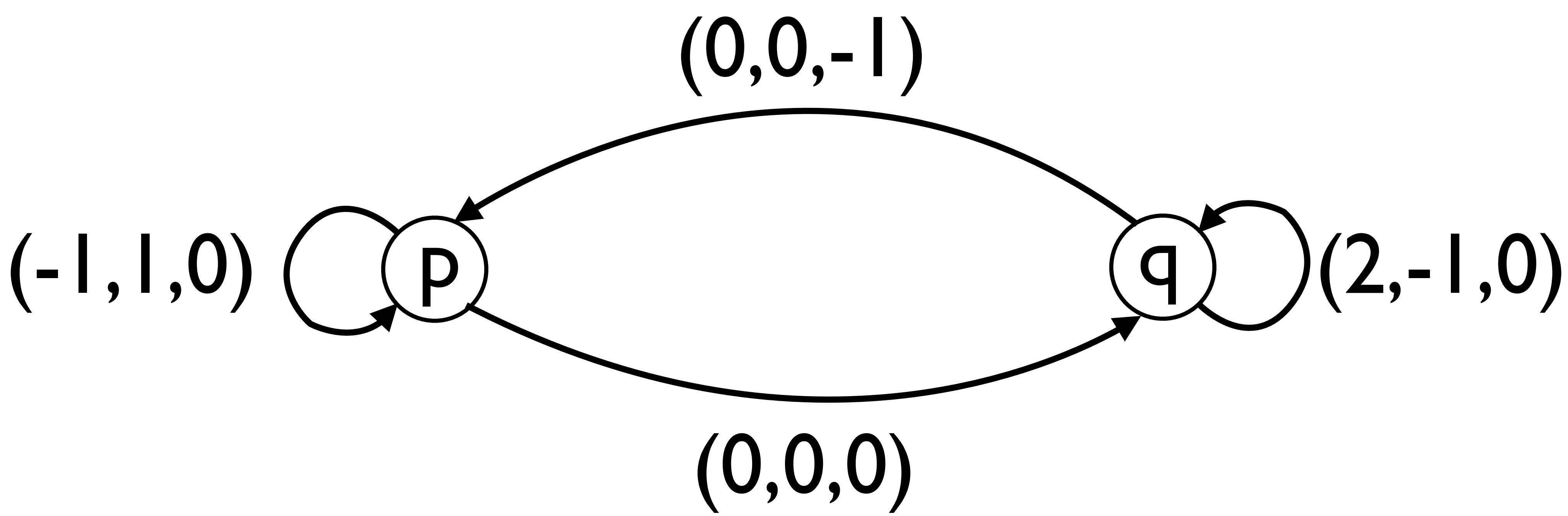

p(1,0,4) ⟶ p(0,1,4) ⟶ q(0,1,4) ⟶ q(2,0,4) ⟶ p(2,0,3)

Short history

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Lipton `76: hardness for exponential space

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Huge complexity gap

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Huge complexity gap

Common feeling: possibly in exponential space

Main result

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Theorem

The Reachability Problem for Vector Addition Systems with States is not elementary.

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Not in time: $2^{2^{2^{\cdot^{\cdot^{\cdot^{2^{n}}}}}}}$

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← fixed height

Common techniques

Common techniques

Upper bounds

Common techniques

Upper bounds

- short paths

Common techniques

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- 2-VASS: exponential

Common techniques

Upper bounds

short paths

2-VASS: exponential

Lower bounds

Common techniques

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short paths

2-VASS: exponential

Lower bounds

long paths?

Common techniques

Upper bounds

- short paths
- 2-VASS: exponential

Lower bounds

- long paths?
- counter machines with big counters

Common techniques

Upper bounds

short paths

2-VASS: exponential

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long paths?

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Lipton 1976: doubly exponential counters

Common techniques

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Hopcroft-Pansiot example

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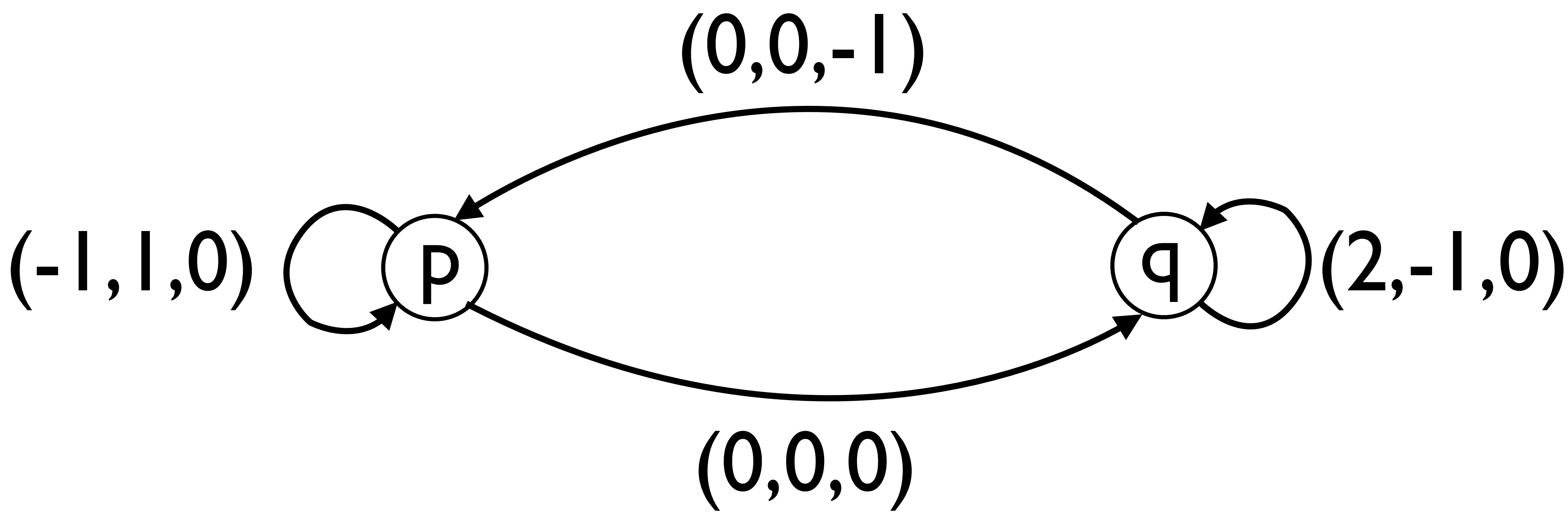

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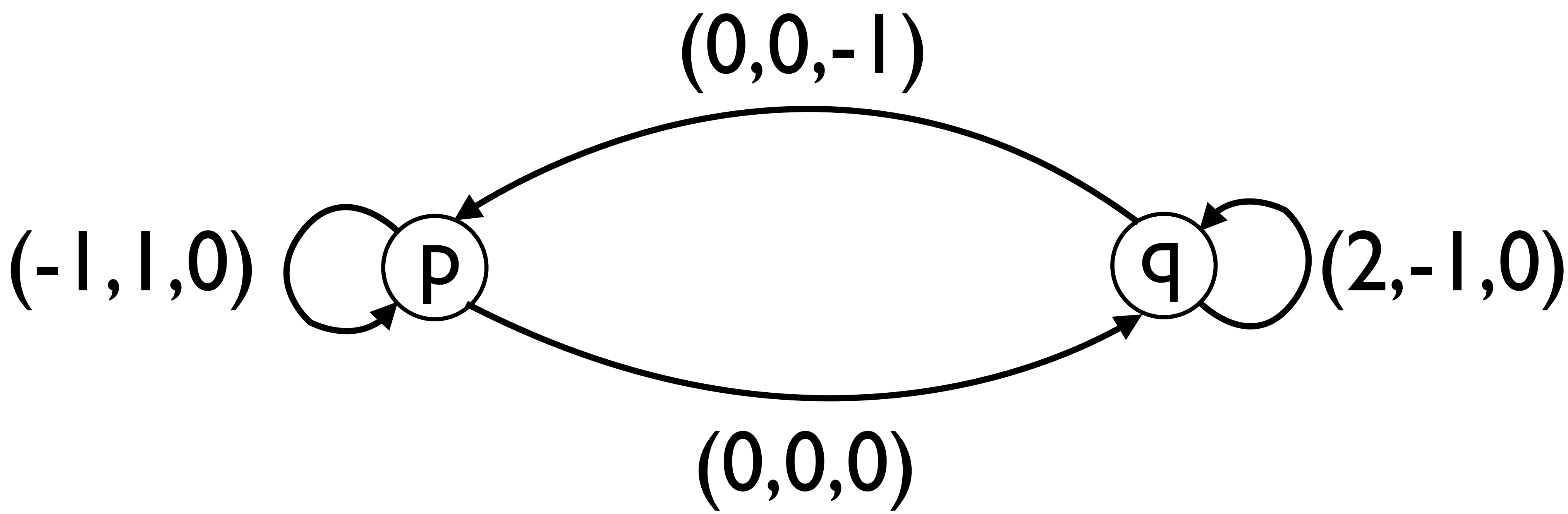
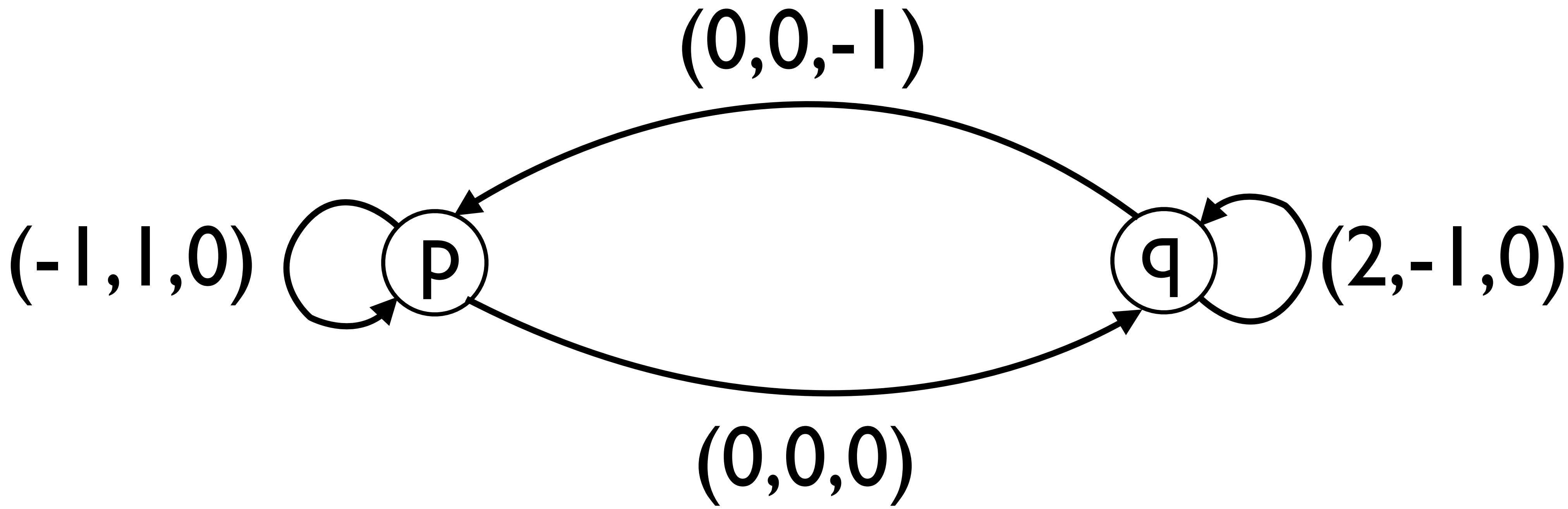


p(k,0,n)

Hopcroft-Pansiot example



p(k,0,n) ⟶ p(0,k,n)

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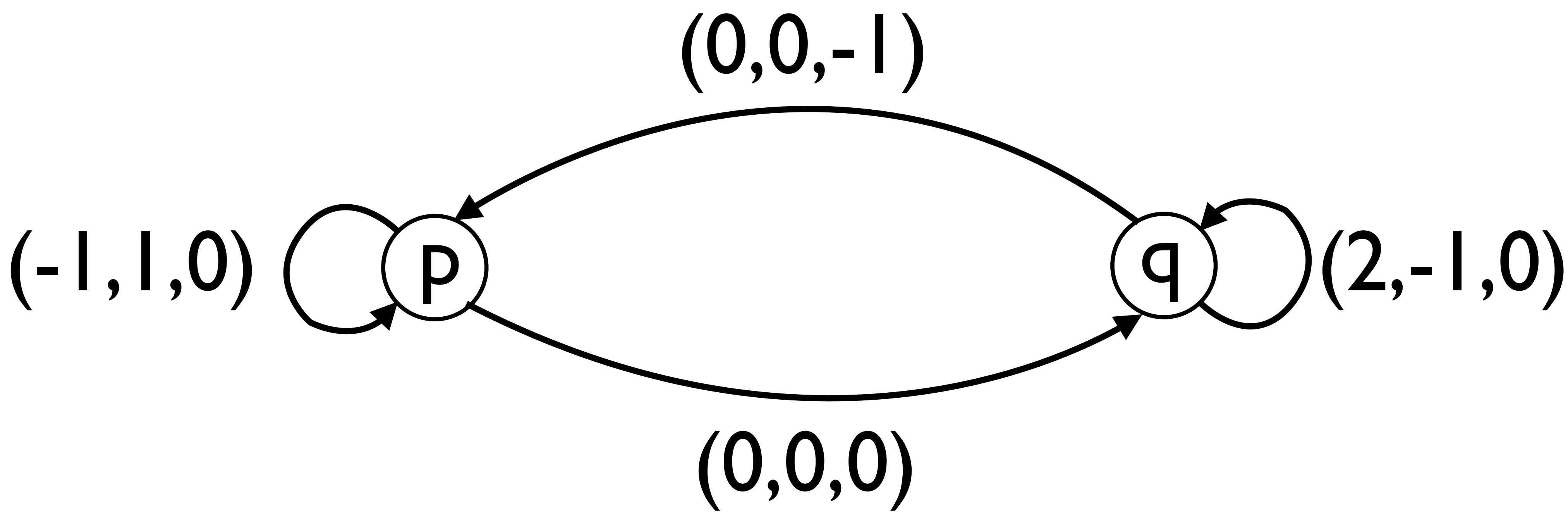

$p(k,0,n) \longrightarrow p(0,k,n) \longrightarrow q(0,k,n)$

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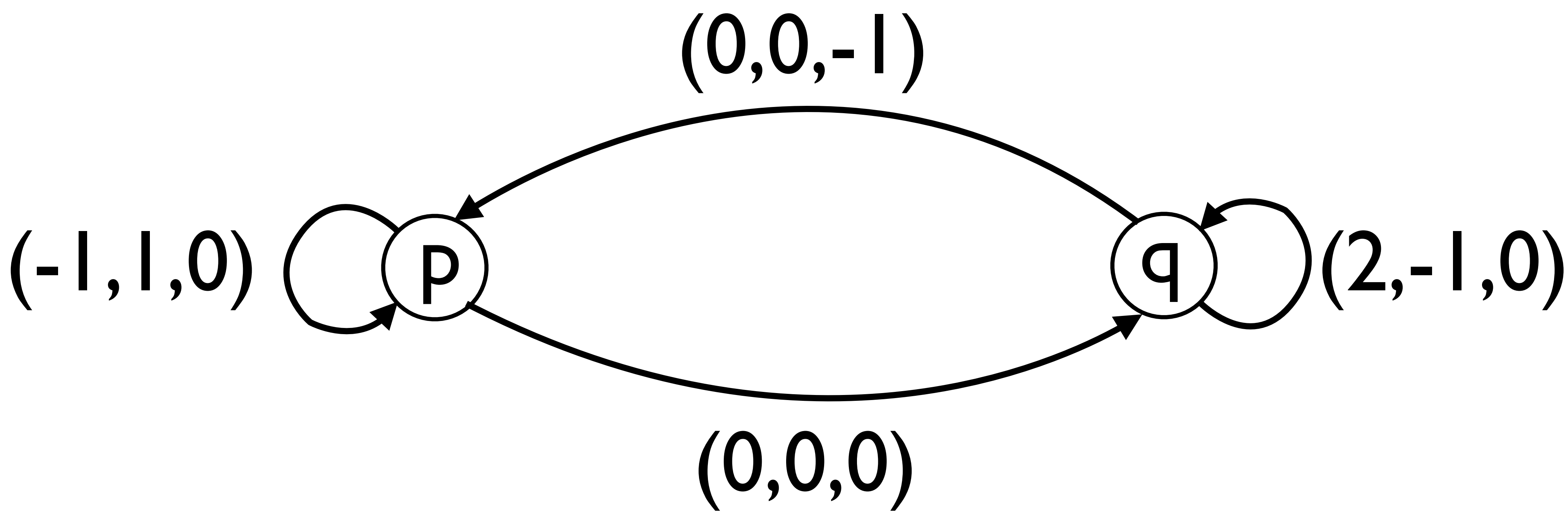

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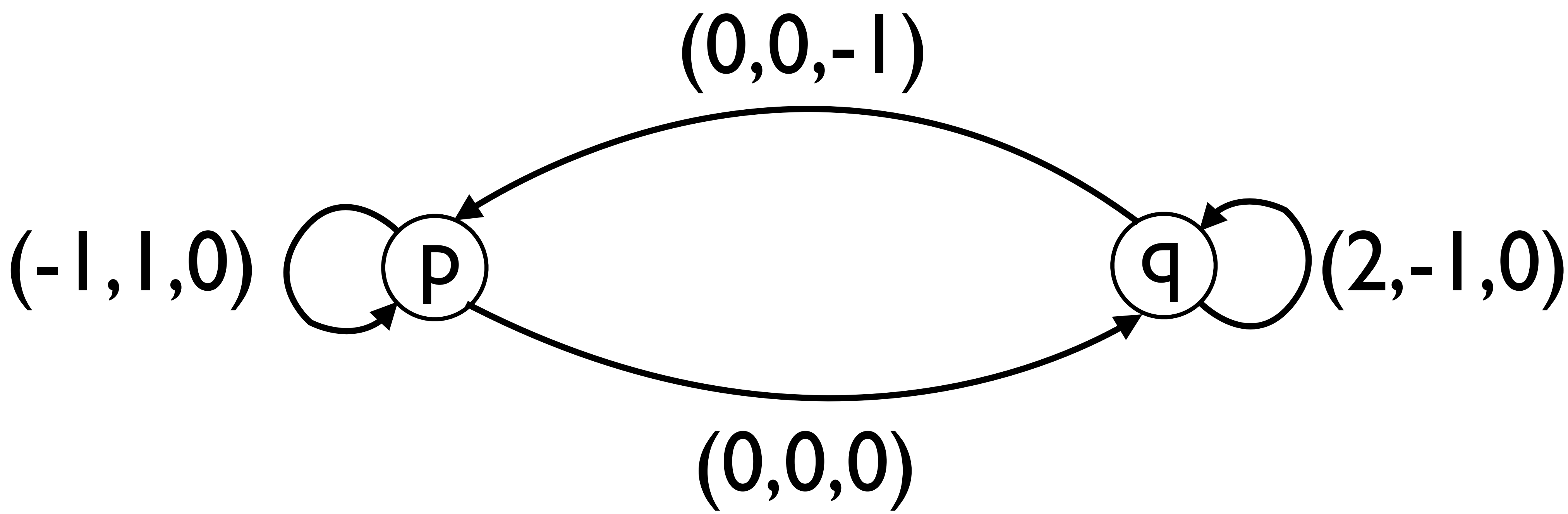

p(k,0,n) ⟶ p(0,k,n) ⟶ q(0,k,n) ⟶ q(2k,0,n) ⟶ p(2k,0,n-1)

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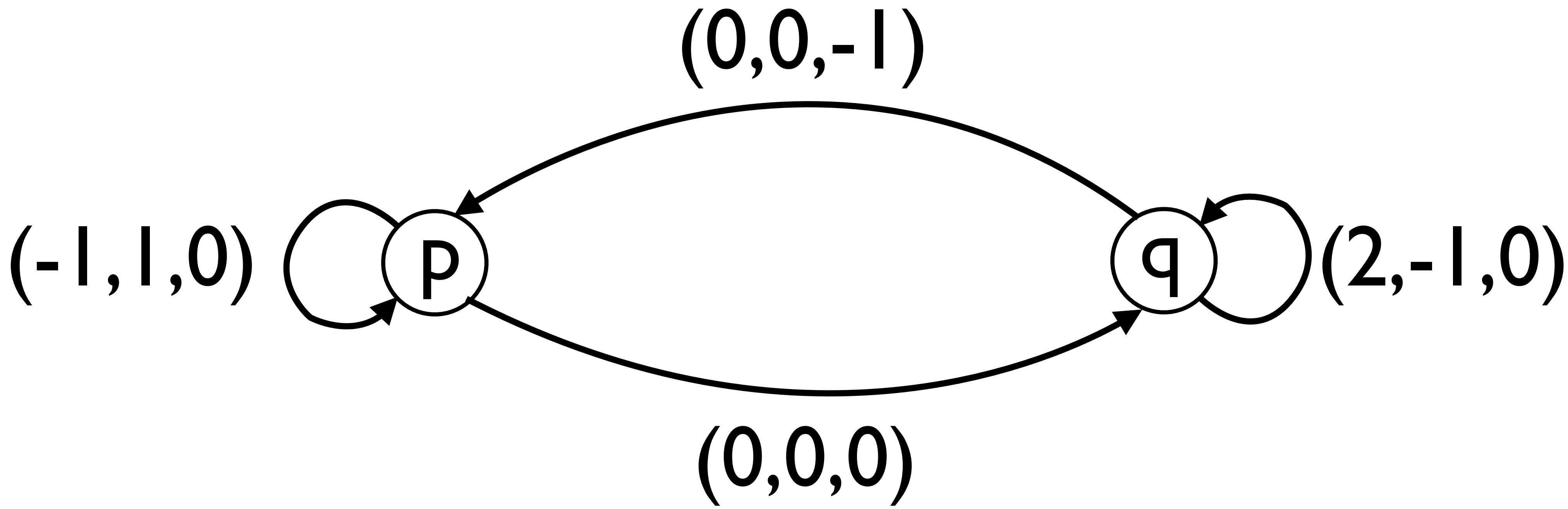

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$p(1,0,n)$

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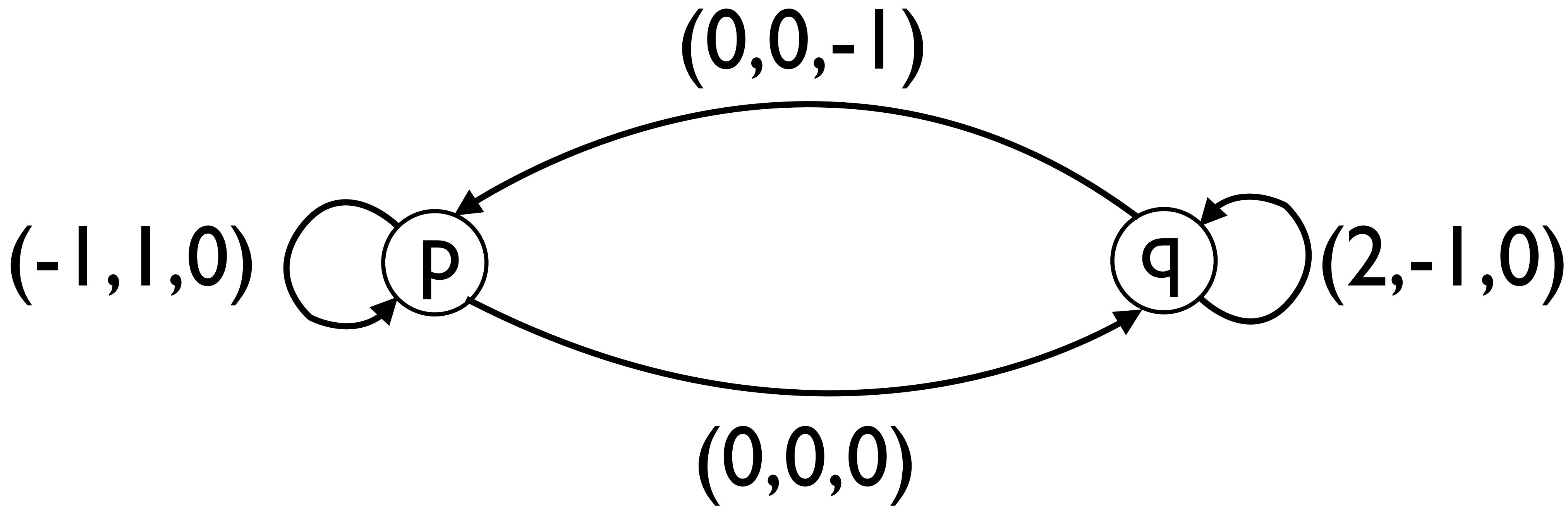

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$p(1,0,n) \longrightarrow p(2,0,n-1)$

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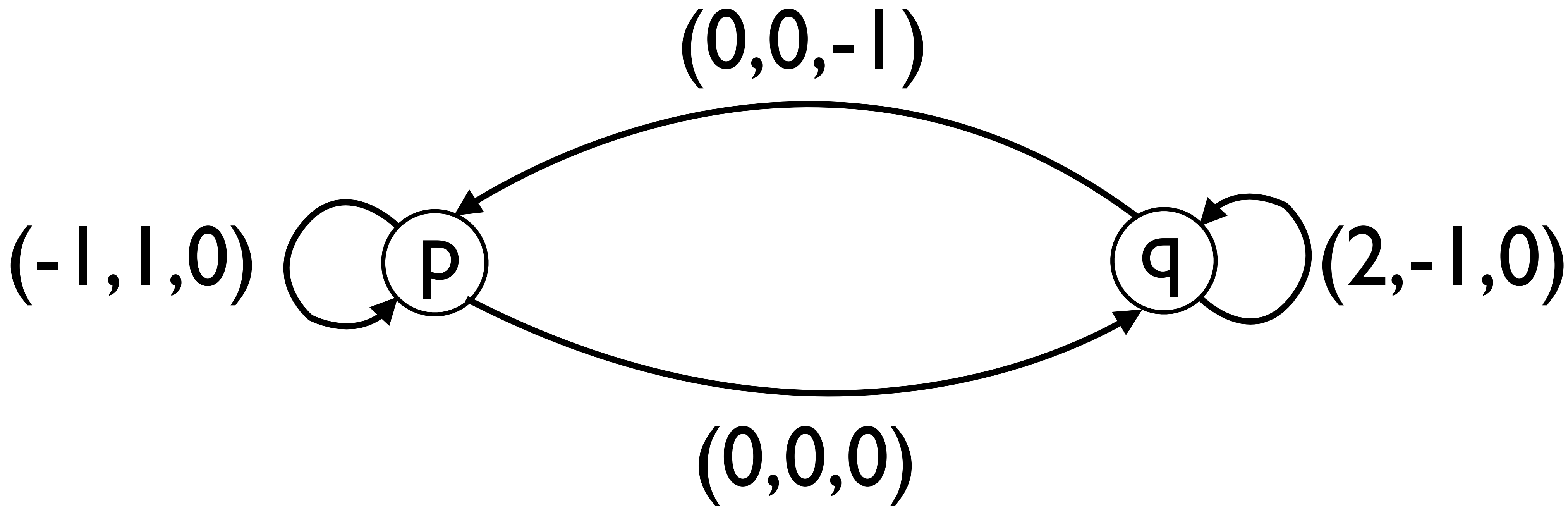

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$$p(1,0,n) \longrightarrow p(2,0,n-1) \longrightarrow$$

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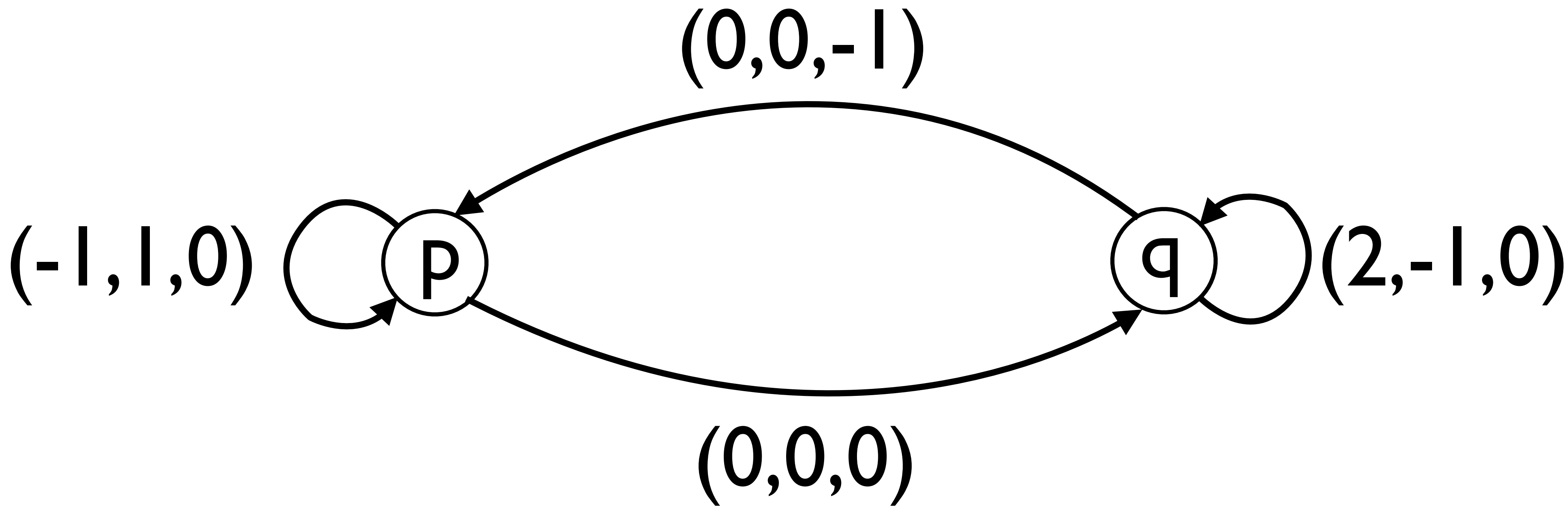

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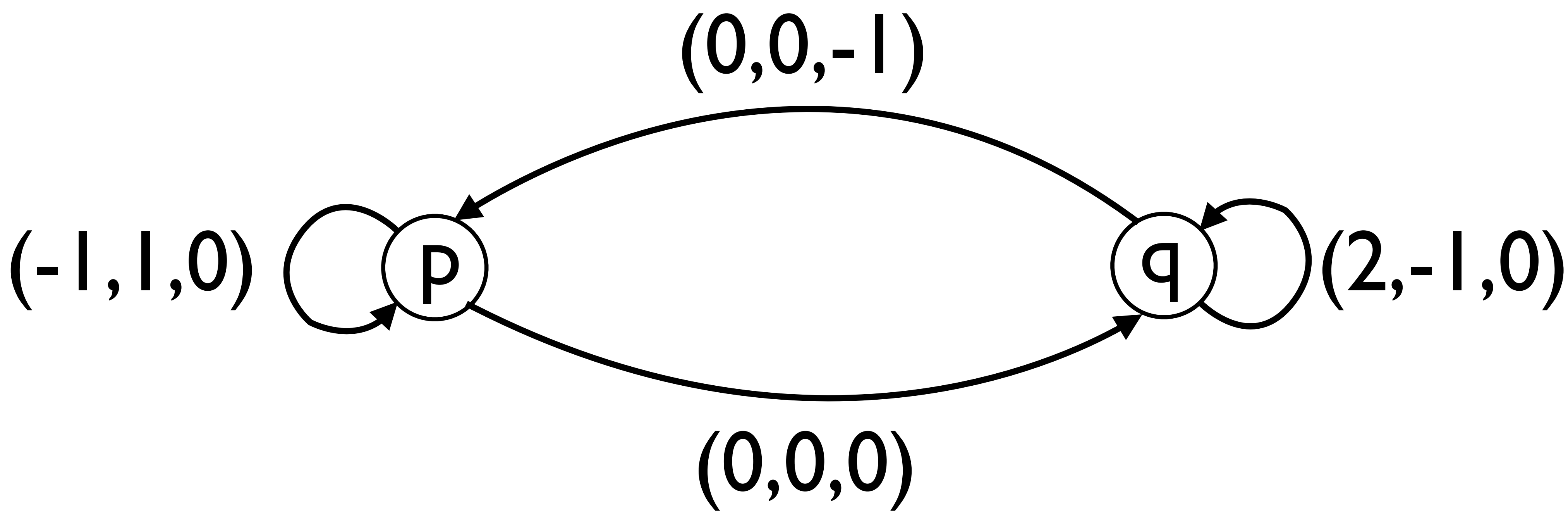
p(1,0,n) ⟶ p(2,0,n-1) ⟶ ...

Hopcroft-Pansiot example



p(k,0,n) ⟶ p(0,k,n) ⟶ q(0,k,n) ⟶ q(2k,0,n) ⟶ p(2k,0,n-1)

p(1,0,n) ⟶ p(2,0,n-1) ⟶ ... ⟶

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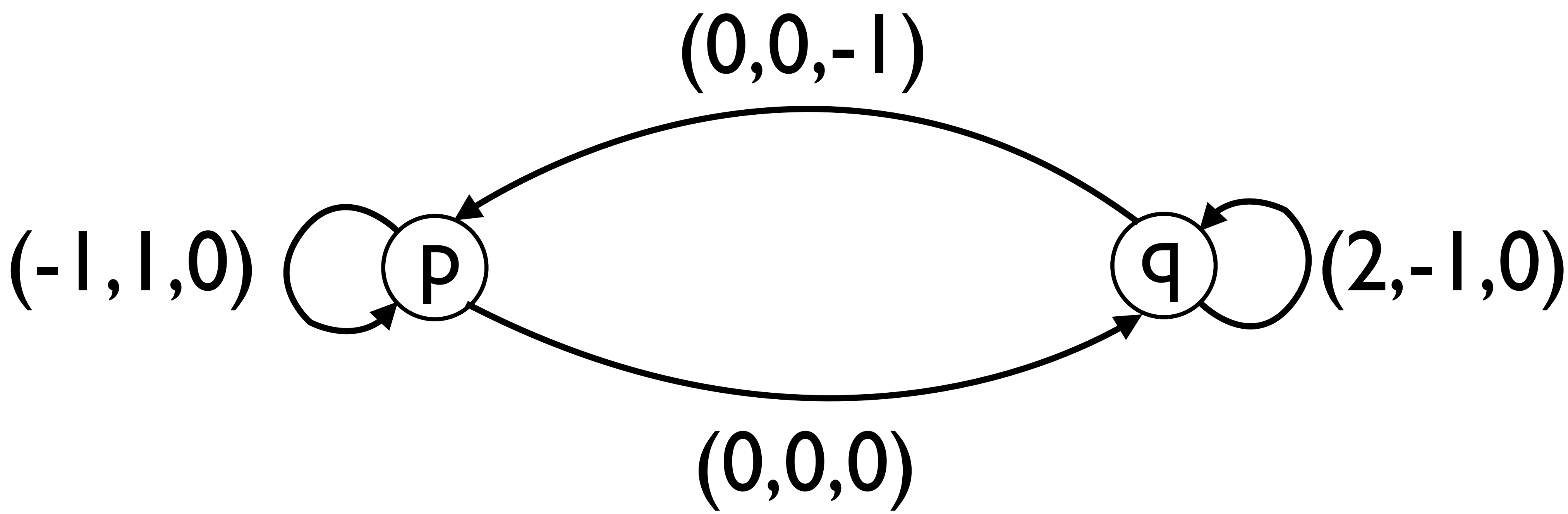

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$$p(1,0,n) \longrightarrow p(2,0,n-1) \longrightarrow \quad \dots \quad \longrightarrow p(2^n,0,0)$$

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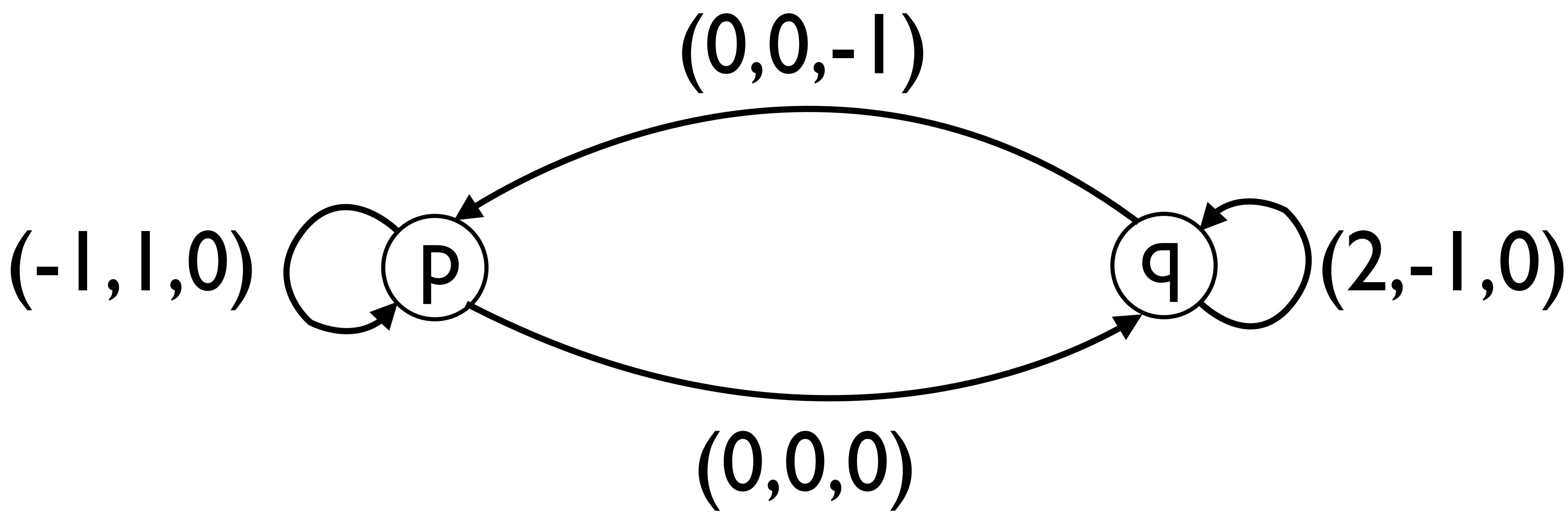

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Weak multiplication by 2/1 at most n times

Fractional equation

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$1 < a_1 / b_1 < a_2 / b_2 < \dots < a_k / b_k$,

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and all a_i, b_i, a and b are at most exponential in k.

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$$(1+2^k / 2^k)^{2^k} \approx e$$

4-VASS

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start in (0, 0, 0, 0)

reach (Kb, K, 0, 0)

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subtract (Ka, K, 0, 0)

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enforces **exact** multiplication!

Problem

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We can get once 2exp path from exp numbers

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Tricky solution

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triples (b, n, bn)

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triples (b, n, bn)

sequence of b operations n times

Triples

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Let $(x, y, z) = (b, n, bn)$

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Performing
exactly b operations
exactly n times

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$x' = 0, x+x' = b$

Triples

Let $(x, y, z) = (b, n, bn)$

loop

Performing exactly b operations exactly n times

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Triples

Let $(x, y, z) = (b, n, bn)$

```
loop
    y--
```

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exactly b operations
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Let (x, y, z) = (b, n, bn)

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    y--
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Triples

Let $(x, y, z) = (b, n, bn)$

```
loop
    y--
    loop
        operation
```

Performing exactly b operations exactly n times

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Let $(x, y, z) = (b, n, bn)$

```
loop
    y--
    loop
        operation
        x--
```

Performing exactly b operations exactly n times

$x' = 0, x + x' = b$

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Let (x, y, z) = (b, n, bn)

```
loop
    y--
    loop
        operation
        x--    x'++
```

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Let (x, y, z) = (b, n, bn)

```
loop
    y--
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```
loop
    y--
    loop
        operation
        x--    x'++
        z--
    swap(x, x')
```

Performing
exactly b operations
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x' = 0, x+x' = b

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Let (x, y, z) = (b, n, bn)

Performing
exactly b operations
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```
loop
    y--
    loop
        operation
        x--    x'++
        z--
    swap(x, x')
reach z = 0
```

x' = 0, x+x' = b

Solution

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From (b, n, bn) obtain $(b!, m, b!\, m)$ for guessed m

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(3, n, 3n)

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Complicated!

$$(3, n, 3n) \xrightarrow{V}$$

Solution

From (b, n, bn) obtain $(b!, m, b!\, m)$ for guessed m

Complicated!

$$(3, n, 3n) \xrightarrow{V} (6, n, 6n)$$

Solution

From (b, n, bn) obtain (b!, m, b! m) for guessed m

Complicated!

$$(3,\ n,\ 3n) \xrightarrow{V} (6,\ n,\ 6n) \xrightarrow{V}$$

Solution

From (b, n, bn) obtain $(b!, m, b!\,m)$ for guessed m

Complicated!

$$(3, n, 3n) \overset{\vee}{\longrightarrow} (6, n, 6n) \overset{\vee}{\longrightarrow} \ldots$$

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From (b, n, bn) obtain (b!, m, b! m) for guessed m

Complicated!

$$(3,\ n,\ 3n) \xrightarrow{V} (6,\ n,\ 6n) \xrightarrow{V} \ldots \xrightarrow{V} (3!\ldots!,\ n,\ 3!\ldots!n)$$

sufficient for simulation of
k-exp bounded counter machines

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k-exp bounded counter machines

Tower-hardness

Thank you!

Code

Code

Algorithm II Factorial Amplifier $\mathcal{F}$.

```
//Untested counters: b, b′, c, c′, d, d′, x, y
//Tested counters: i, i′
i += 1   b += 1   c += 1   d += 1   x += 1   y += 1
loop
    c += 1   d += 1   x += 1   y += 1
loop
    loop
        c −= i   c′ += 1
        loop at most b times
            d −= i   x −= i   d′ += i + 1
    loop
        b −= 1   b′ += i + 1
    loop
        b′ −= 1   b += 1
    loop
        c′ −= 1   c += 1
        loop at most b times
            d′ −= 1   d += 1   x += 1
    i += 1
max? i
loop
    x −= i   y −= 1
halt if y = 0
```