Reachability
in Vector Addition Systems
by Examples

Wojciech Czerwiński

CONCUR 2022

Plan

Plan

- basic notions

Plan

- basic notions
- short history

Plan

- basic notions
- short history
- motivation

Plan

- basic notions
- short history
- motivation
- big reachability sets

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- only long paths

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  - telescopic equations

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- basic notions
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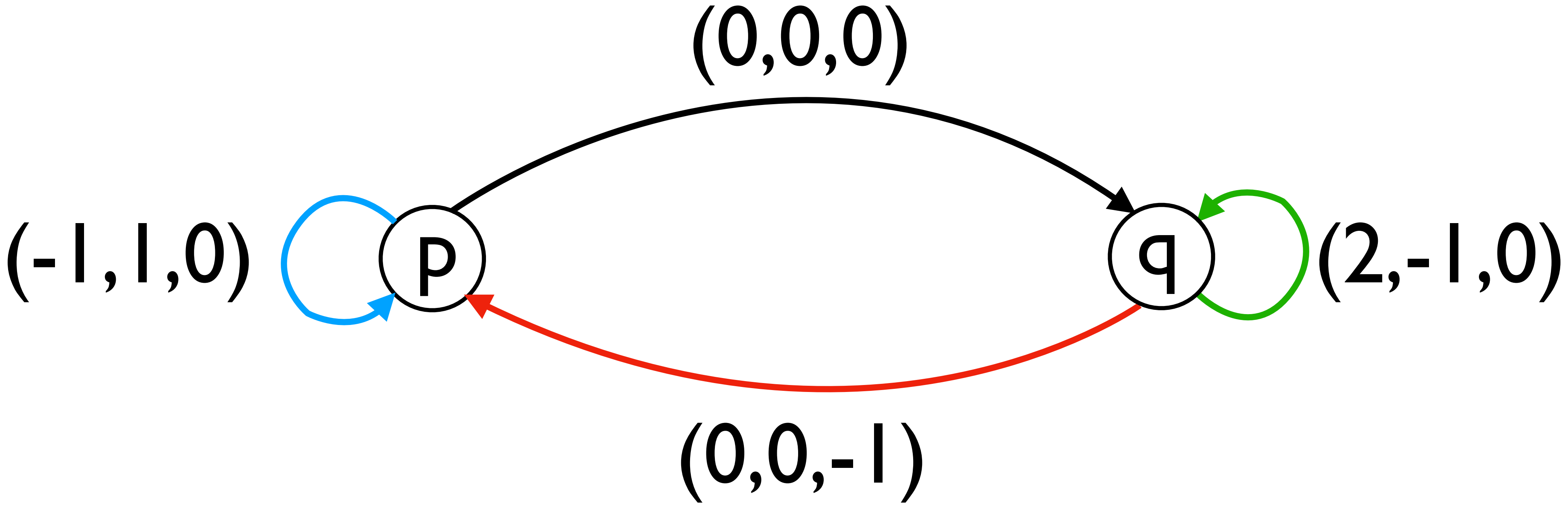
- only long paths
  - telescopic equations
  - controlling-counter

Vector Addition Systems with States

Vector Addition Systems with States



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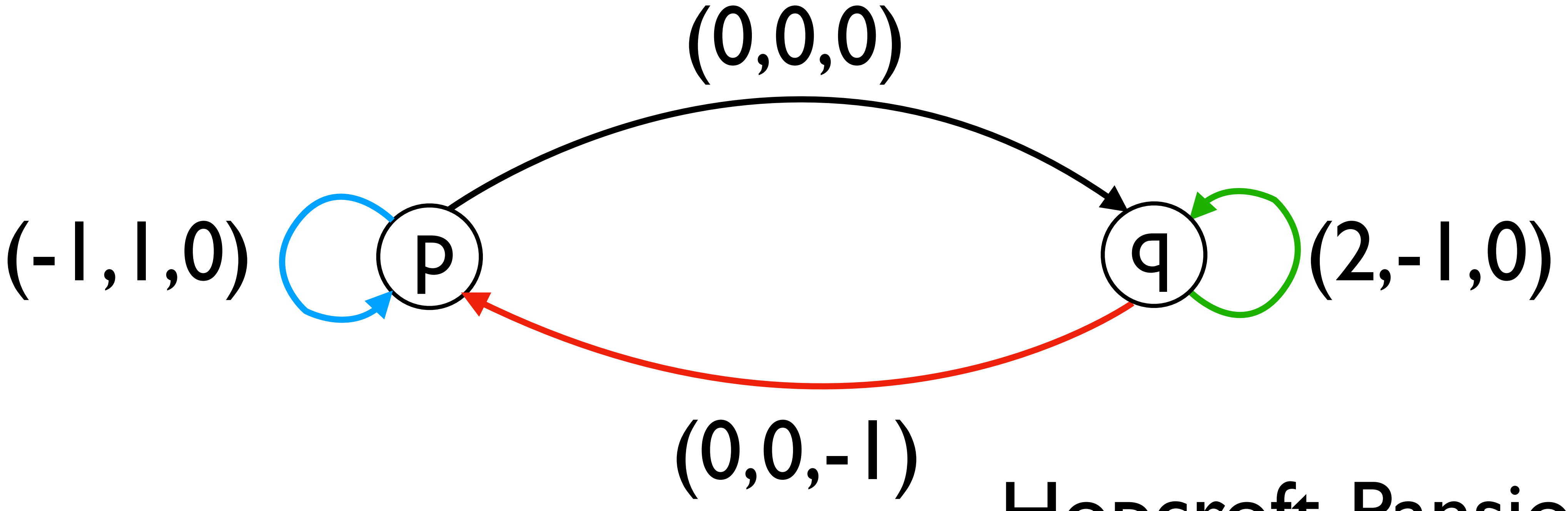
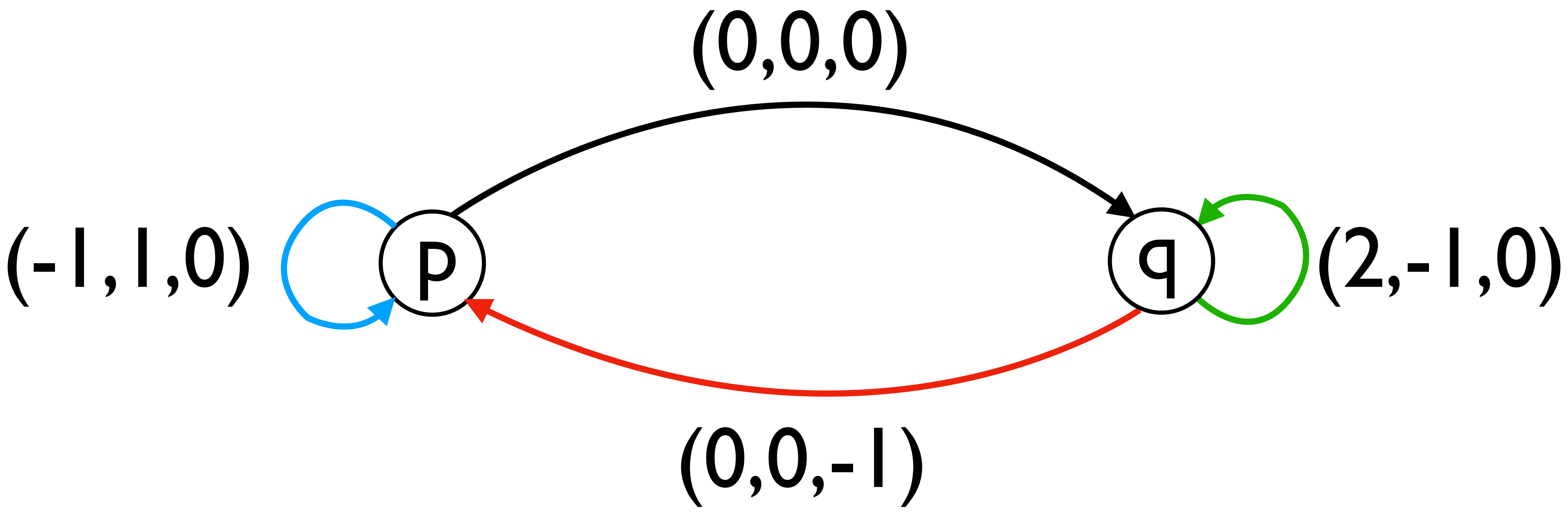


Hopcroft-Pansiot `78

Vector Addition Systems with States



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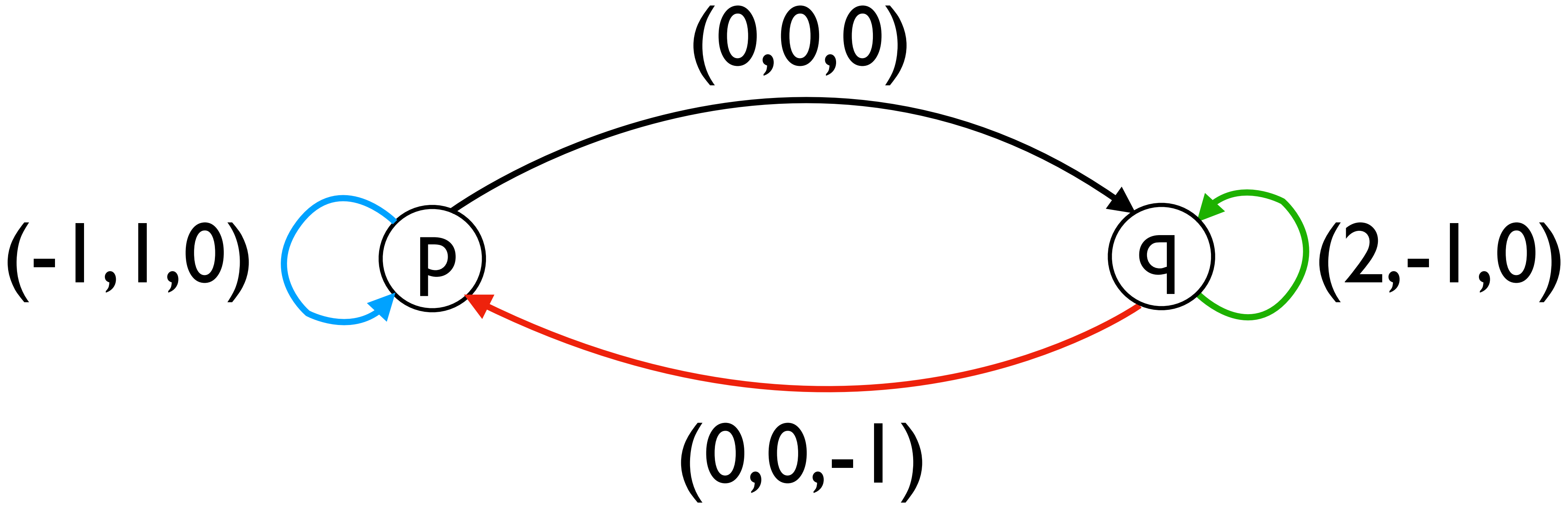
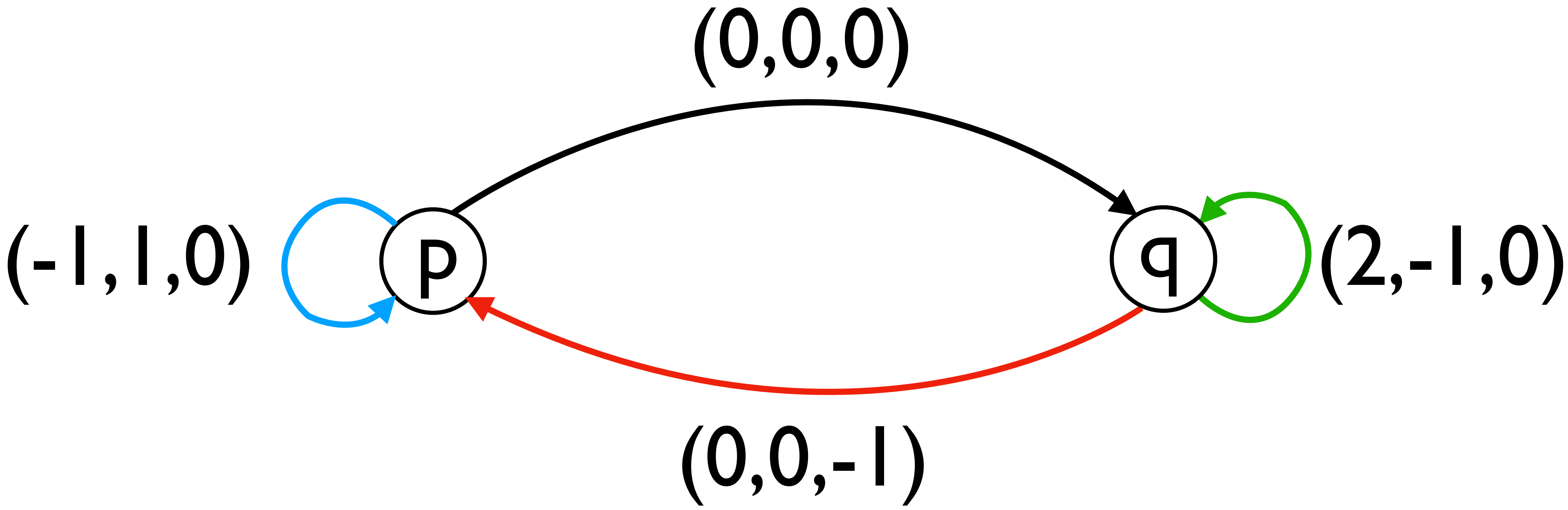


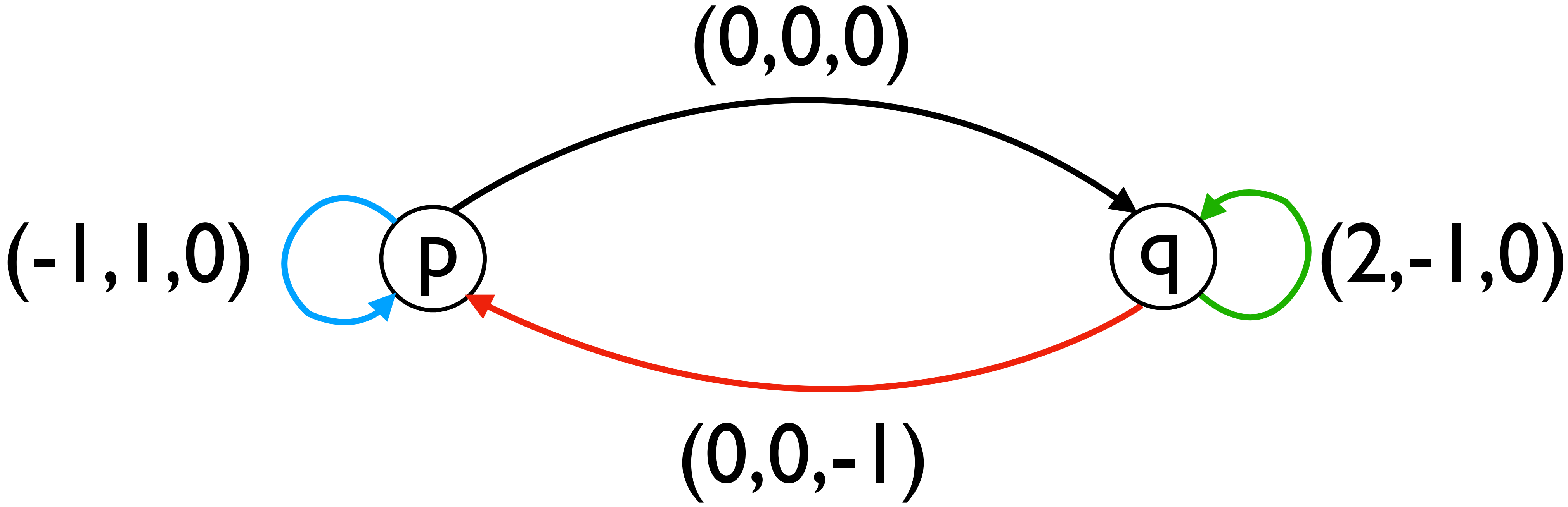
p(2,0,7)

Vector Addition Systems with States



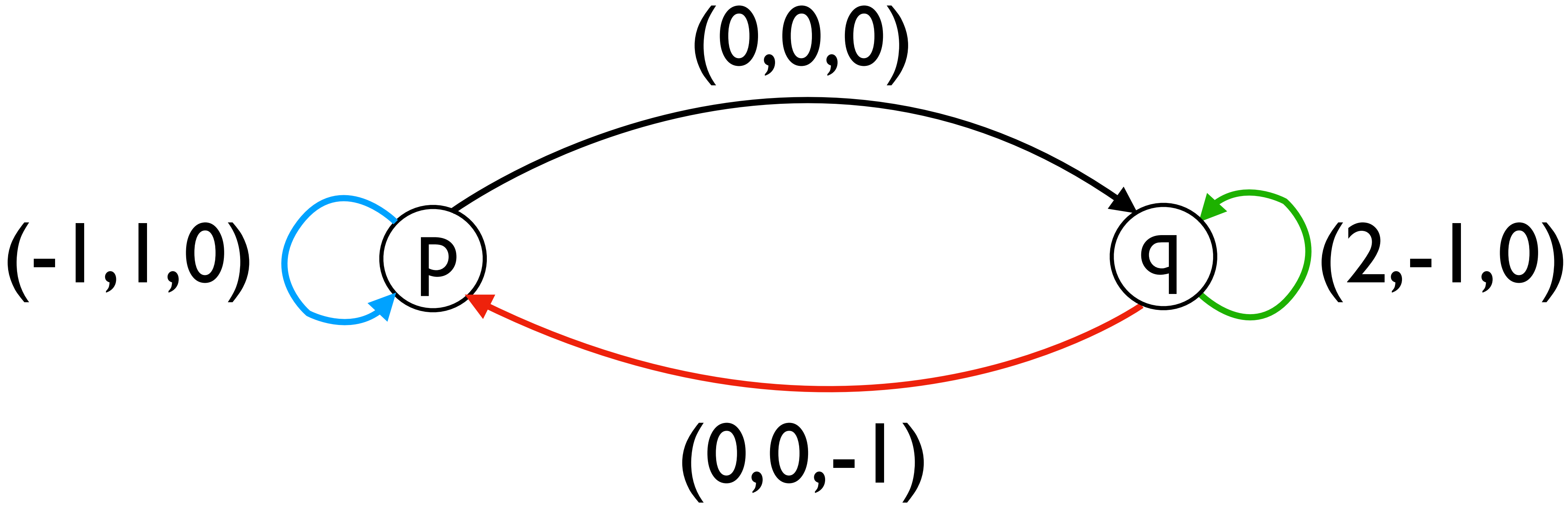
$p(2,0,7) \longrightarrow p(1,1,7)$

Vector Addition Systems with States



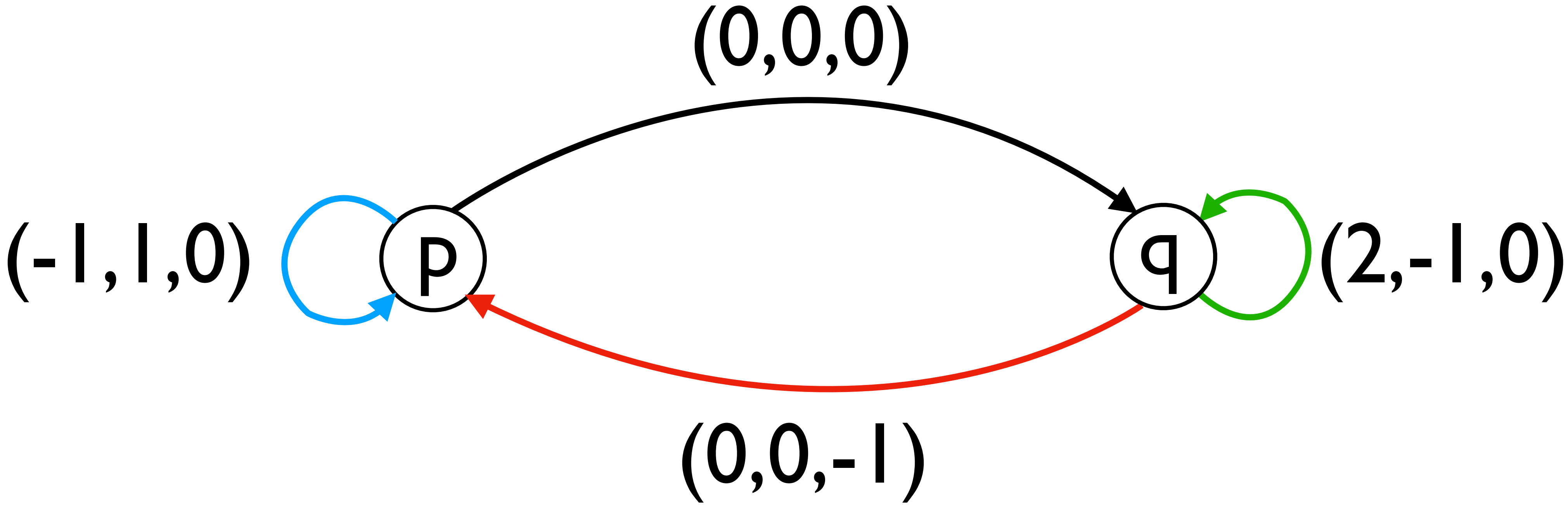
$p(2,0,7) \longrightarrow p(1,1,7) \longrightarrow p(0,2,7)$

Vector Addition Systems with States



$$p(2,0,7) \longrightarrow p(1,1,7) \longrightarrow p(0,2,7) \longrightarrow q(0,2,7)$$

Vector Addition Systems with States



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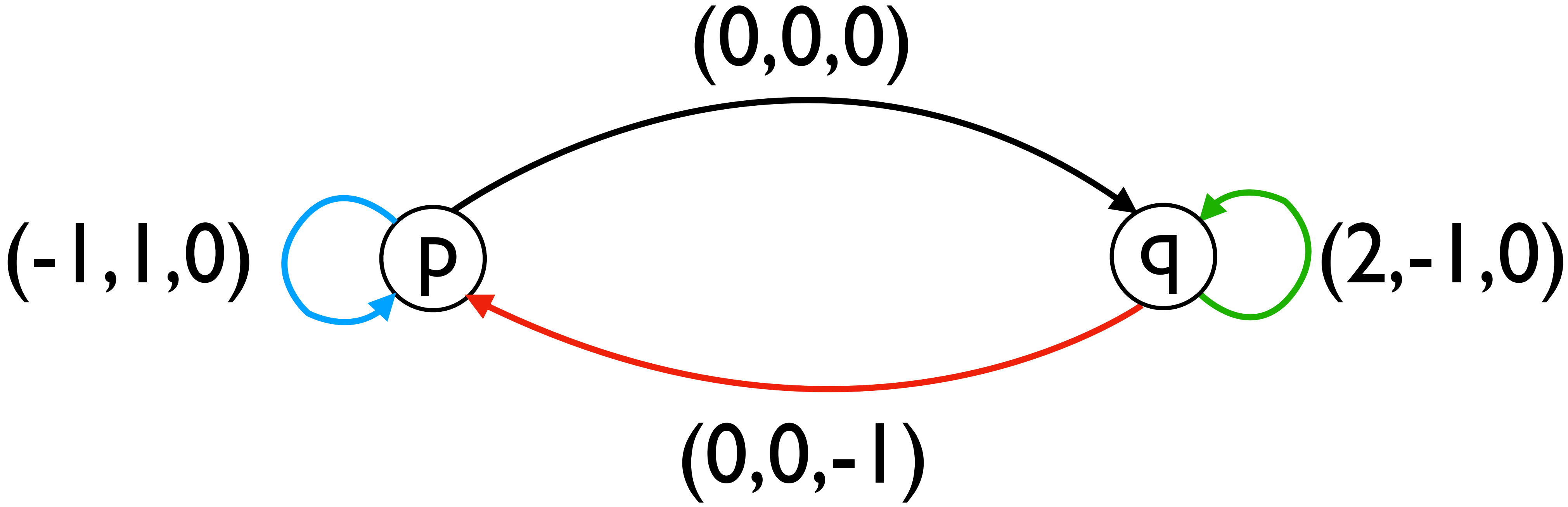
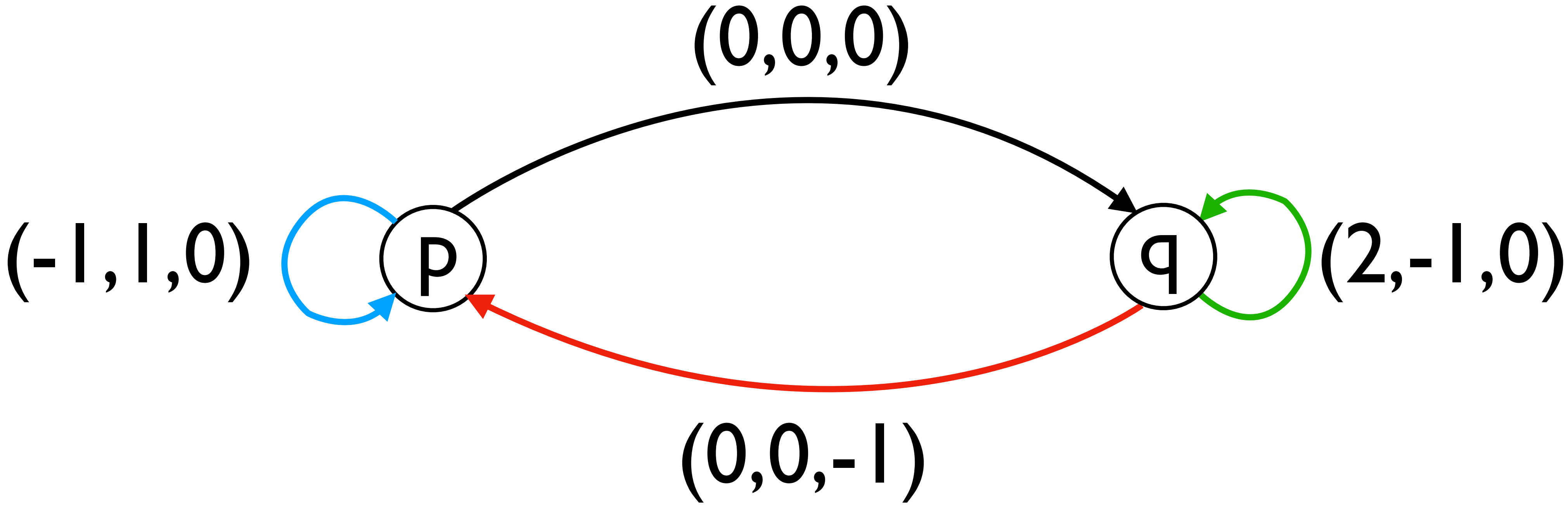


$$p(2,0,7) \longrightarrow p(1,1,7) \longrightarrow p(0,2,7) \longrightarrow q(0,2,7) \longrightarrow q(2,1,7)$$

$$\longrightarrow q(4,0,7)$$

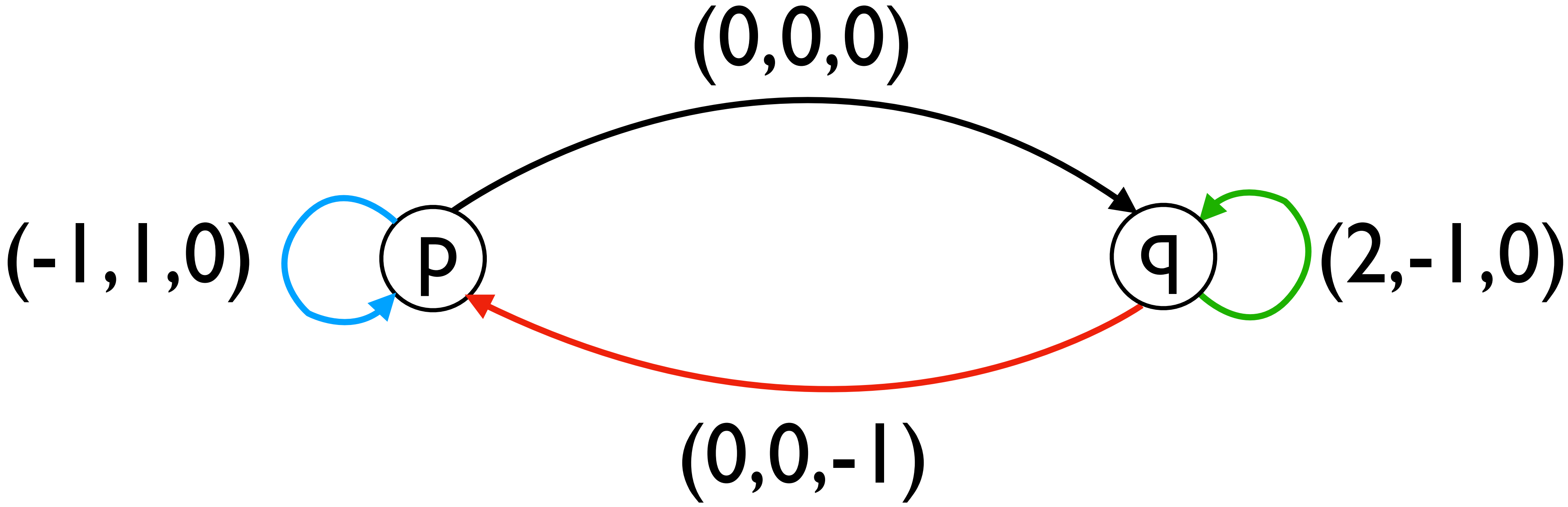
Vector Addition Systems with States



$$p(2,0,7) \longrightarrow p(1,1,7) \longrightarrow p(0,2,7) \longrightarrow q(0,2,7) \longrightarrow q(2,1,7)$$

$$\longrightarrow q(4,0,7) \longrightarrow p(4,0,6)$$

Vector Addition Systems with States



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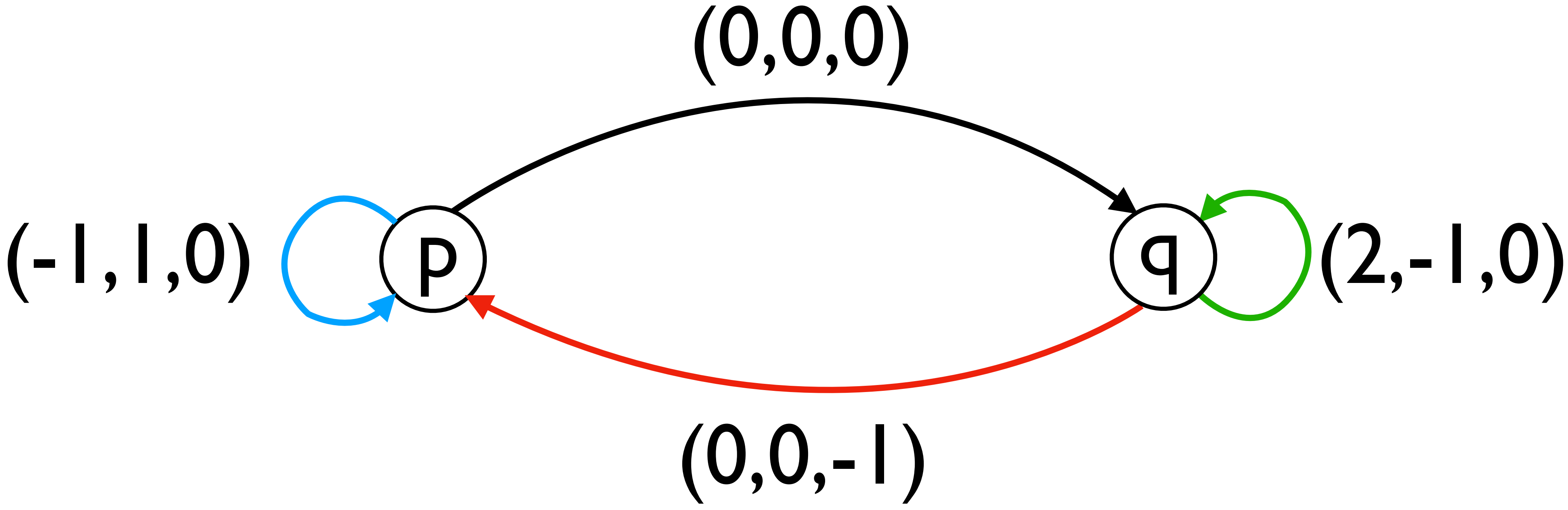
$$\longrightarrow q(4,0,7) \longrightarrow p(4,0,6)$$

Petri nets

Vector Addition Systems with States



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Vector Addition Systems
with States

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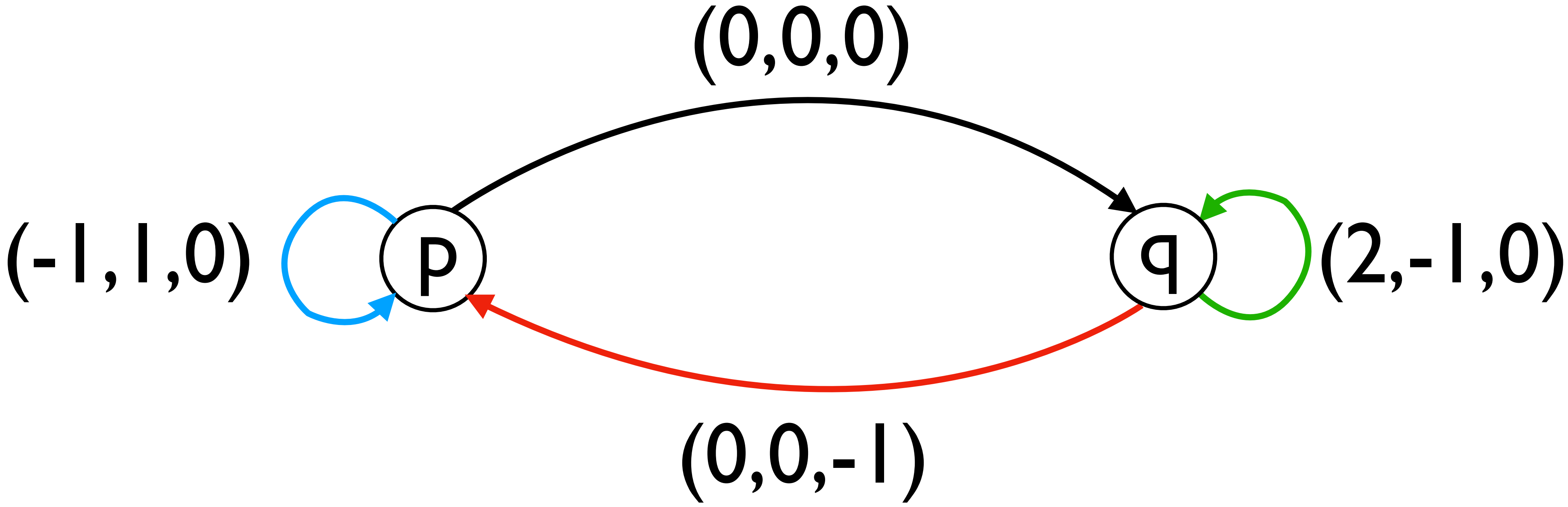

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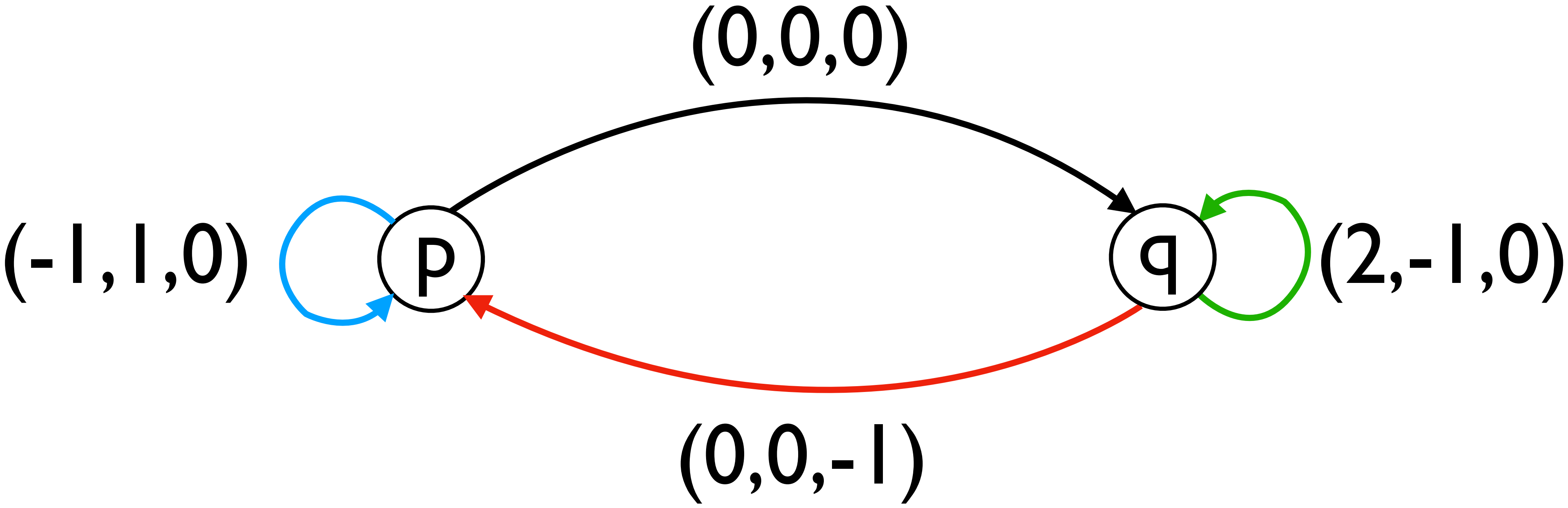
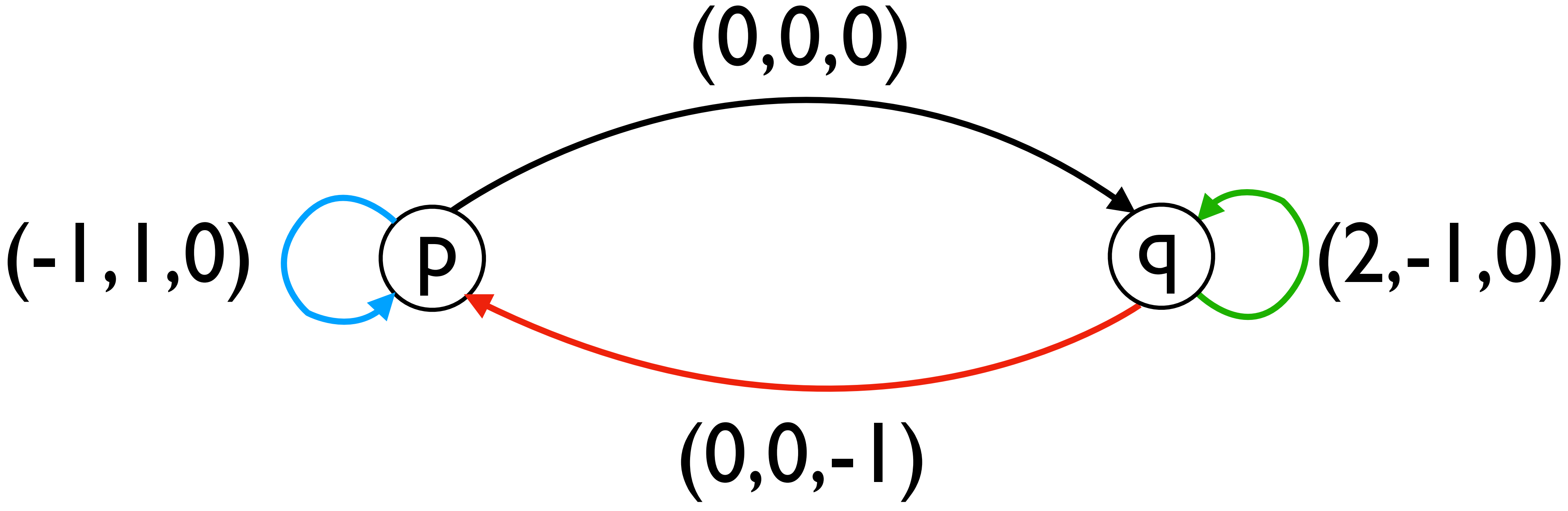


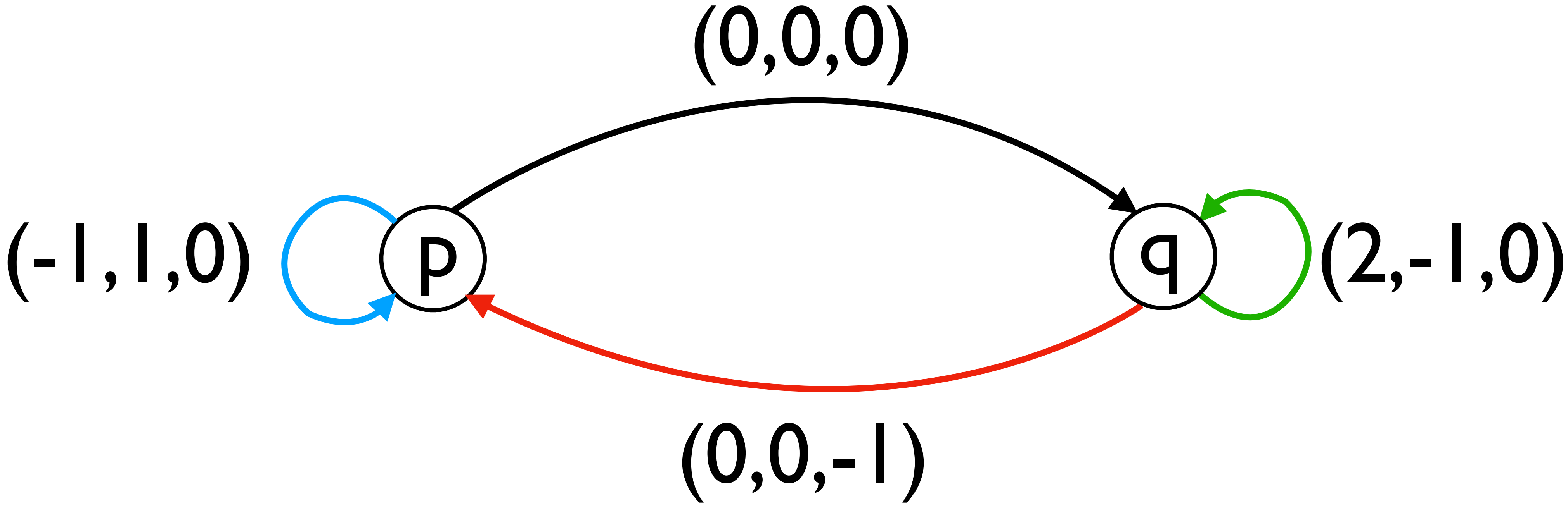
p(k,0,n)

Vector Addition Systems with States



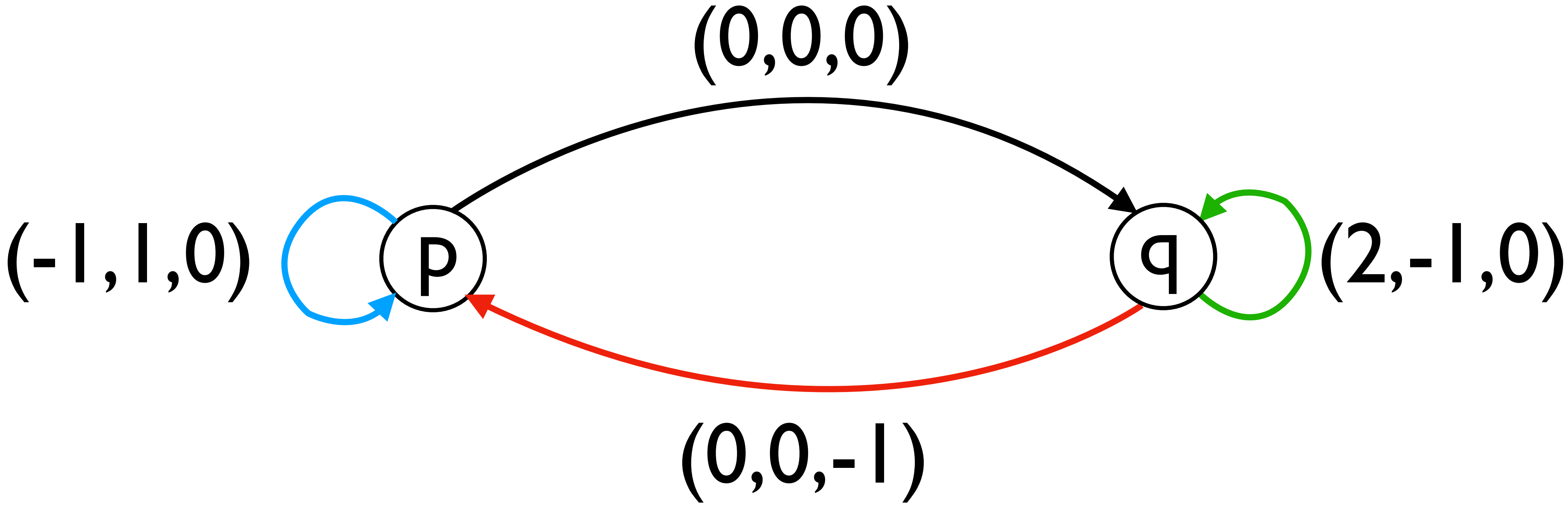
p(k,0,n) ⟶ p(0,k,n)

Vector Addition Systems with States



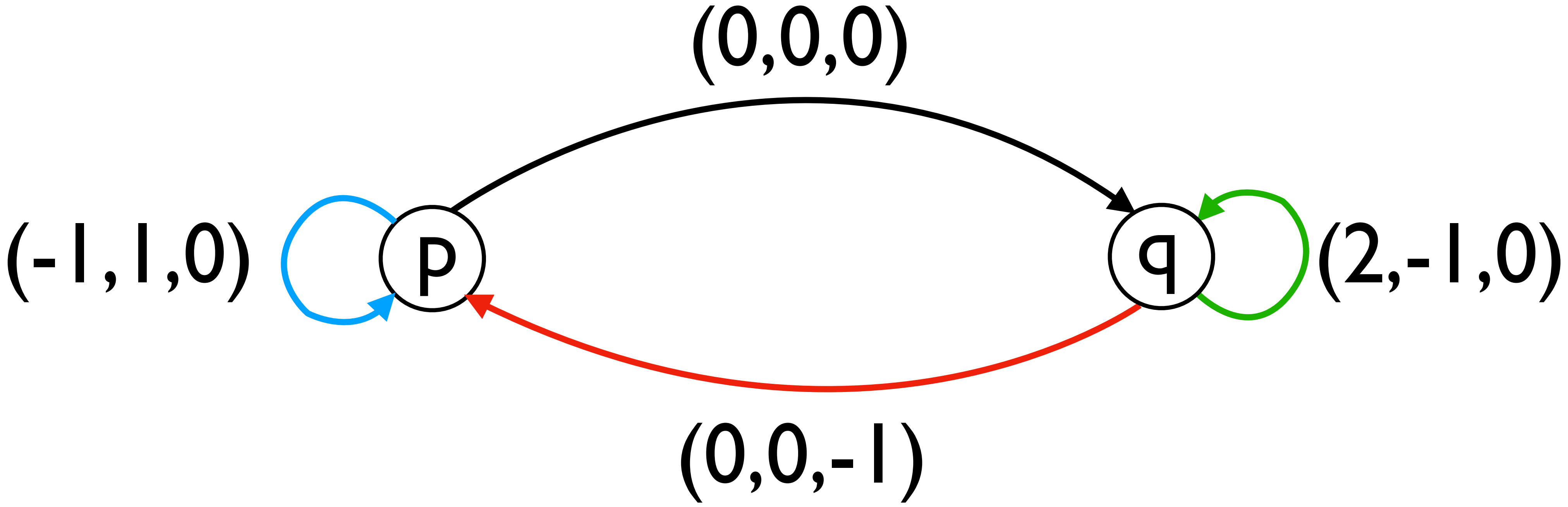
$$p(k,0,n) \longrightarrow p(0,k,n) \longrightarrow q(0,k,n)$$

Vector Addition Systems with States



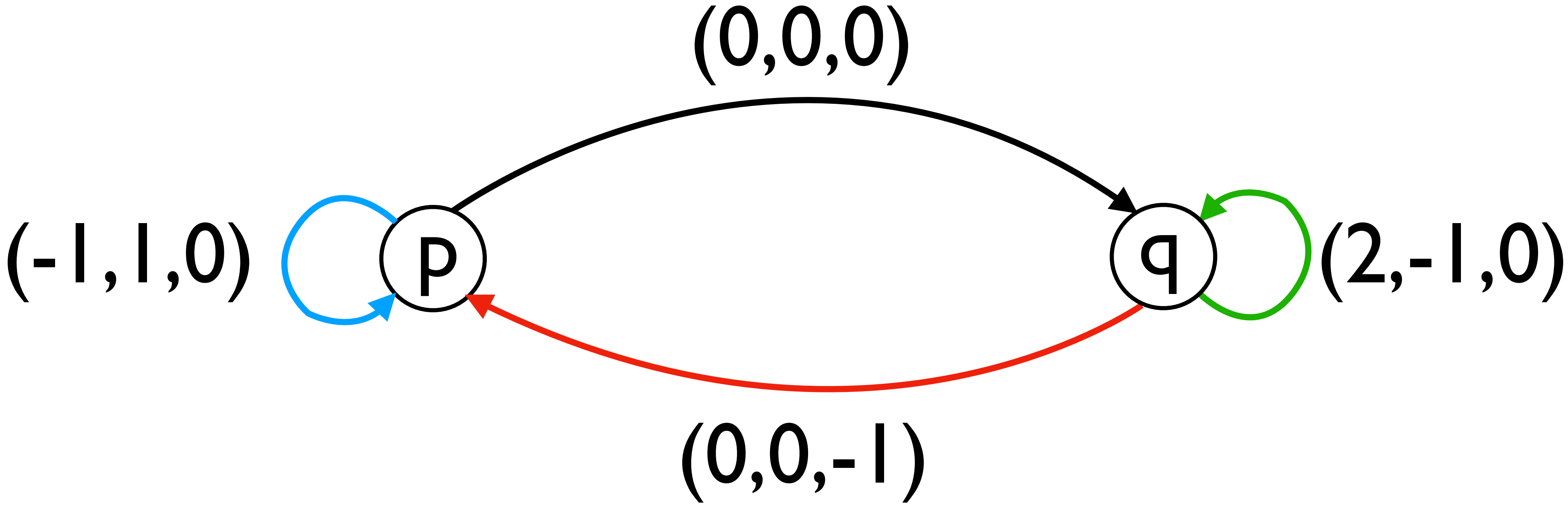
$$p(k,0,n) \longrightarrow p(0,k,n) \longrightarrow q(0,k,n) \longrightarrow q(2k,0,n)$$

Vector Addition Systems with States



$$p(k,0,n) \longrightarrow p(0,k,n) \longrightarrow q(0,k,n) \longrightarrow q(2k,0,n) \longrightarrow p(2k,0,n-1)$$

Vector Addition Systems with States



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$p(1,0,n)$

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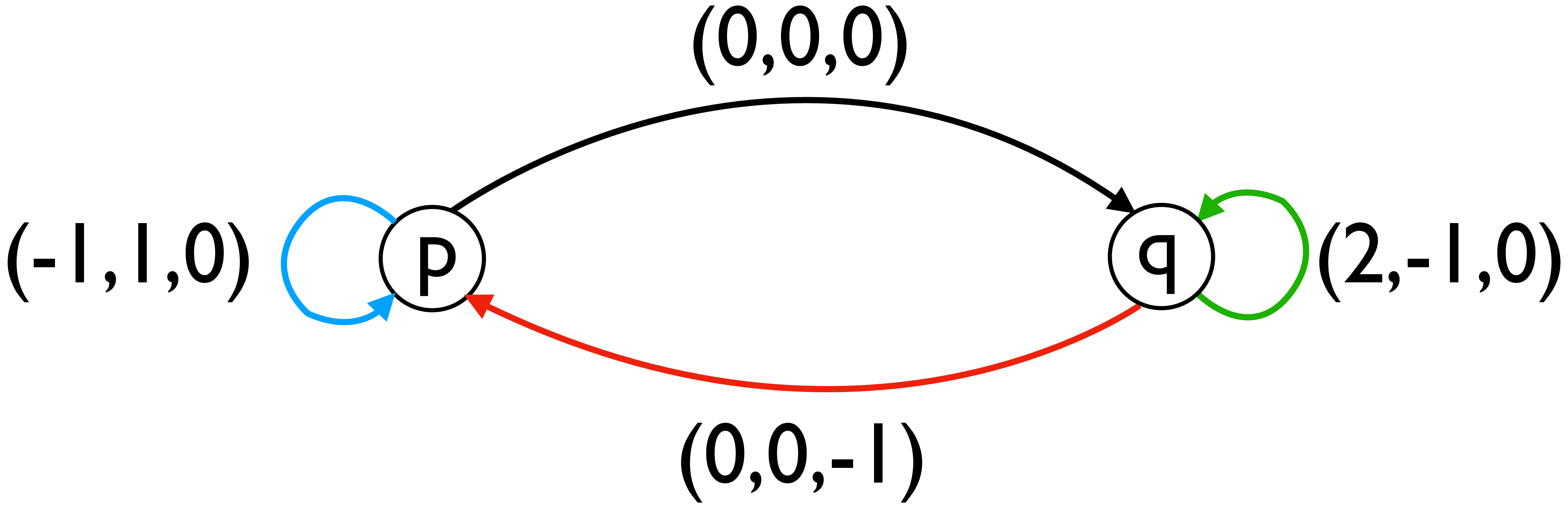

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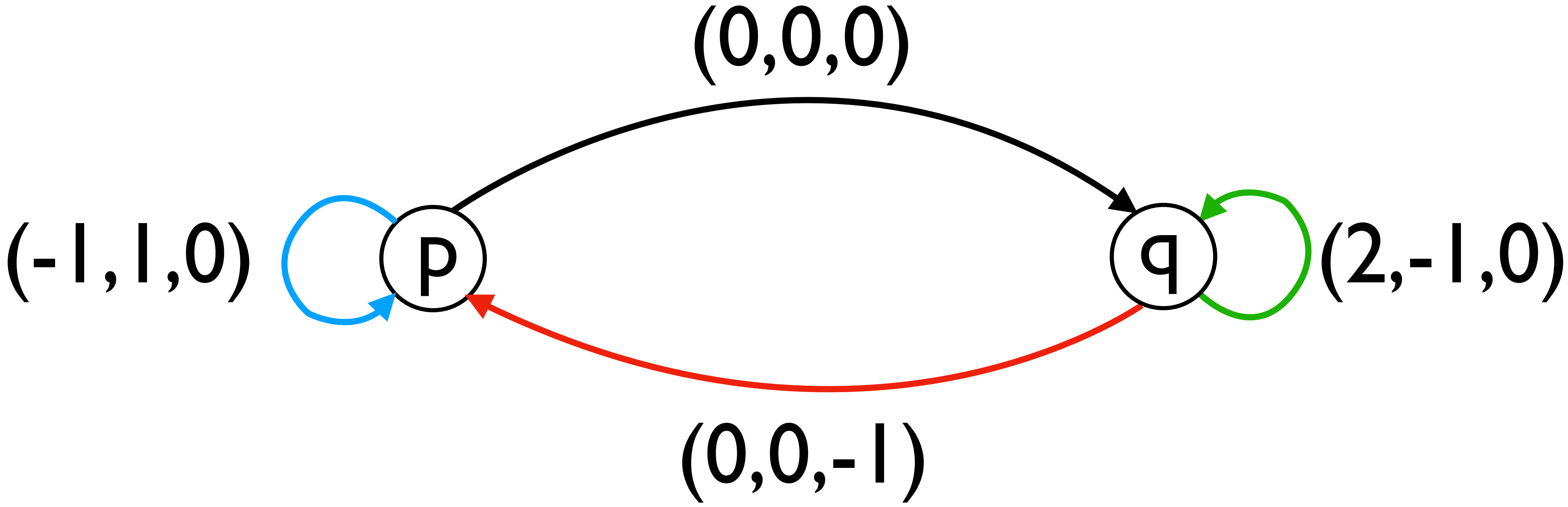
$$p(1,0,n) \longrightarrow p(2,0,n-1)$$

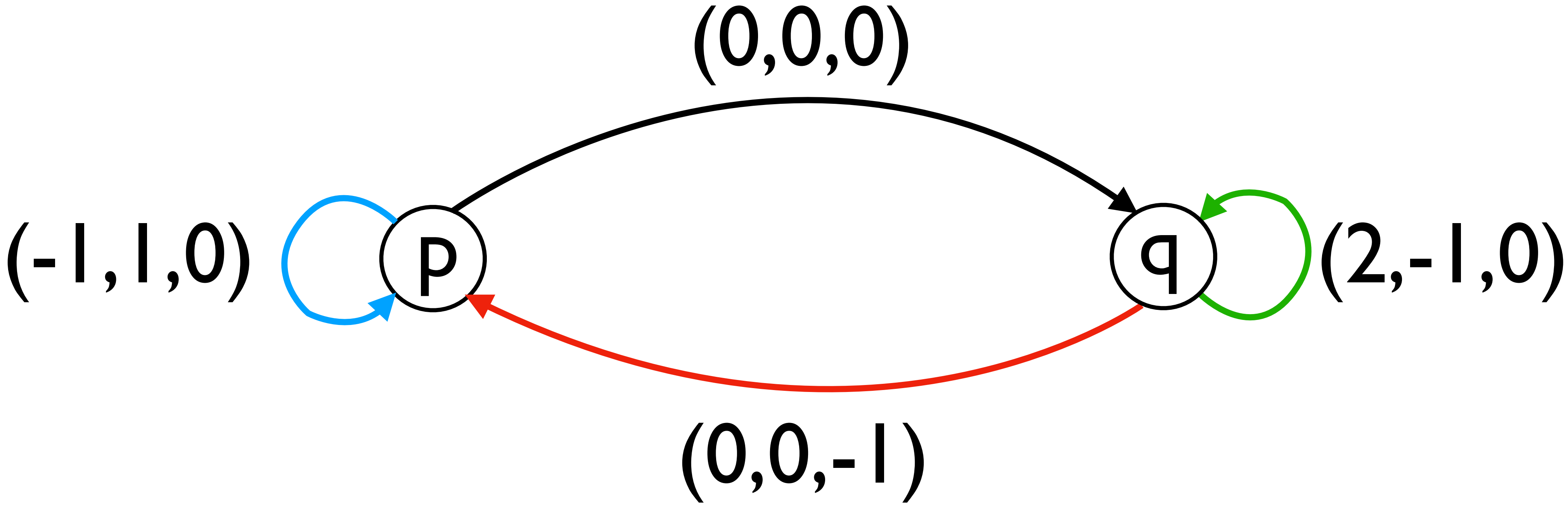
Vector Addition Systems with States



$$p(k,0,n) \longrightarrow p(0,k,n) \longrightarrow q(0,k,n) \longrightarrow q(2k,0,n) \longrightarrow p(2k,0,n-1)$$

$$p(1,0,n) \longrightarrow p(2,0,n-1) \ \ldots$$

Vector Addition Systems with States



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$$p(1,0,n) \longrightarrow p(2,0,n-1) \;\ldots\; \longrightarrow p(2^n,0,0)$$

Vector Addition Systems with States

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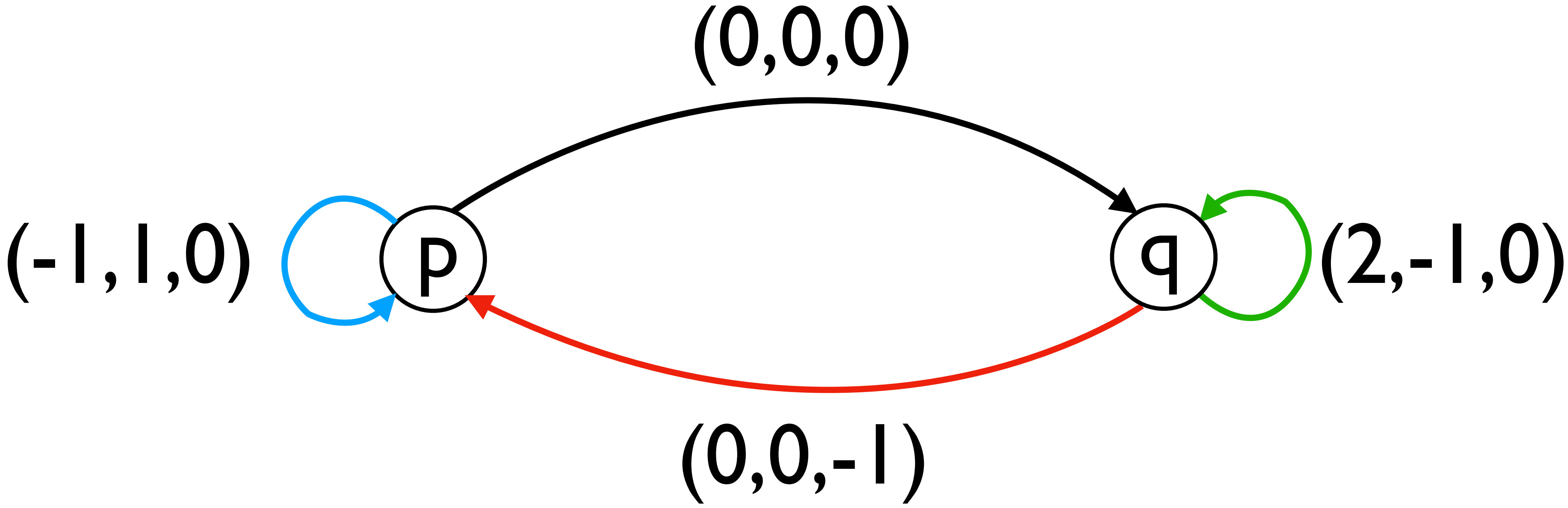

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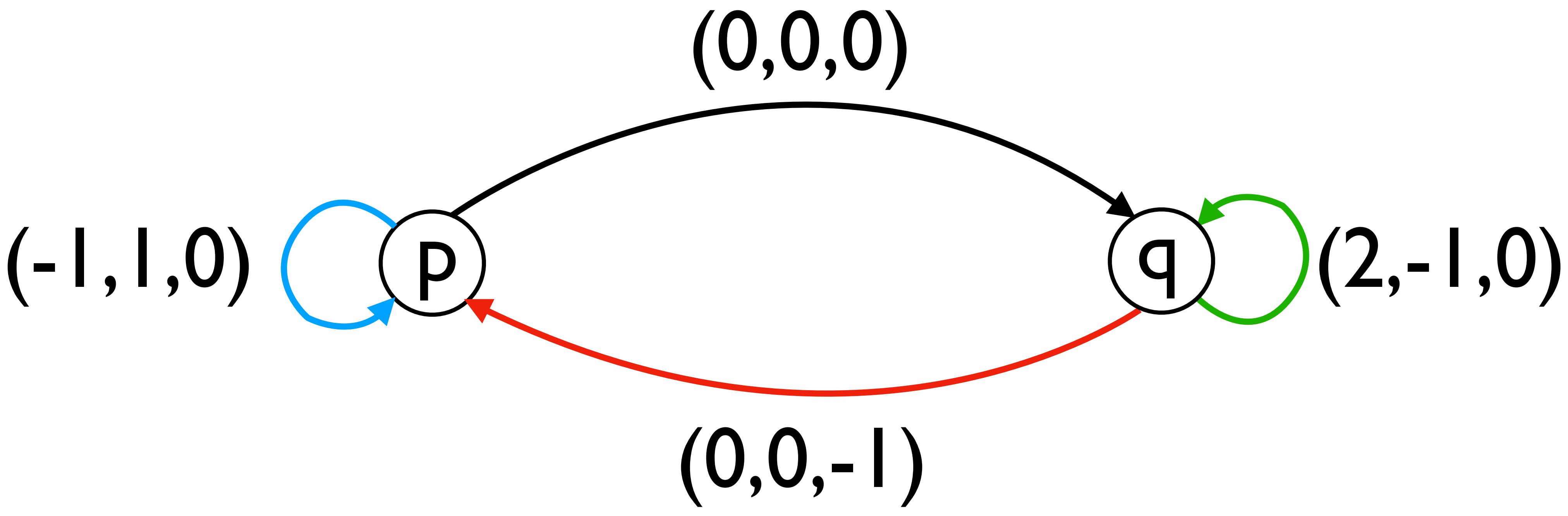
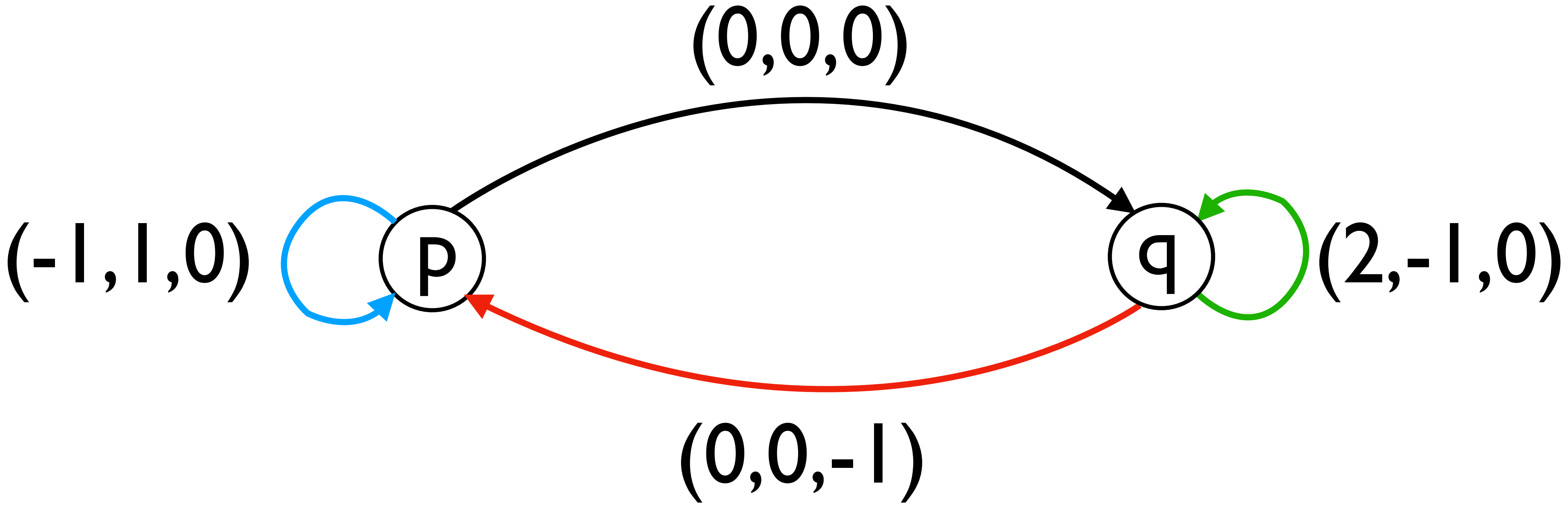


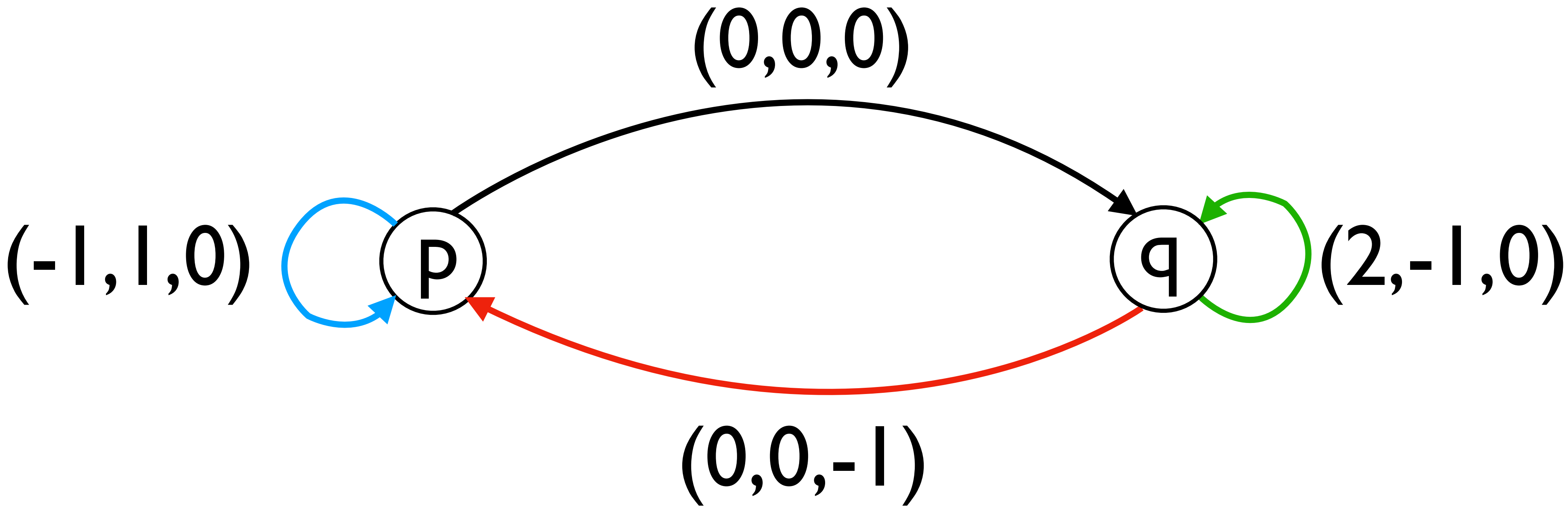
p(1,0,n)

Vector Addition Systems with States



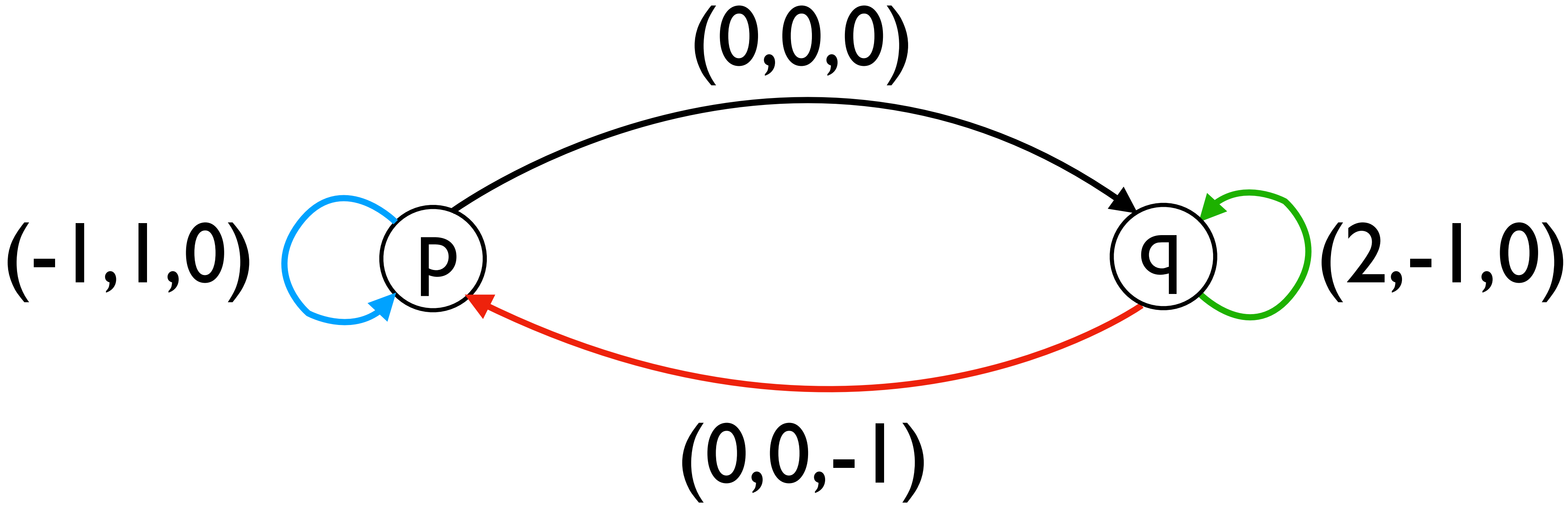
$p(1,0,n) \longrightarrow p(2,0,n-1)$

Vector Addition Systems with States



$p(1,0,n) \longrightarrow p(2,0,n-1)$ …

Vector Addition Systems with States



$$p(1,0,n) \longrightarrow p(2,0,n-1) \;\ldots\; \longrightarrow p(2^n,0,0)$$

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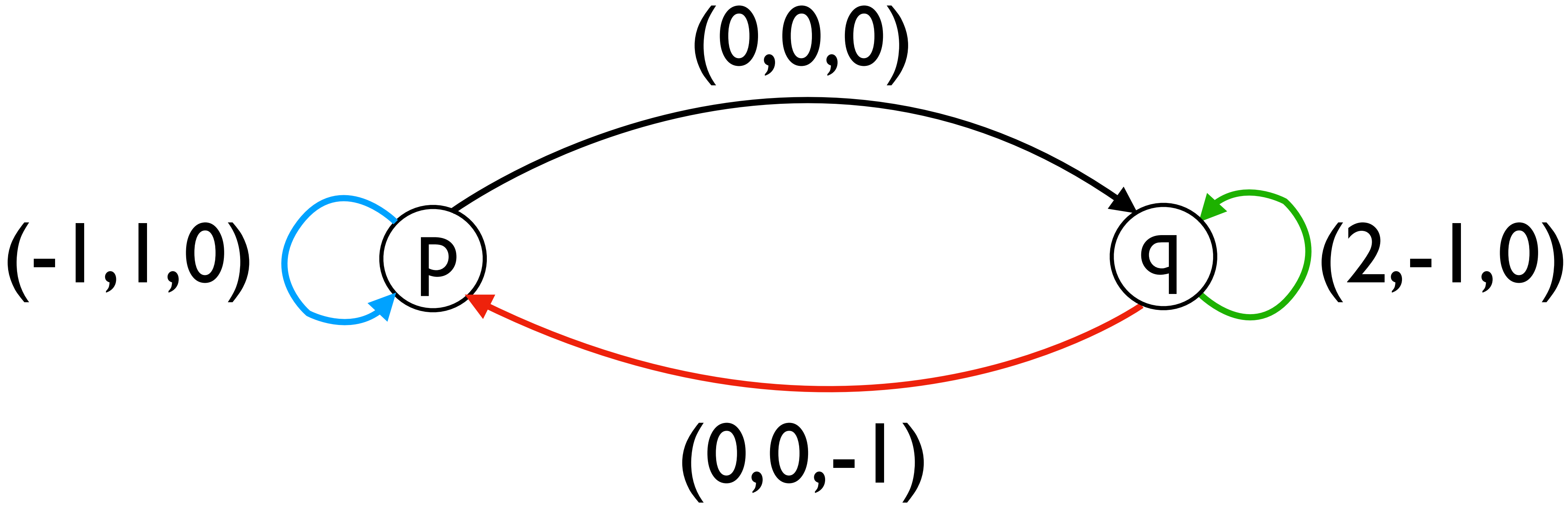

$p(1,0,n) \longrightarrow p(2,0,n-1)\ \ldots \longrightarrow p(2^n,0,0) \longrightarrow p(2^n-a, a, 0)$

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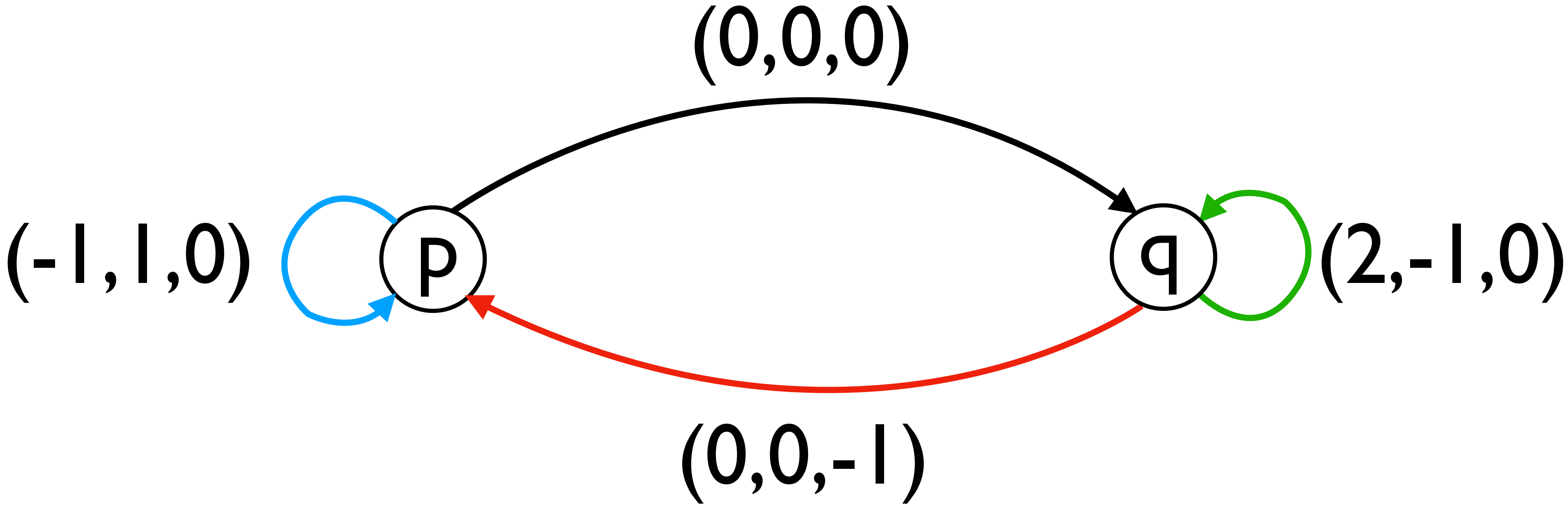

$p(1,0,n) \longrightarrow p(2,0,n-1) \ \dots \ \longrightarrow p(2^n,0,0) \longrightarrow p(2^n\text{-}a, a, 0)$

finite exponential reachability set

Reachability problem

Reachability problem

Given: a VASS, two its configurations s and t

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Question: is there a run from s to t?

Reachability problem

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coverability problem

Short history

Short history

Lipton `76: ExpSpace-hardness of coverability

Short history

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Rackoff `78: coverability in ExpSpace

Short history

Lipton `76: ExpSpace-hardness of coverability

Rackoff `78: coverability in ExpSpace

doubly-exponential length paths

Short history

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Rackoff `78: coverability in ExpSpace

Short history

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Mayr `81: decidability of reachability

Short history

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Blondin at el. `15: reachability PSpace-complete
for 2-VASSes

Short history

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Short history

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Leroux, Schmitz `15: cubic-Ackermann

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Leroux, Schmitz `19: Ackermann

Short history

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Conjecture: reachability in ExpSpace

Short history

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Conjecture: reachability in ExpSpace

Cz., Lasota, Lazic, Leroux, Mazowiecki `19:
Tower-hardness

Short history

Leroux, Schmitz `15: cubic-Ackermann

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Conjecture: reachability in ExpSpace

Cz., Lasota, Lazic, Leroux, Mazowiecki `19:
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Leroux & Cz., Orlikowski`21: Ackermann-hardness

Functions F_k

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$F_1(n) = 2n$

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$F_{k+1}(n) = F_k \circ \ldots \circ F_k(1)$

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$\mathrm{Ack}(n) = F_\omega(n) = F_n(n)$

Ackermann-hardness

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Theorem

The Reachability Problem for (2k+4)-VASSes is $\mathbb{F}_k$-hard.

Ackermann-hardness

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Jerome Leroux

Ackermann-hardness

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Cz., Łukasz Orlikowski: 6k

Ackermann-hardness

Theorem

The Reachability Problem for (2k+4)-VASSes is $\mathbb{F}_k$-hard.

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Cz., Łukasz Orlikowski: 6k

open: is k+C enough?

Everything solved?

Everything solved?

reachability for 3-VASSes

Everything solved?

reachability for 3-VASSes (Tower? PSpace?)

Everything solved?

reachability for 3-VASSes (Tower? PSpace?)

reachability for fixed VASSes

Everything solved?

reachability for 3-VASSes (Tower? PSpace?)

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Everything solved?

reachability for 3-VASSes (Tower? PSpace?)

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reachability for VASS extensions

Everything solved?

reachability for 3-VASSes	(Tower? PSpace?)
reachability for fixed VASSes	(Ackermann? PSpace?)
reachability for VASS extensions	(decidable?)

Hard examples

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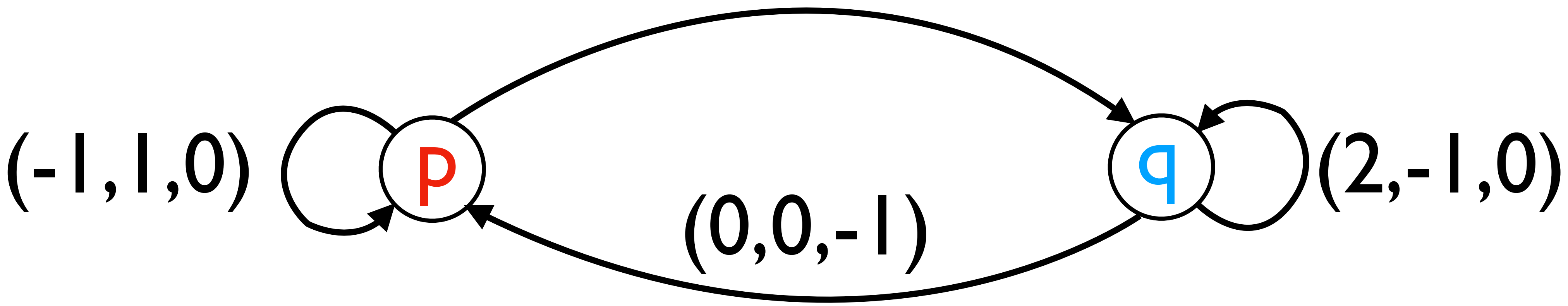

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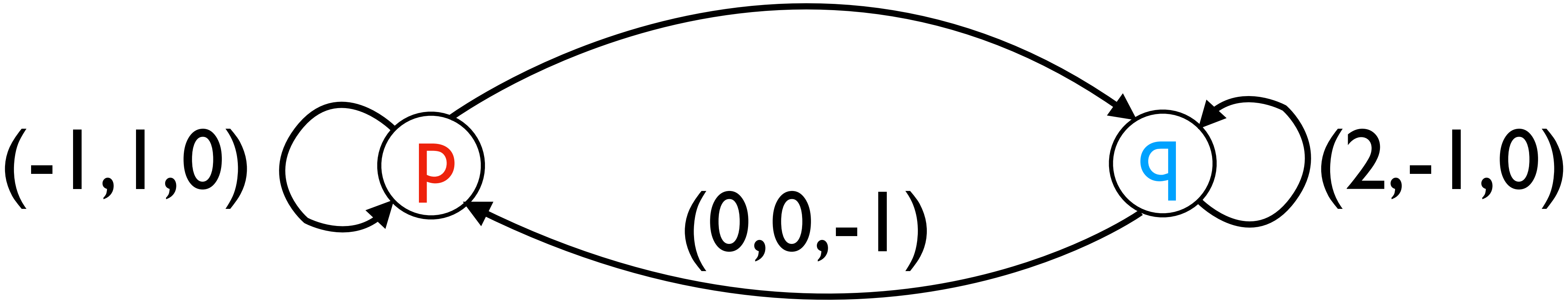

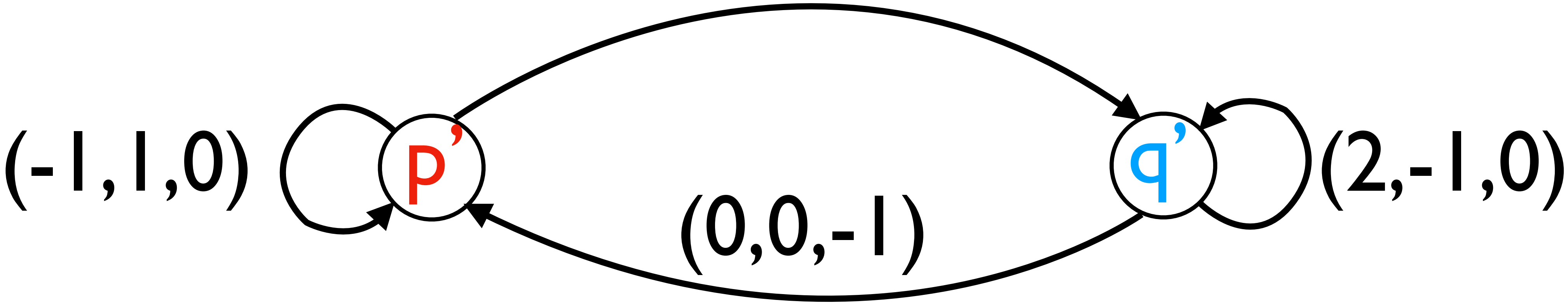

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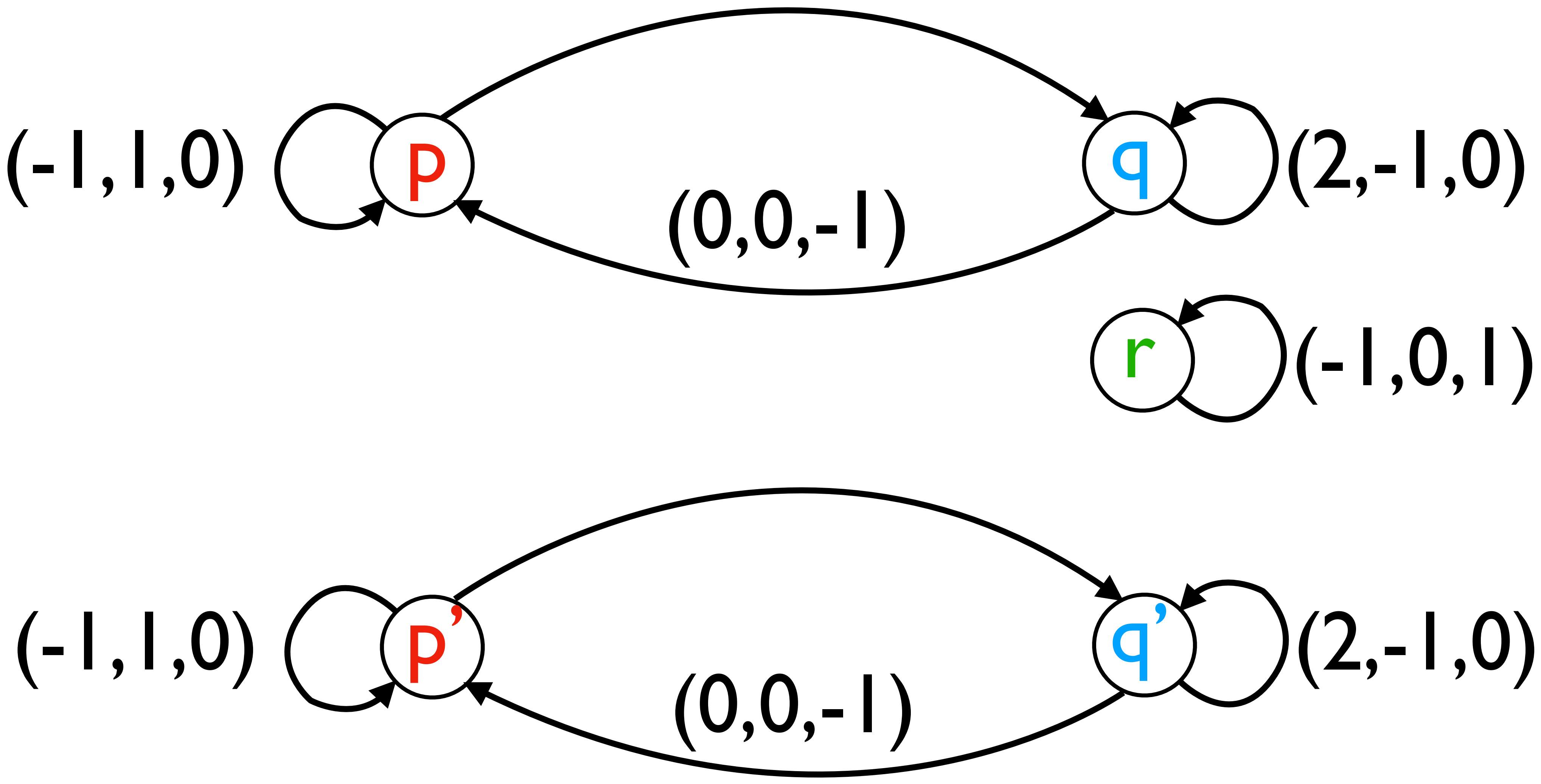

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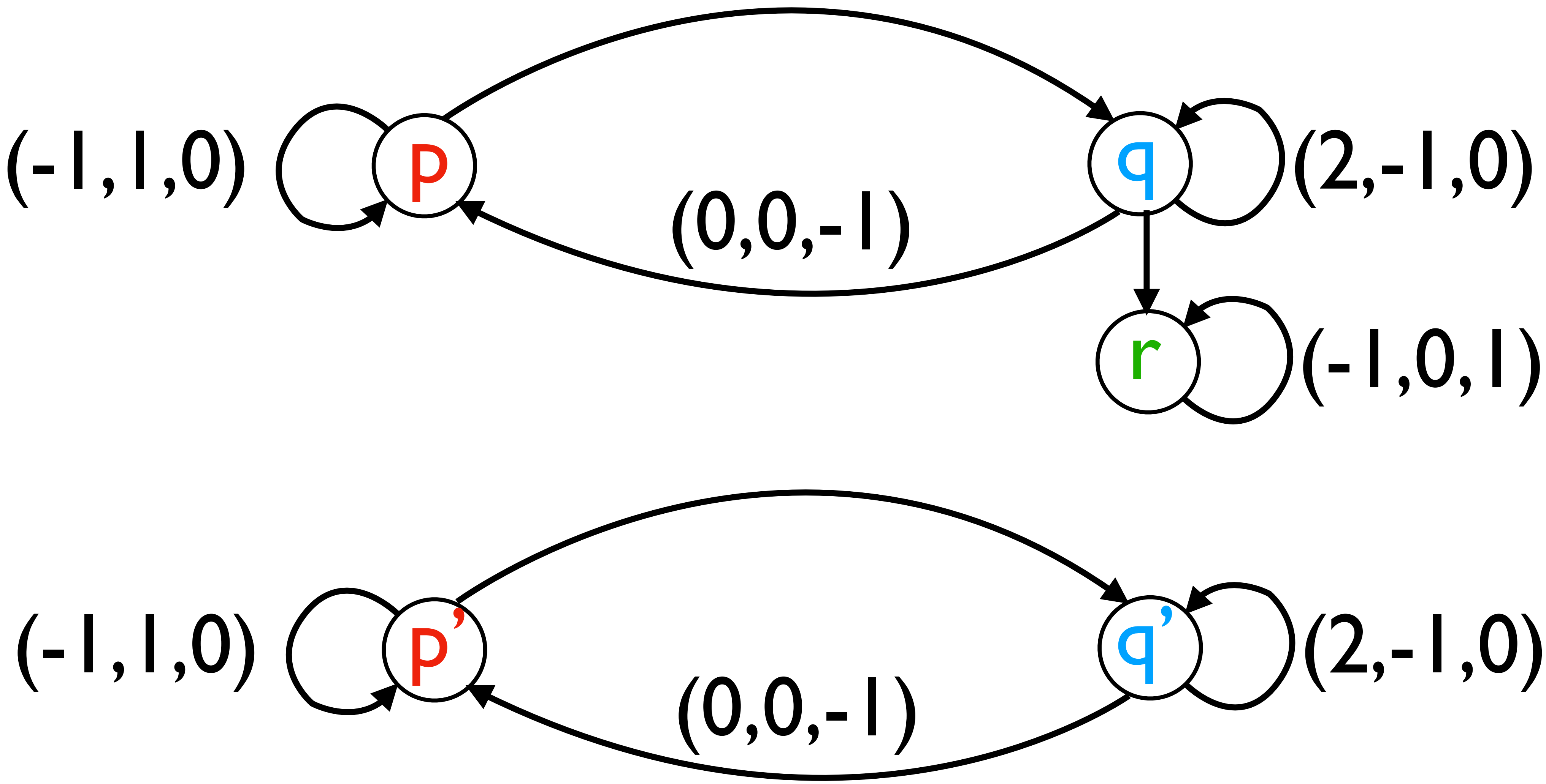

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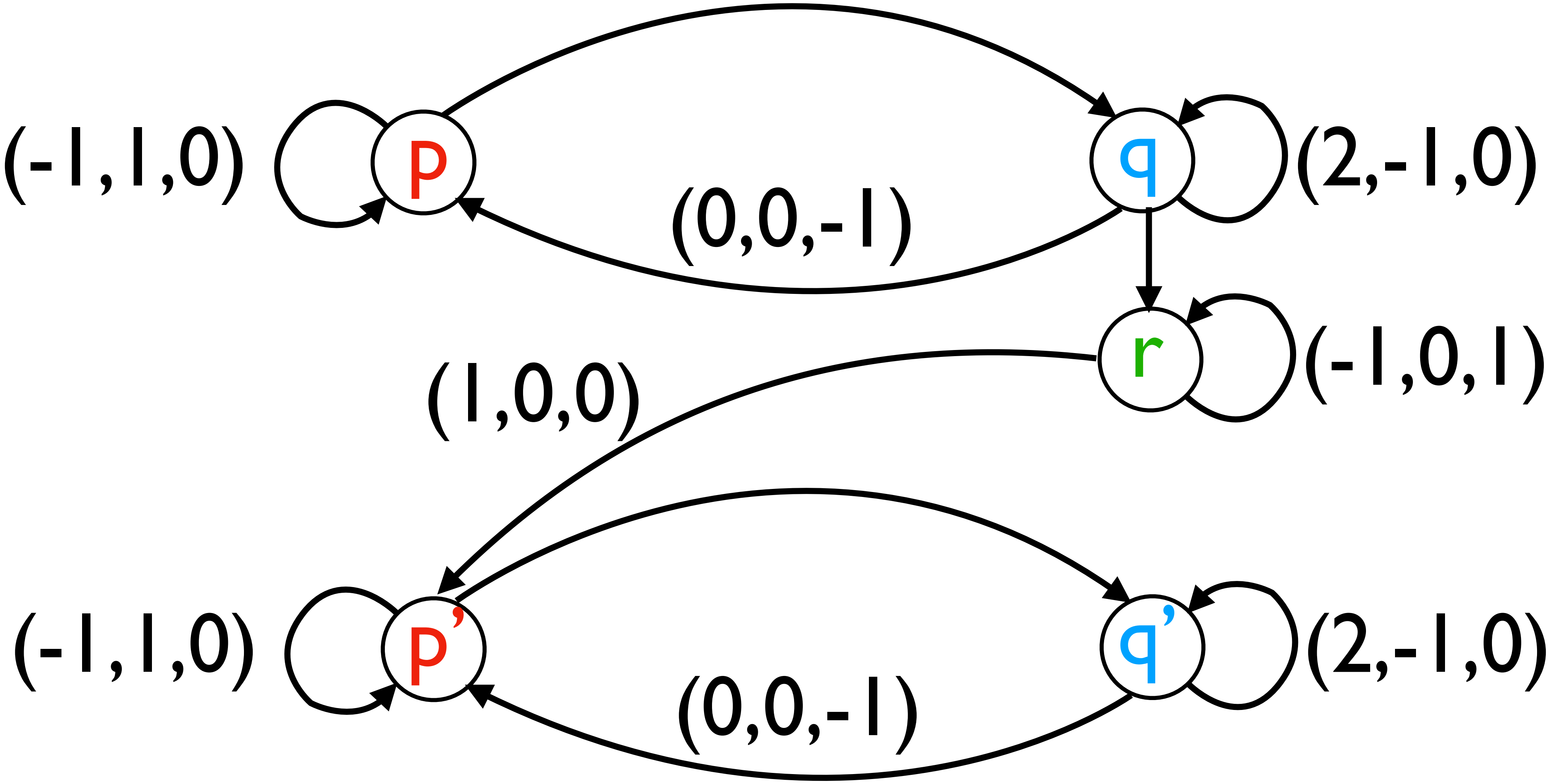

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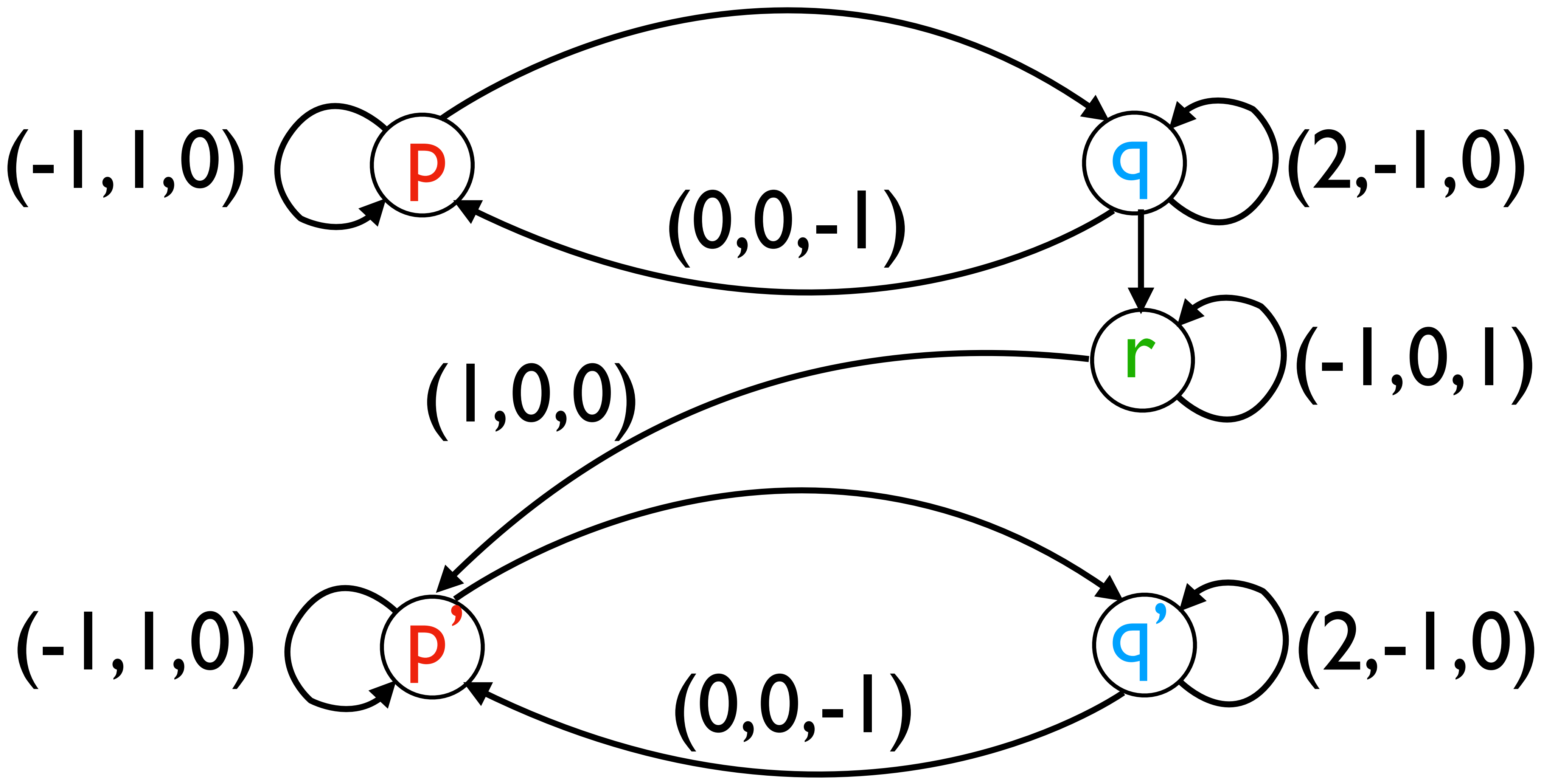
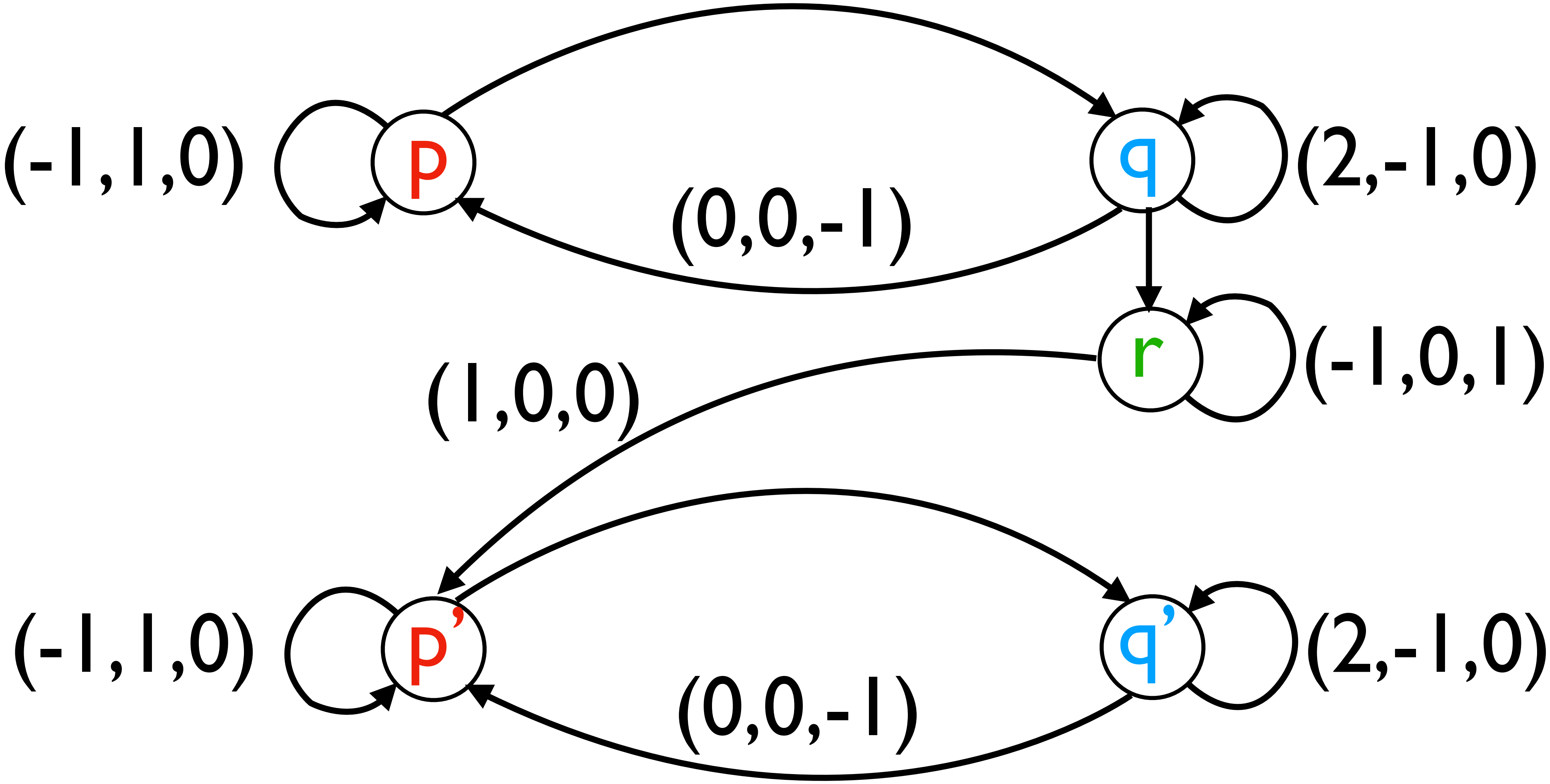


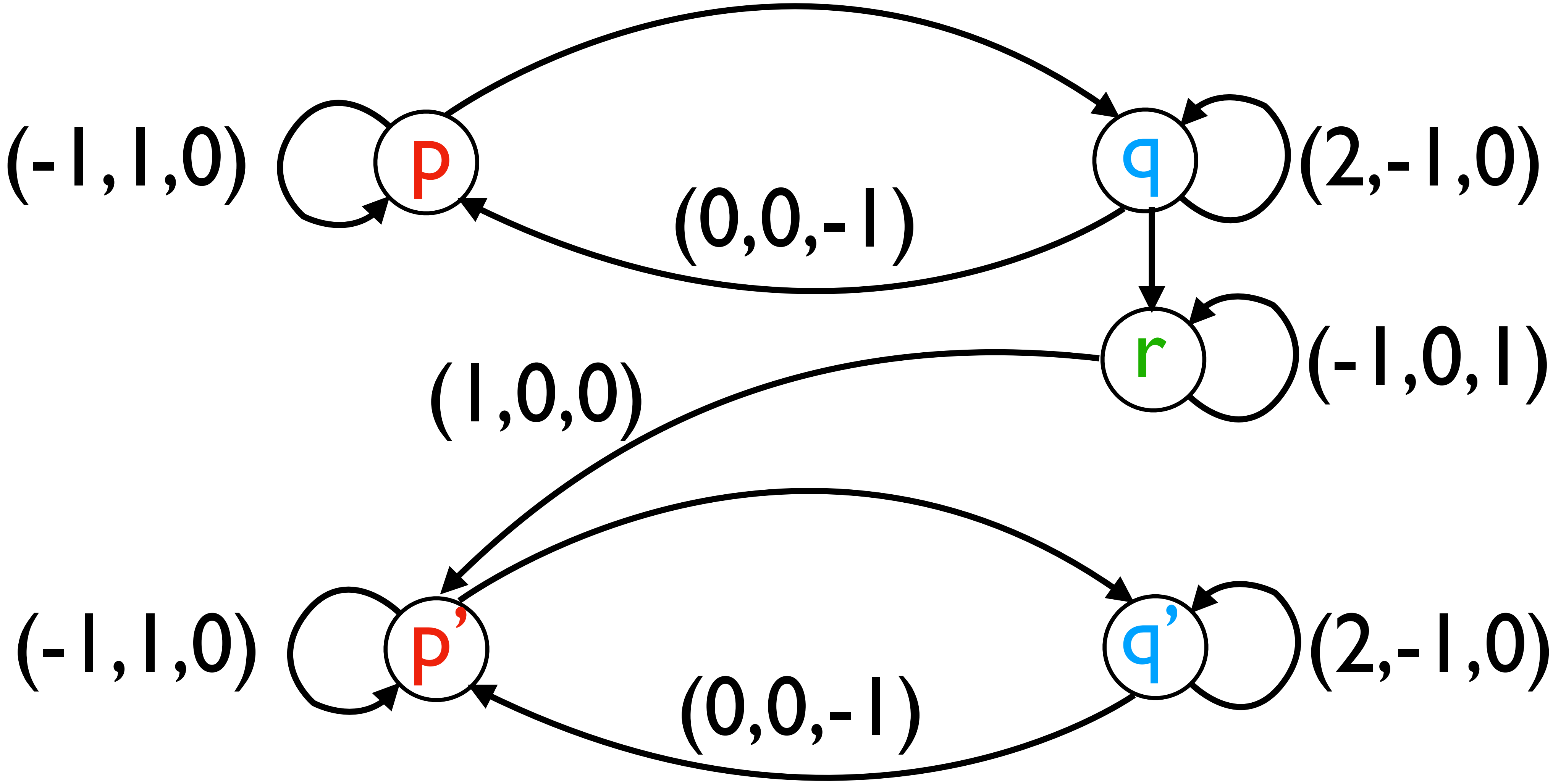
p(1,0,n)

Hard examples



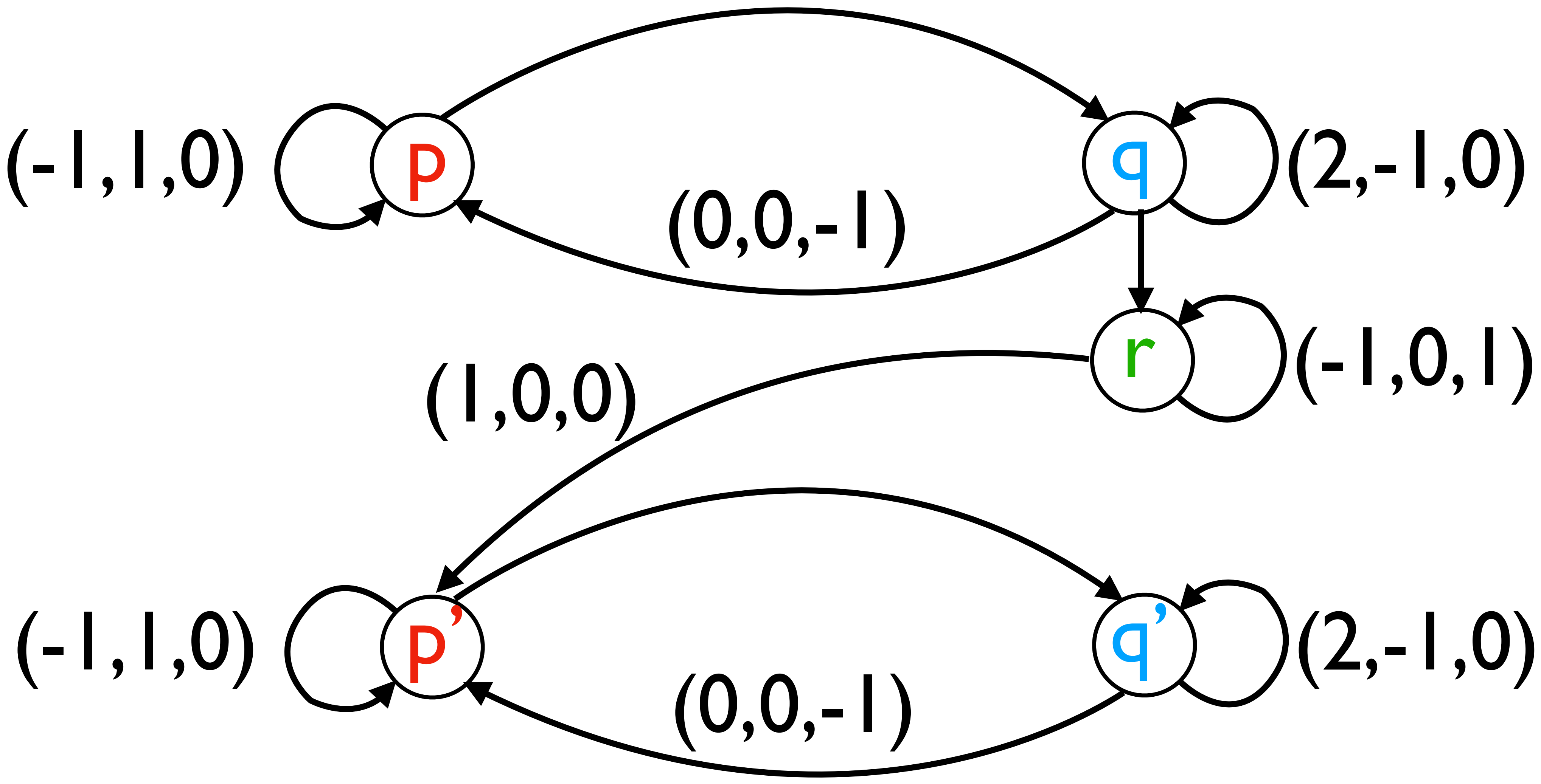
$p(1,0,n) \longrightarrow q(2^n,0,0)$

Hard examples



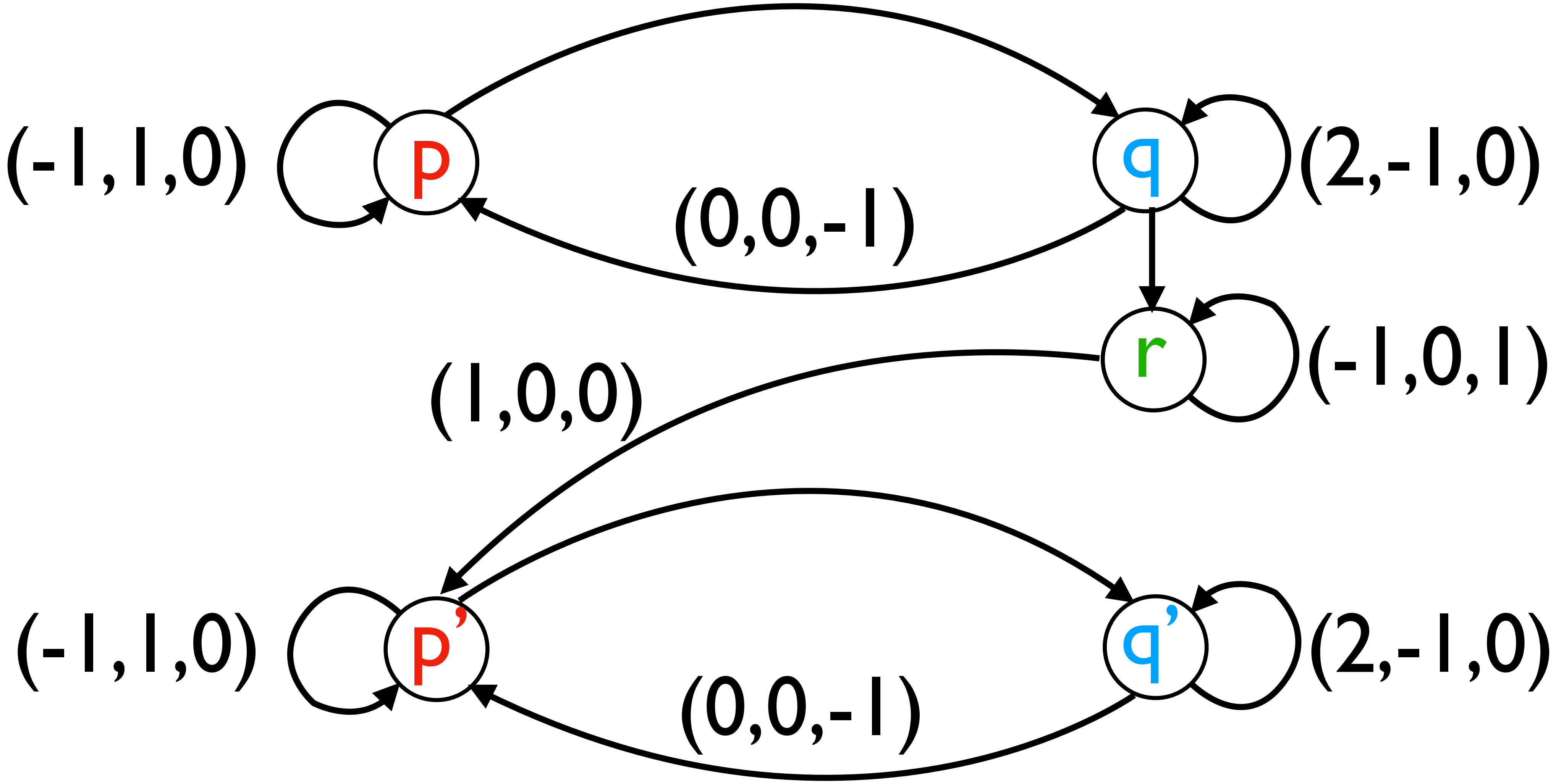
$p(1,0,n) \longrightarrow q(2^n,0,0) \longrightarrow r(2^n,0,0)$

Hard examples



$$p(1,0,n) \longrightarrow q(2^n,0,0) \longrightarrow r(2^n,0,0) \longrightarrow r(0,0,2^n)$$

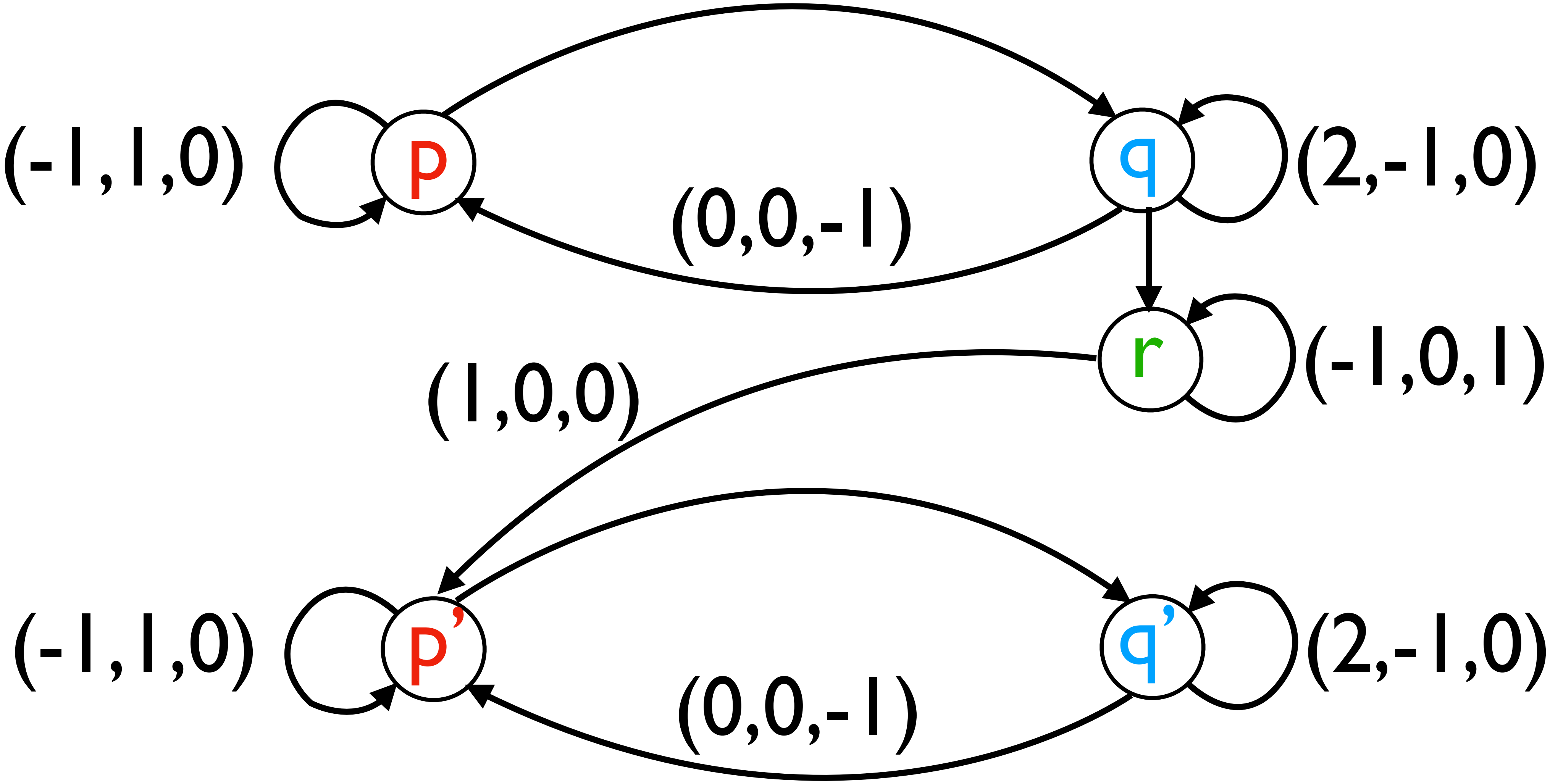
Hard examples



$p(1,0,n) \longrightarrow q(2^n,0,0) \longrightarrow r(2^n,0,0) \longrightarrow r(0,0,2^n)$

$\longrightarrow p'(1,0,2^n)$

Hard examples



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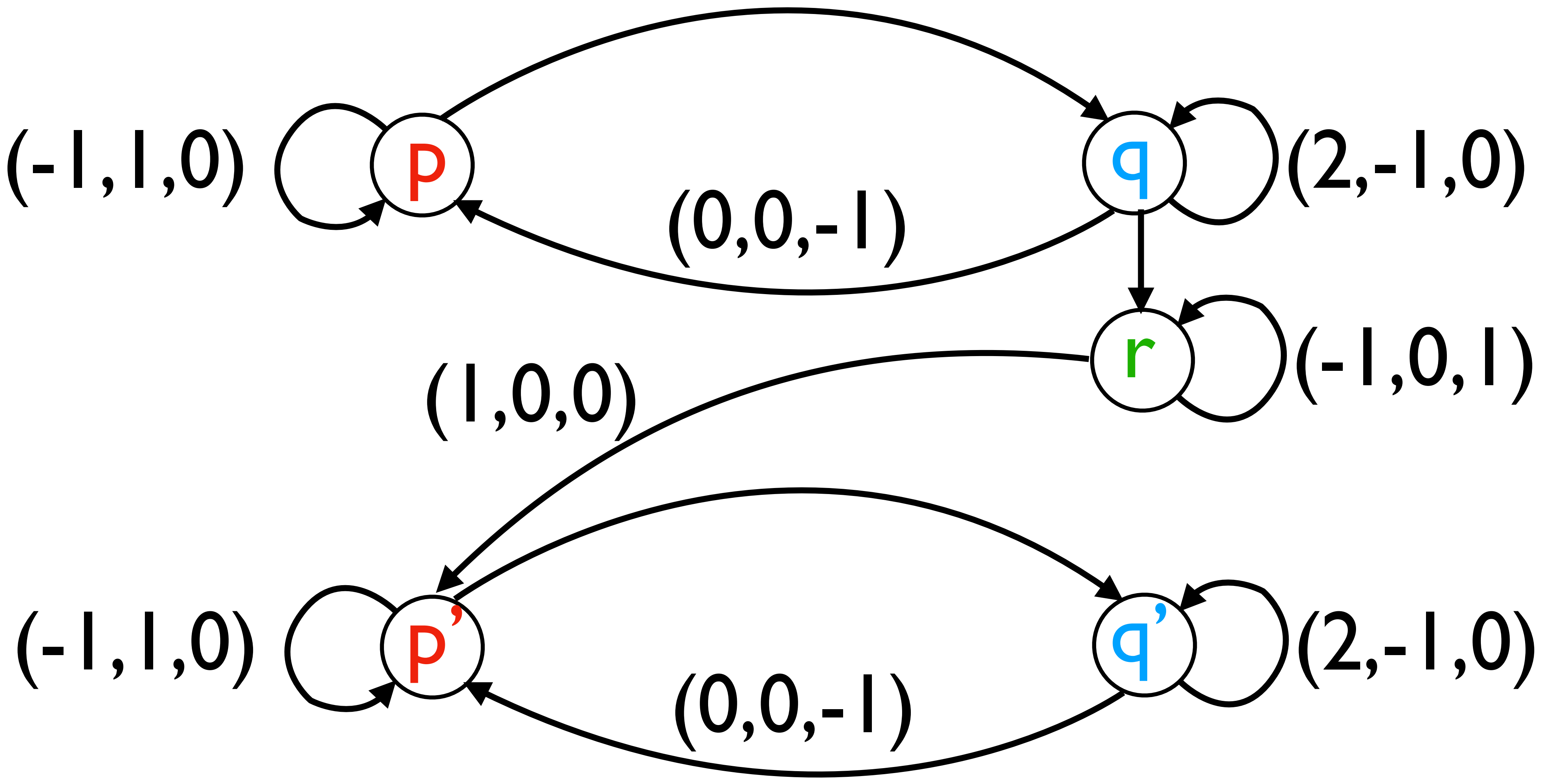

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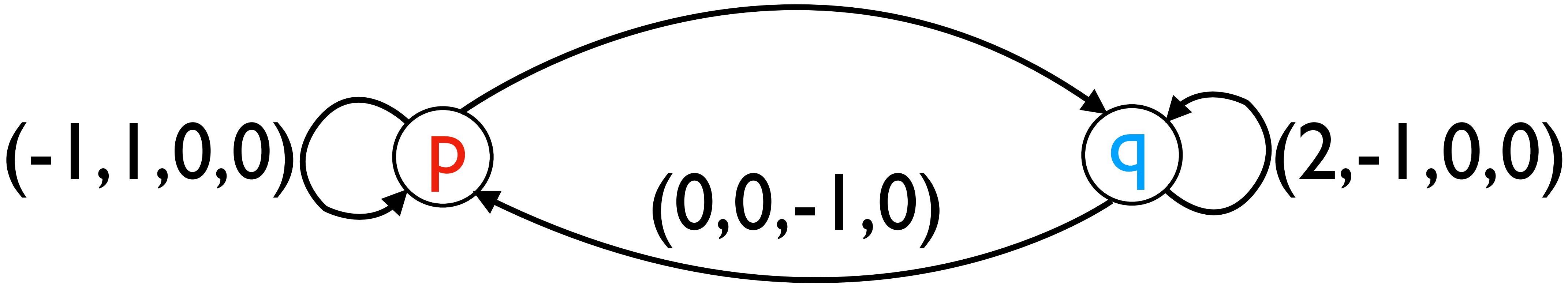
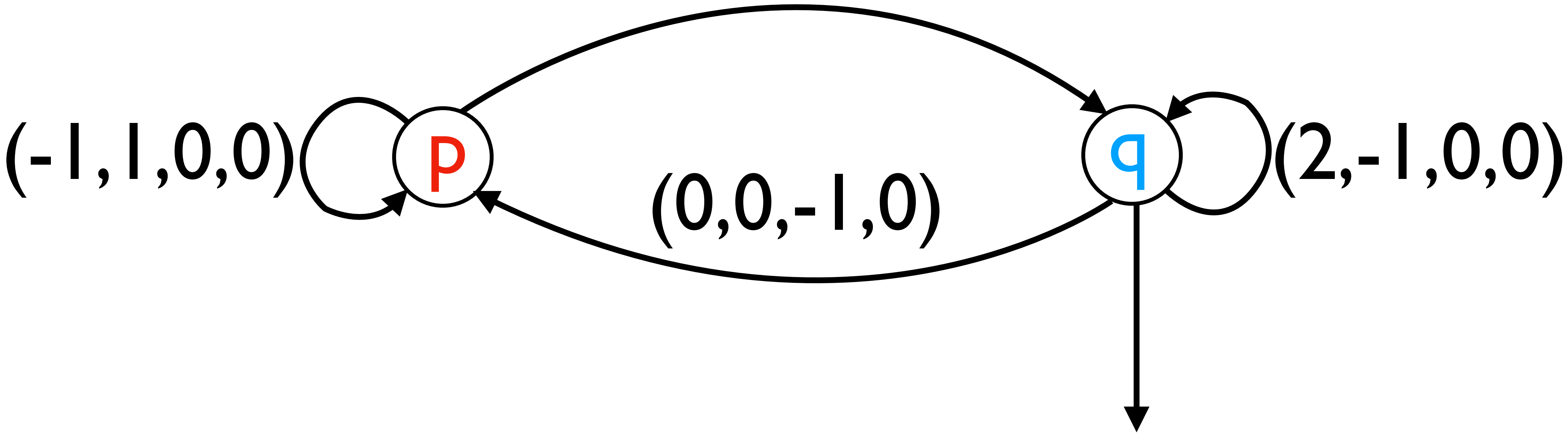
finite doubly-exponential reachability set

Hard examples

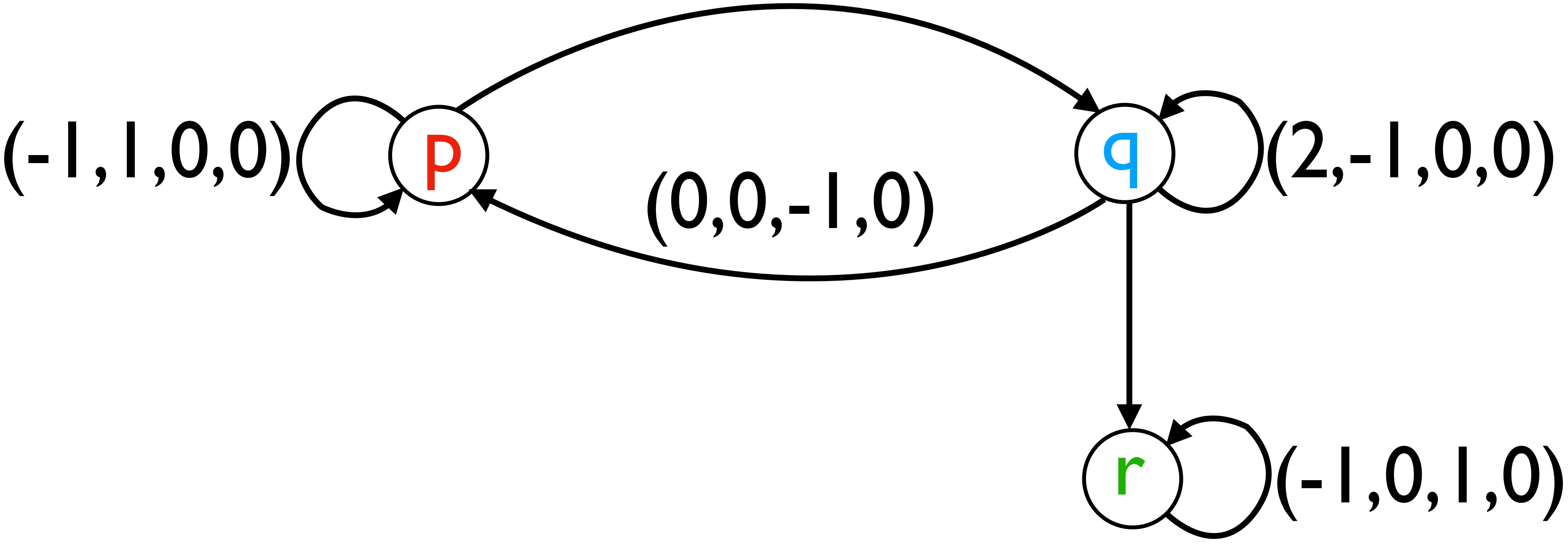
Hard examples



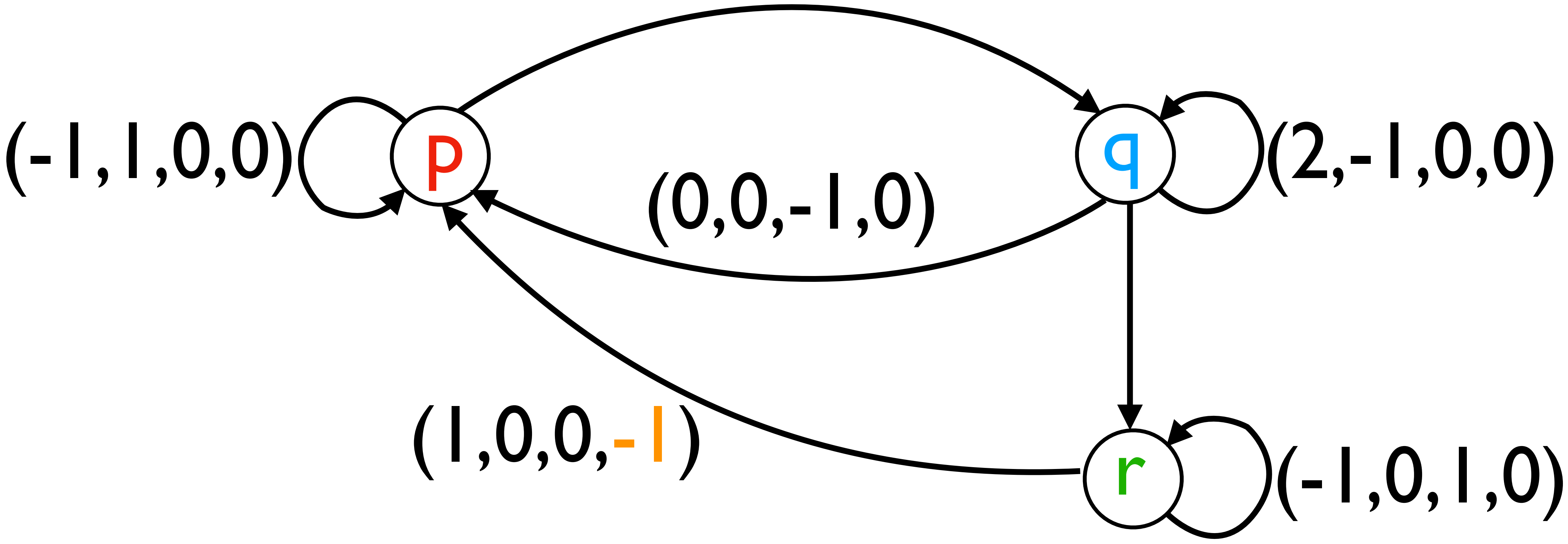
Hard examples



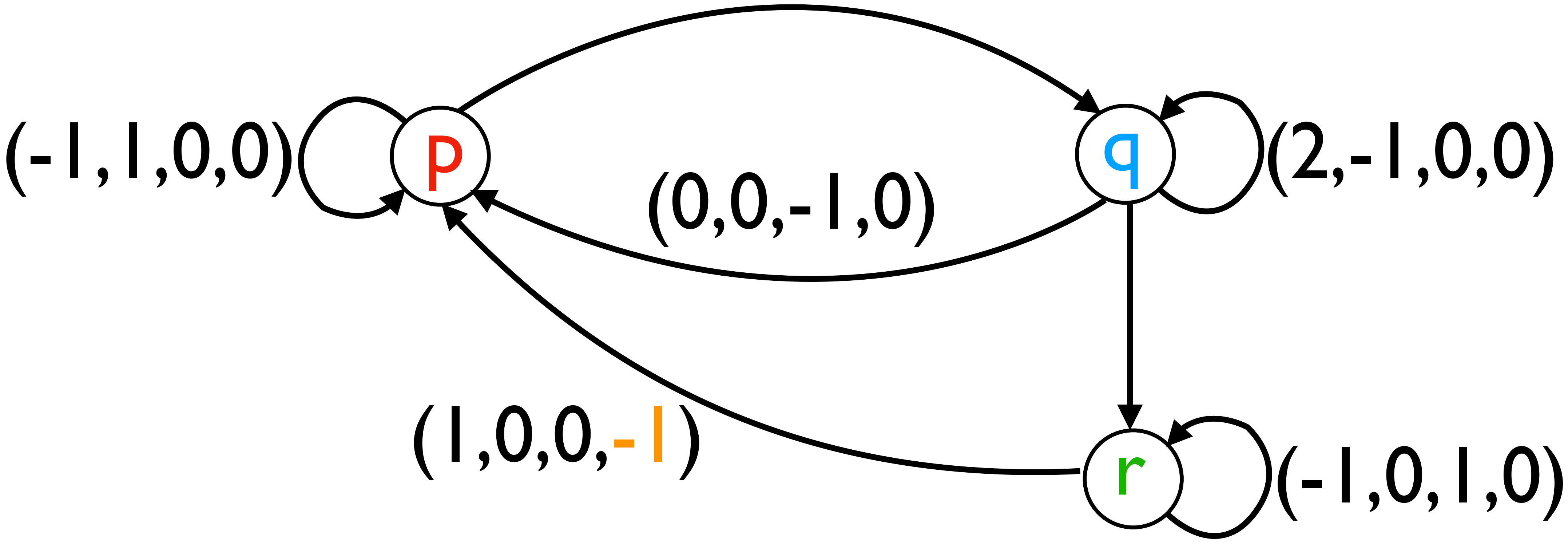
Hard examples



Hard examples

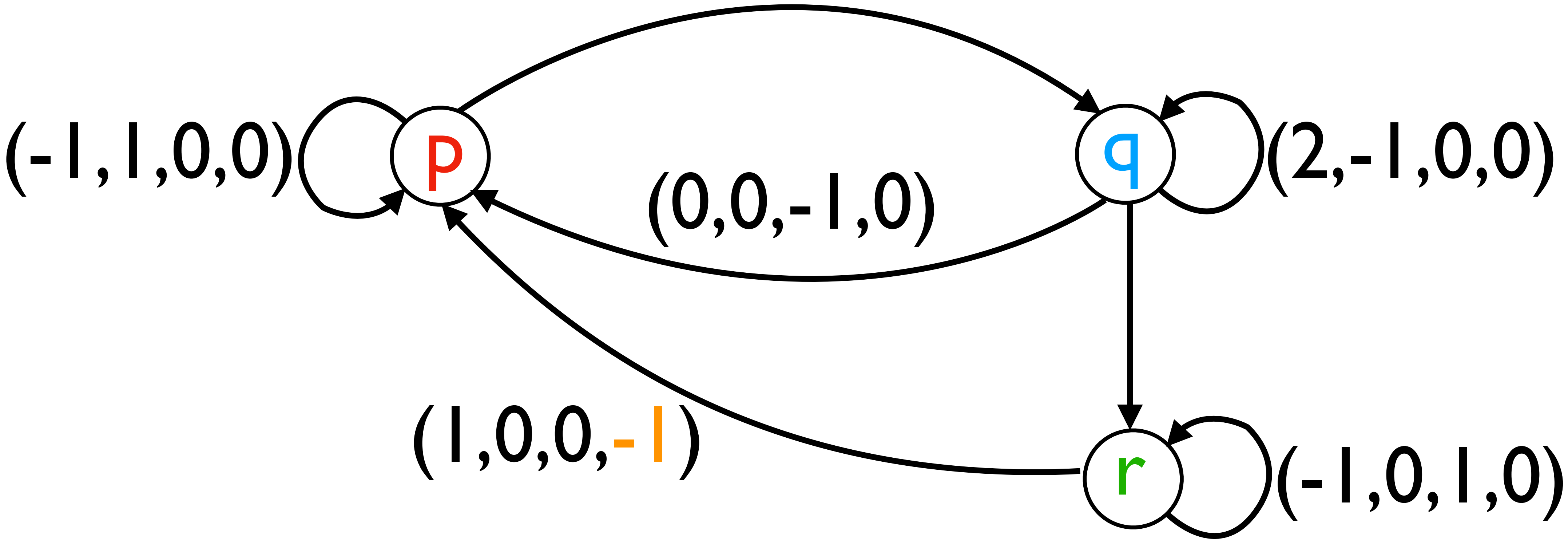


Hard examples



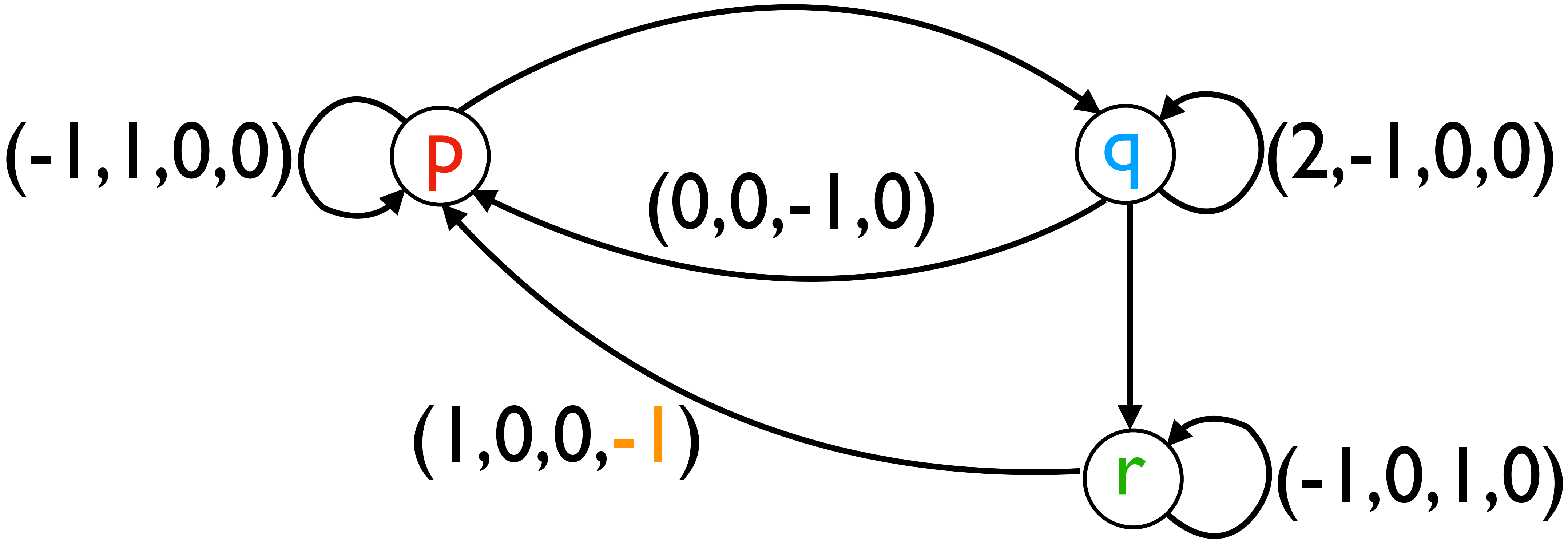
p(1,0,1,n)

Hard examples



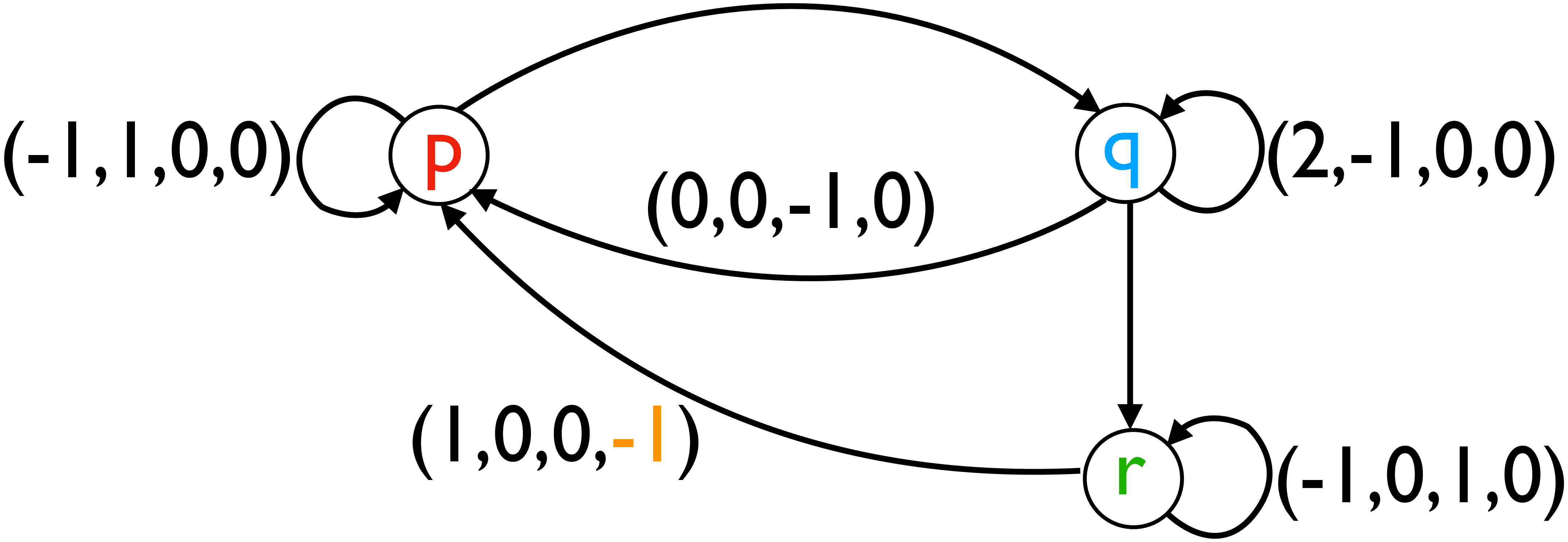
$$p(1,0,1,n) \longrightarrow p(1,0,2^1,n-1)$$

Hard examples



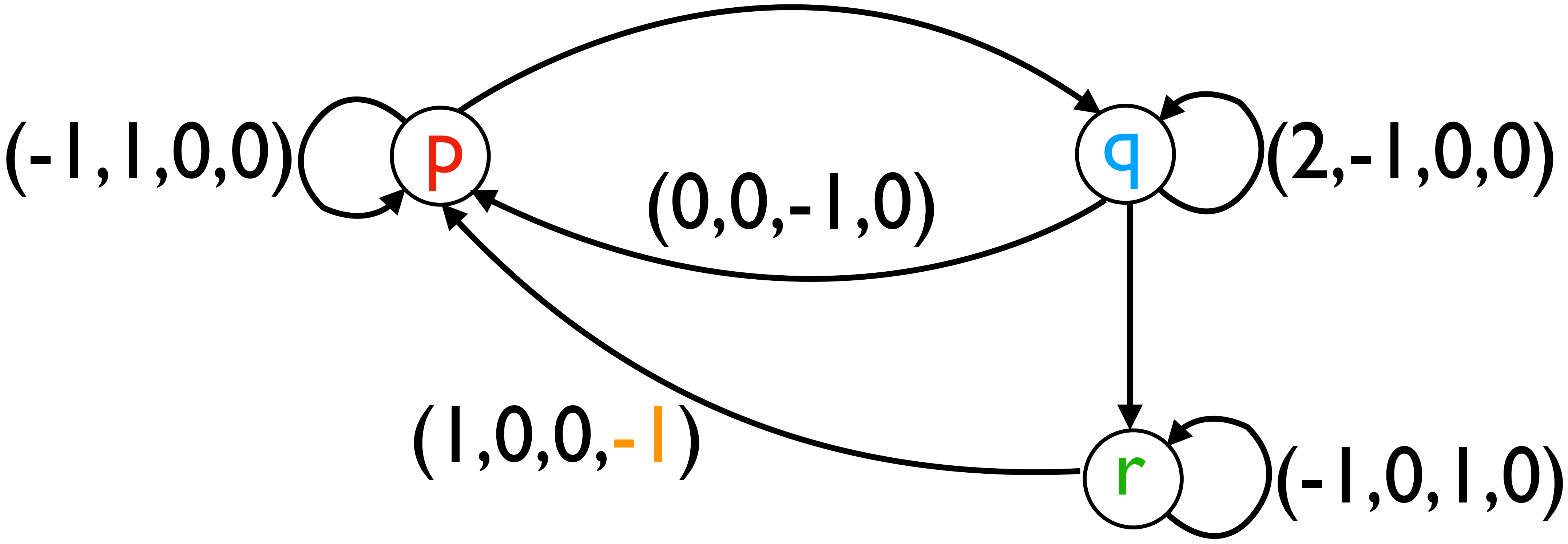
$p(1,0,1,n) \longrightarrow p(1,0,2^1,n-1) \ldots$

Hard examples



$p(1,0,1,n) \longrightarrow p(1,0,2^1,n-1) \ldots \longrightarrow p(1,0,\mathrm{Tower}(n),0)$

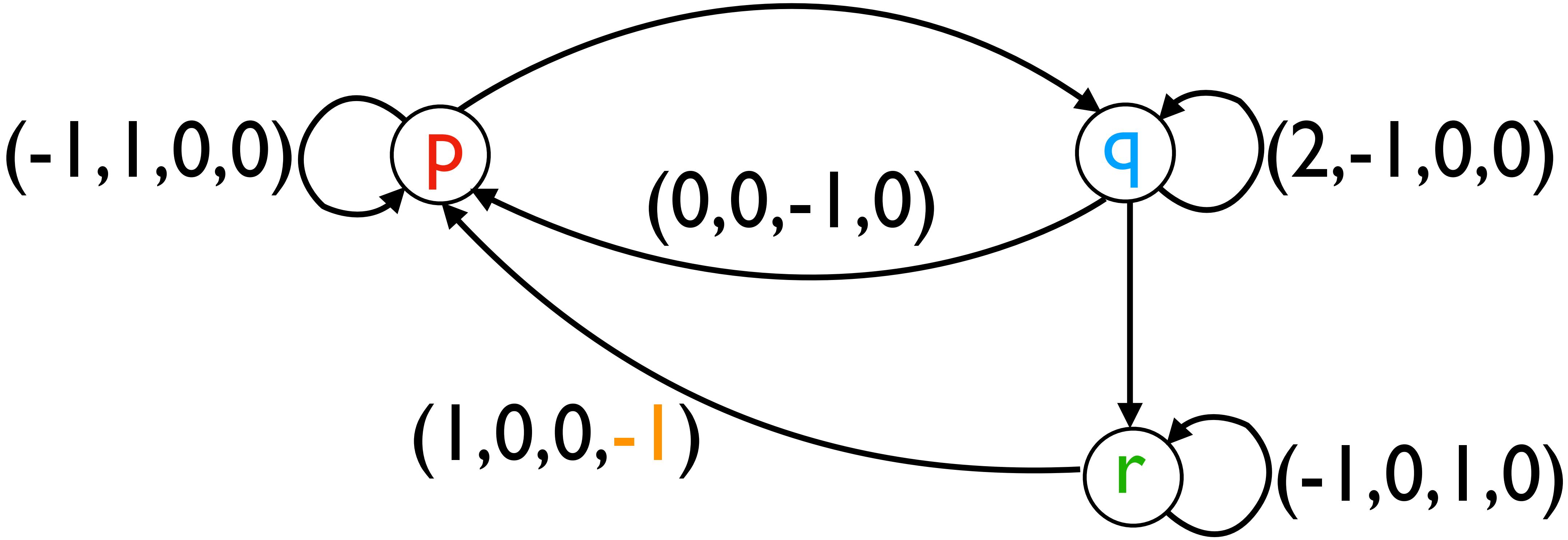
Hard examples



$$p(1,0,1,n) \longrightarrow p(1,0,2^1,n-1) \dots \longrightarrow p(1,0,\mathrm{Tower}(n),0)$$

finite tower-size reachability set

Hard examples



$$p(1,0,1,n) \longrightarrow p(1,0,2^1,n-1) \ldots \longrightarrow p(1,0,\mathrm{Tower}(n),0)$$

finite tower-size reachability set

finite F_d-size reachability set

Enforcing

Enforcing

$\mathbb{F}_d$-hardness implies no $\mathbb{F}_d$-short run

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Presented examples not sufficient

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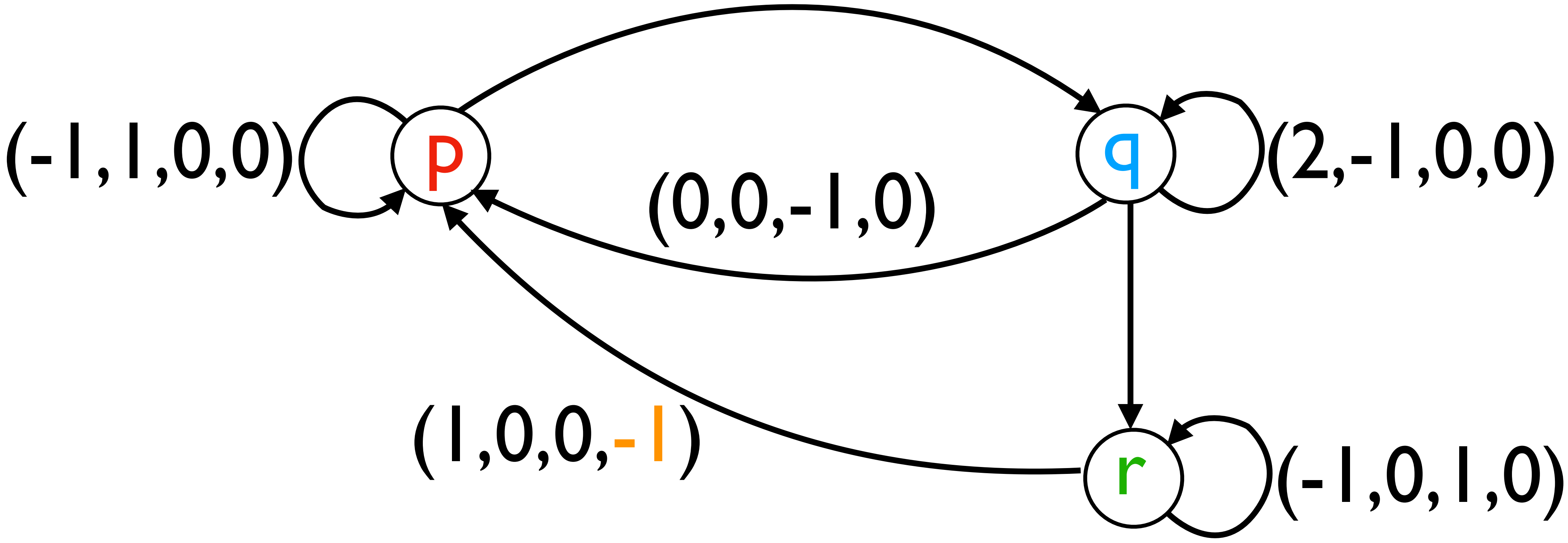
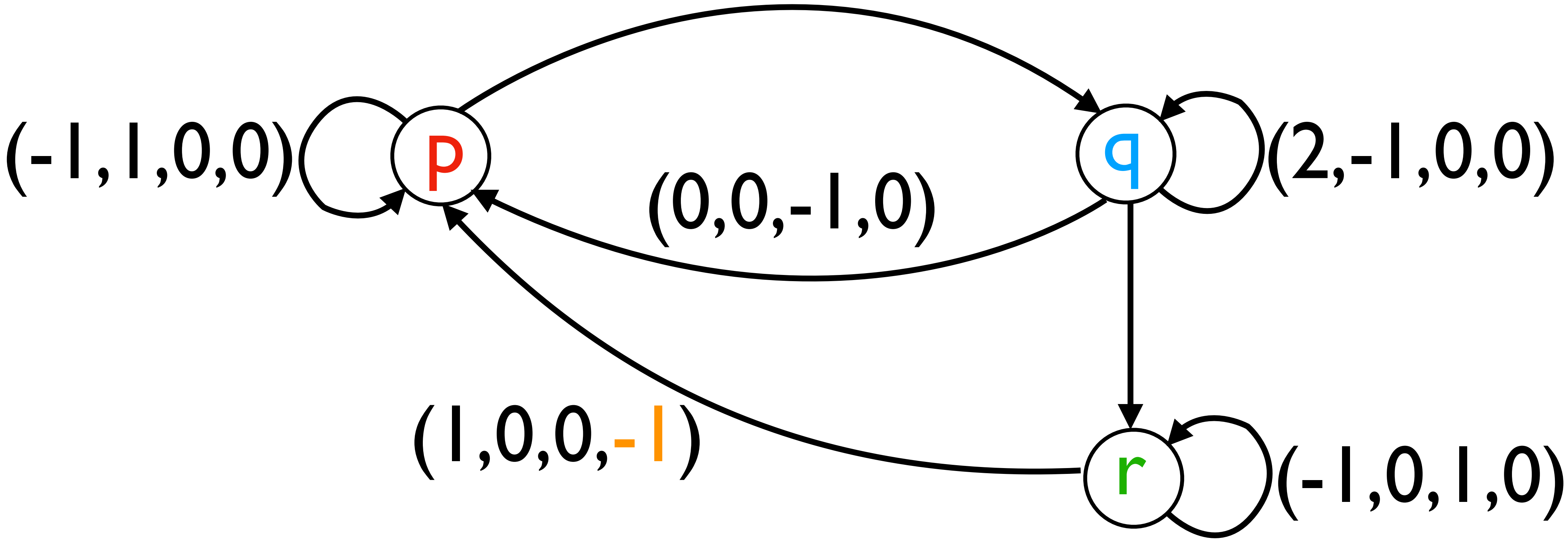


Enforcing

$\mathbb{F}_d$-hardness implies no $\mathbb{F}_d$-short run

Presented examples not sufficient

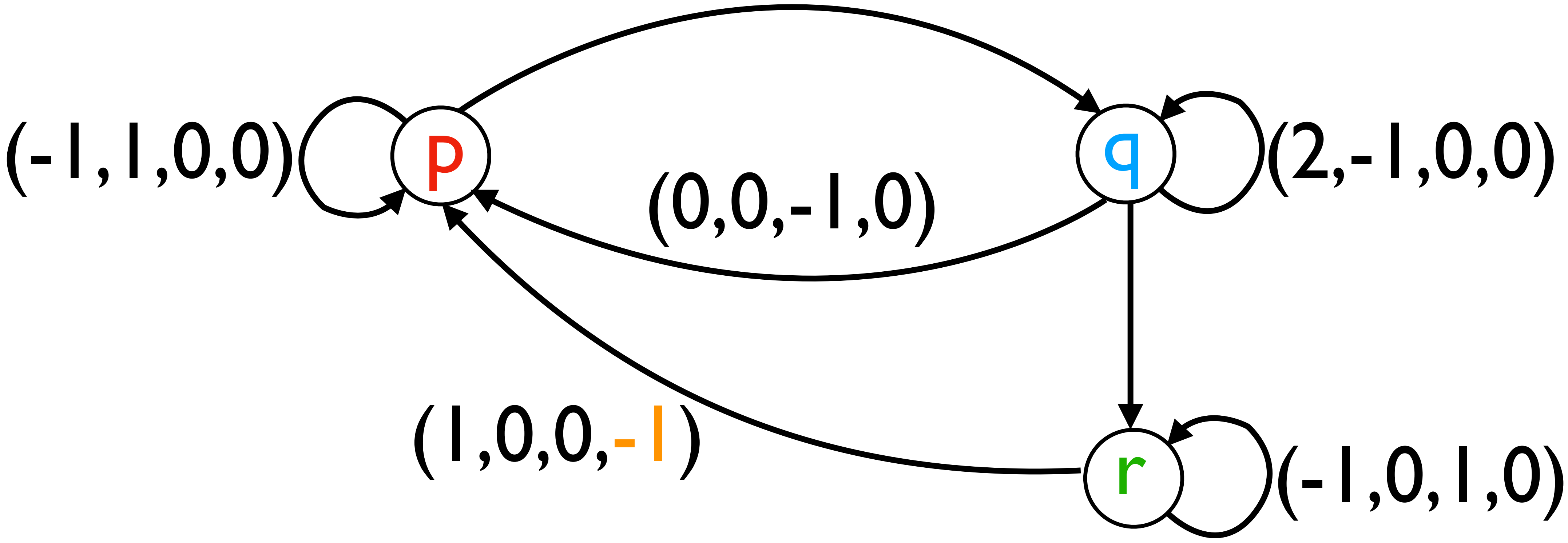


p(1,0,1,n)

Enforcing

$\mathbb{F}_d$-hardness implies no $\mathbb{F}_d$-short run

Presented examples not sufficient

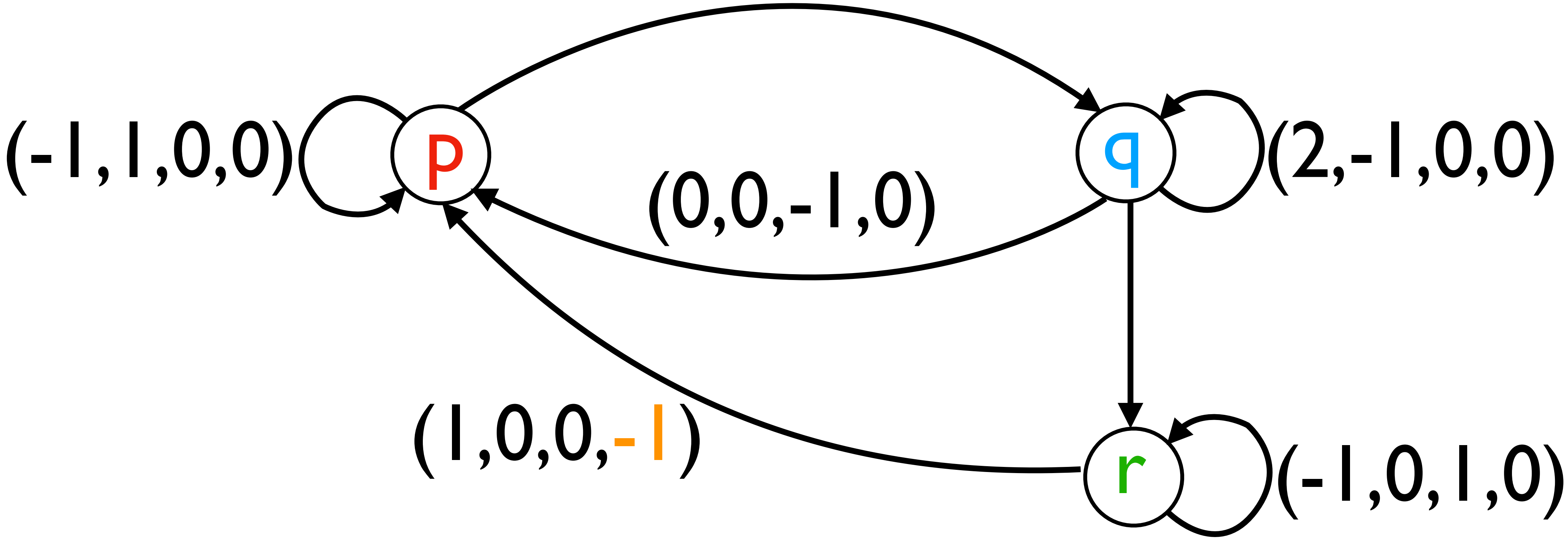


$p(1,0,1,n) \longrightarrow p(1,0,2^{1},n-1)$

Enforcing

$\mathbb{F}_d$-hardness implies no $\mathbb{F}_d$-short run

Presented examples not sufficient



$p(1,0,1,n) \longrightarrow p(1,0,2^1,n-1) \ldots$

Enforcing

$\mathbb{F}_d$-hardness implies no $\mathbb{F}_d$-short run

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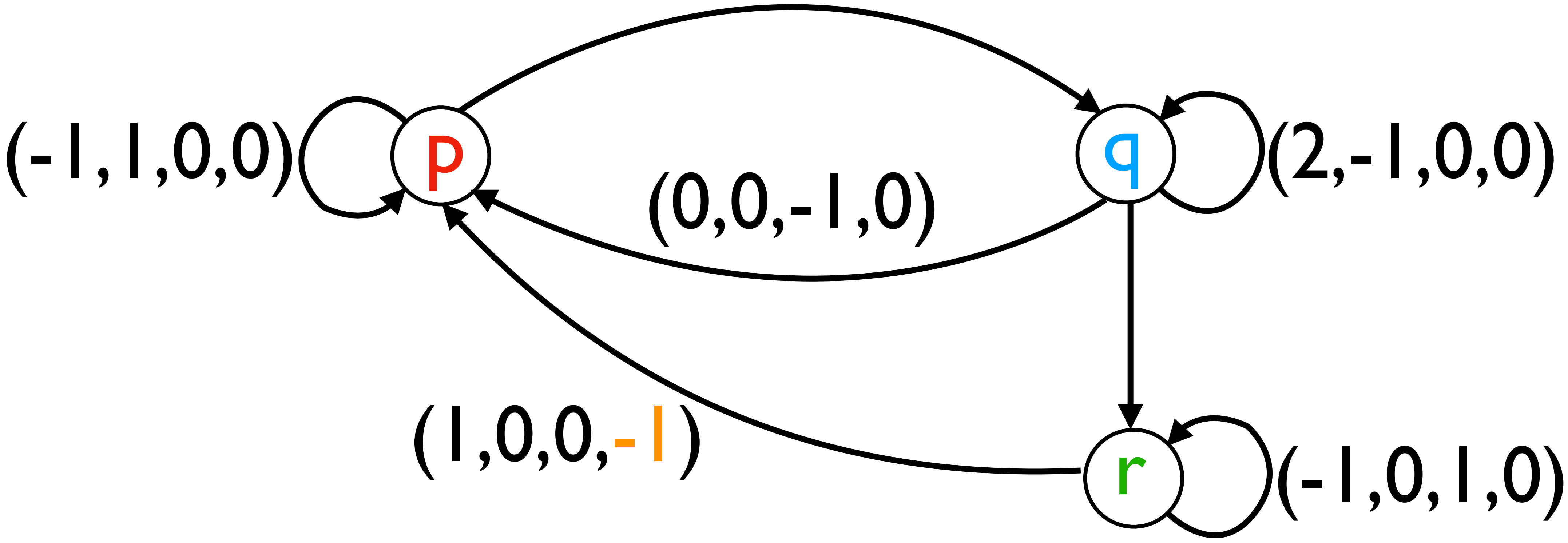

$$p(1,0,1,n) \longrightarrow p(1,0,2^1,n-1) \ldots \longrightarrow p(1,0,\mathrm{Tower}(n),0)$$

Enforcing

$\mathbb{F}_d$-hardness implies no $\mathbb{F}_d$-short run

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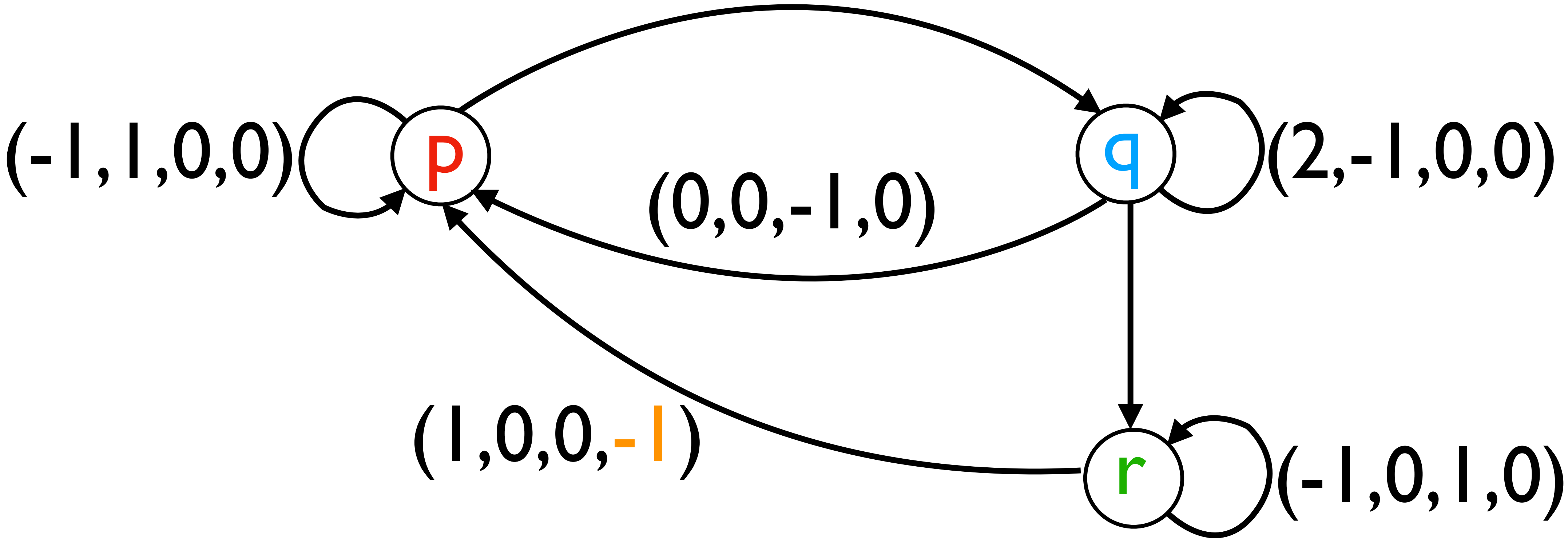


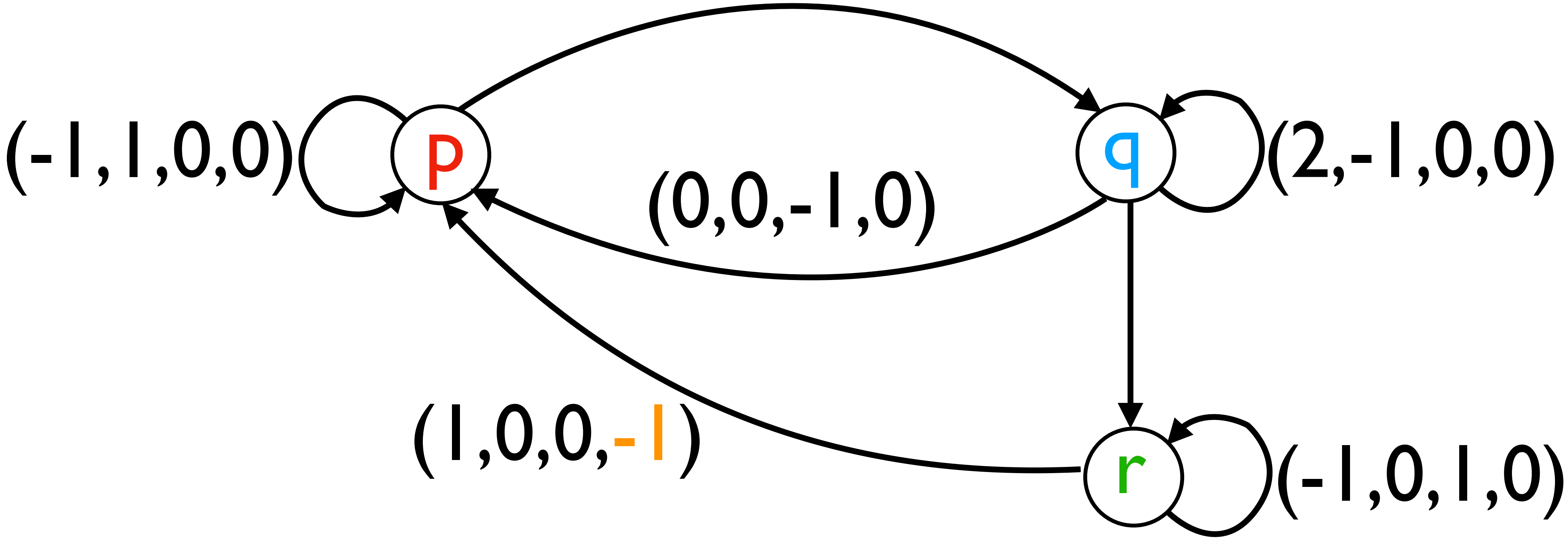
$p(1,0,1,n) \longrightarrow p(1,0,2^1,n-1) \ldots \longrightarrow p(1,0,\text{Tower}(n),0)$

Need enforcing techniques!

Enforcing

$\mathbb{F}_d$-hardness implies no $\mathbb{F}_d$-short run

Presented examples not sufficient

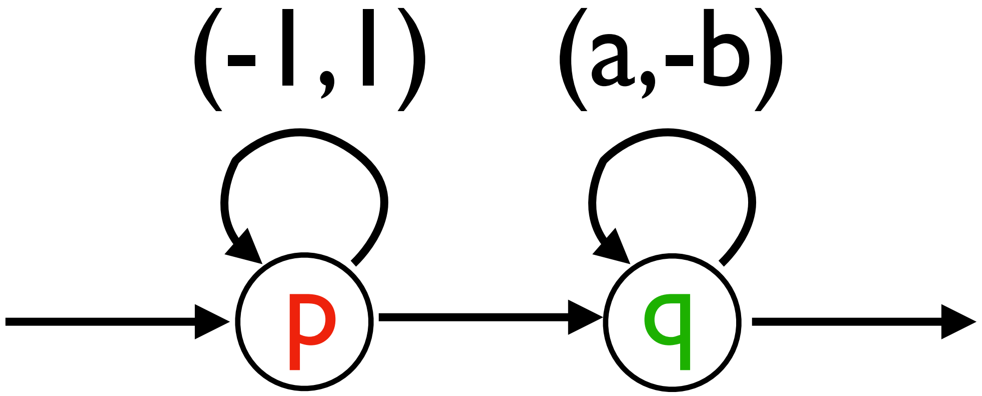


$p(1,0,1,n) \longrightarrow p(1,0,2^1,n-1) \ldots \longrightarrow p(1,0,\text{Tower}(n),0)$

Need enforcing techniques! Consider small dimensions!

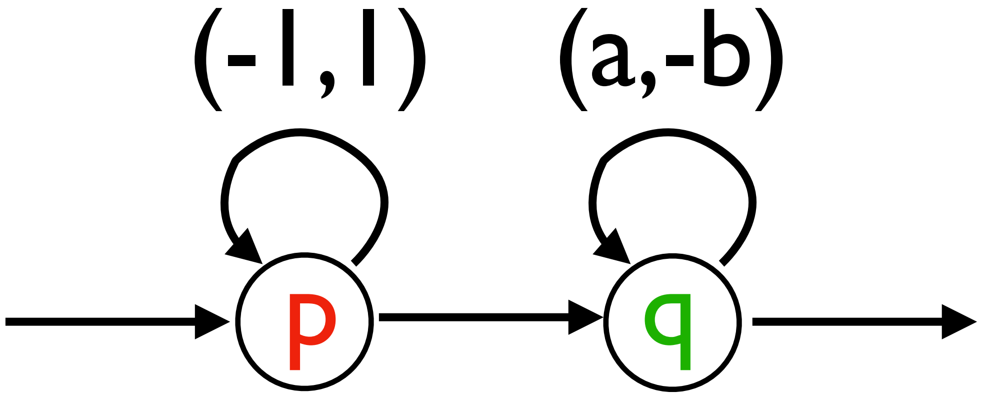
Telescopic equations

Telescopic equations

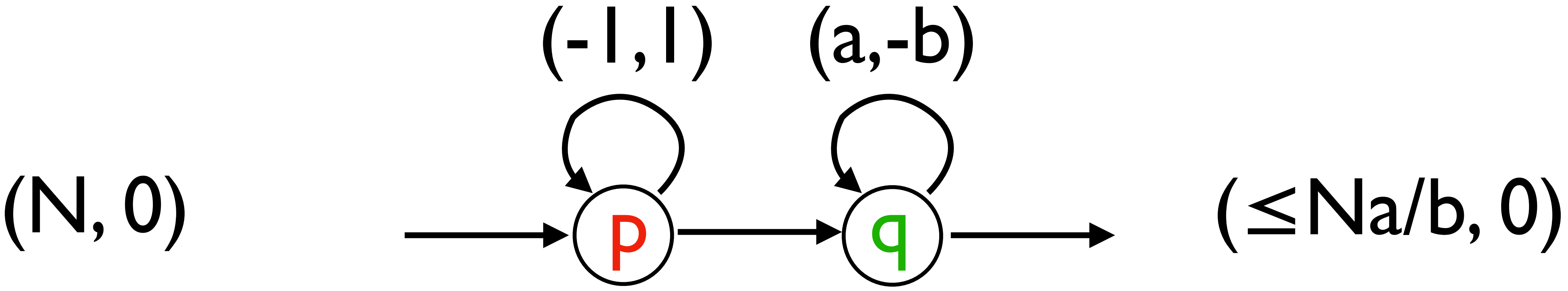


Telescopic equations

(N, 0)



Telescopic equations



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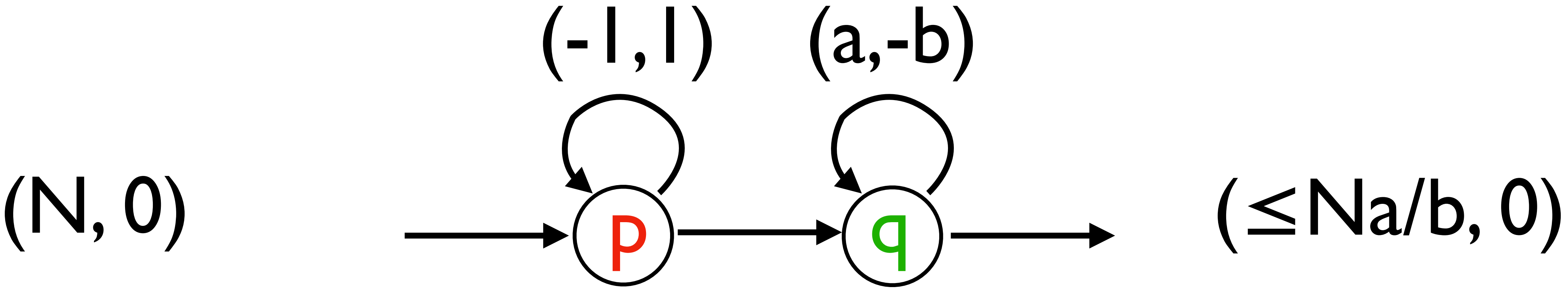
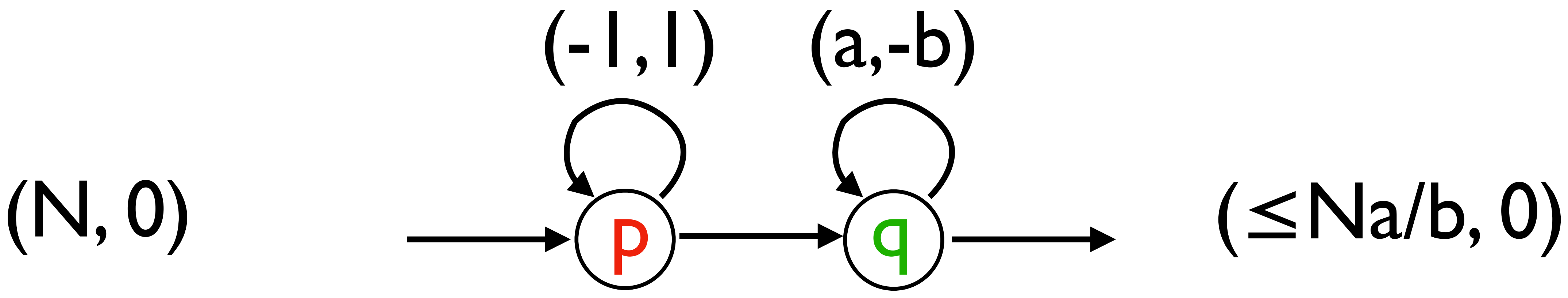


$$k/k\text{-}1 \cdot k\text{-}1/k\text{-}2 \cdot \ldots \cdot 3/2 \cdot 2/1 = k/1$$

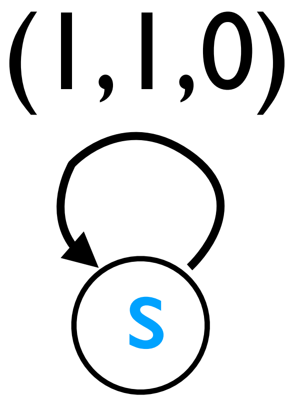
Telescopic equations



(-1,1) (a,-b)

(N, 0) p q ($\leq$Na/b, 0)

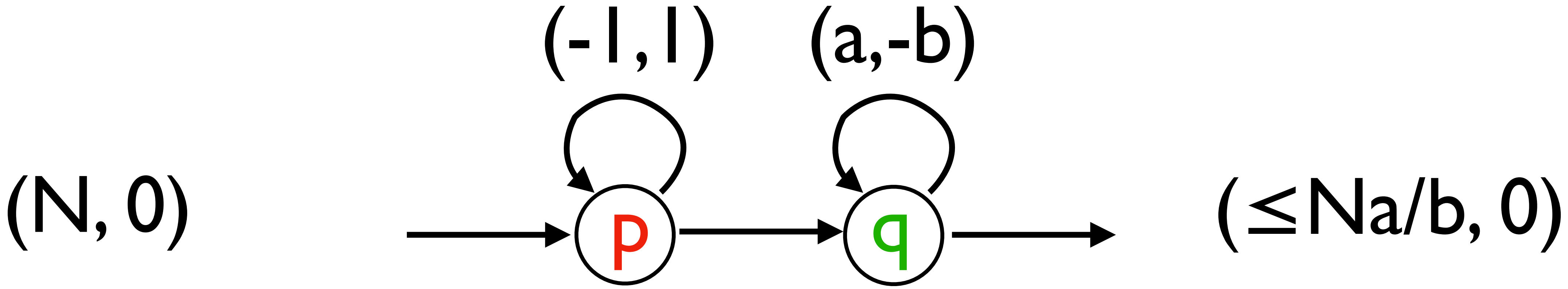
$$k/k-1 \cdot k-1/k-2 \cdot \ldots \cdot 3/2 \cdot 2/1 = k/1$$



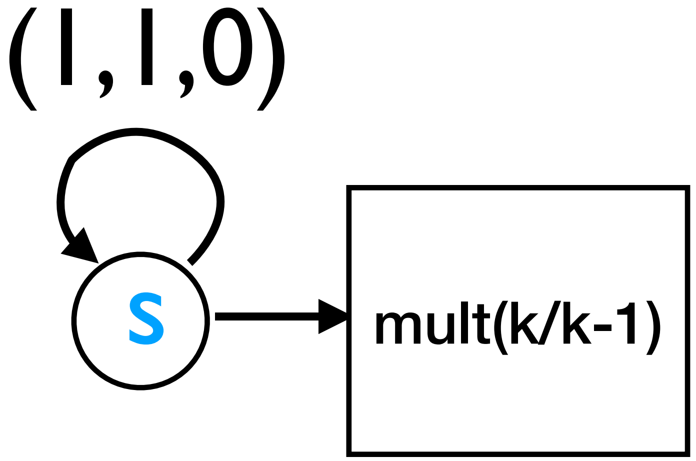
(1,1,0)

s

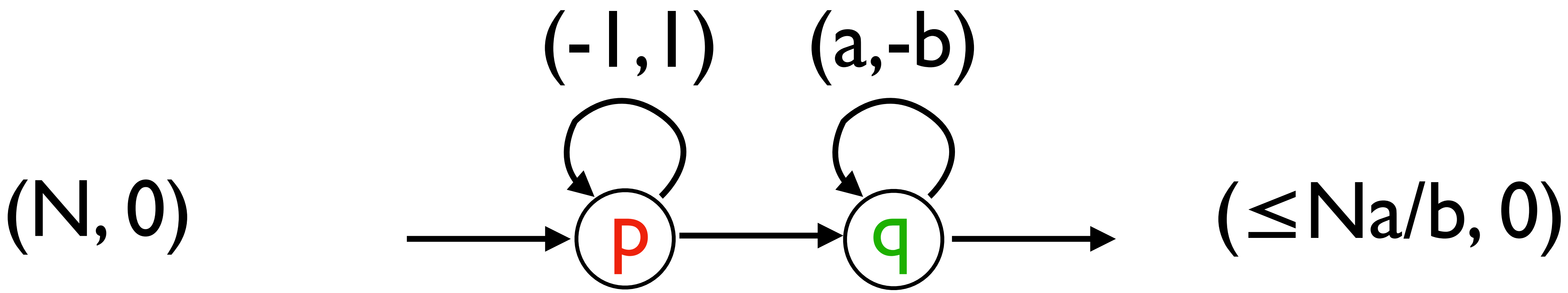
Telescopic equations



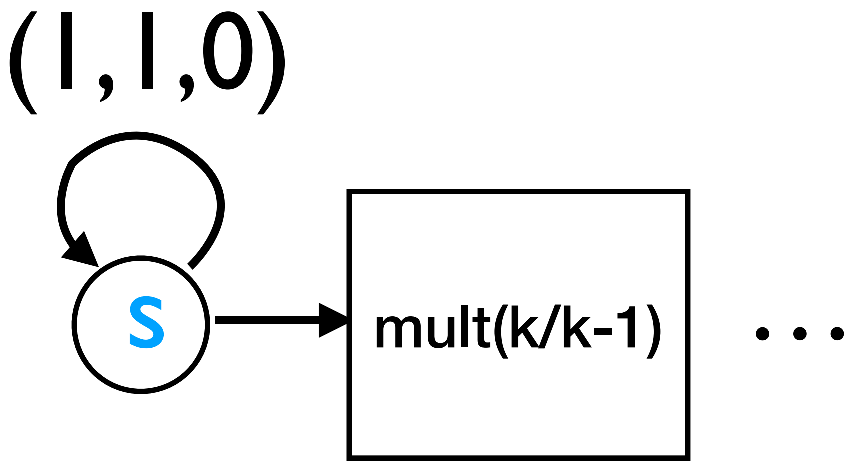
$$k/k-1 \cdot k-1/k-2 \cdot \ldots \cdot 3/2 \cdot 2/1 = k/1$$



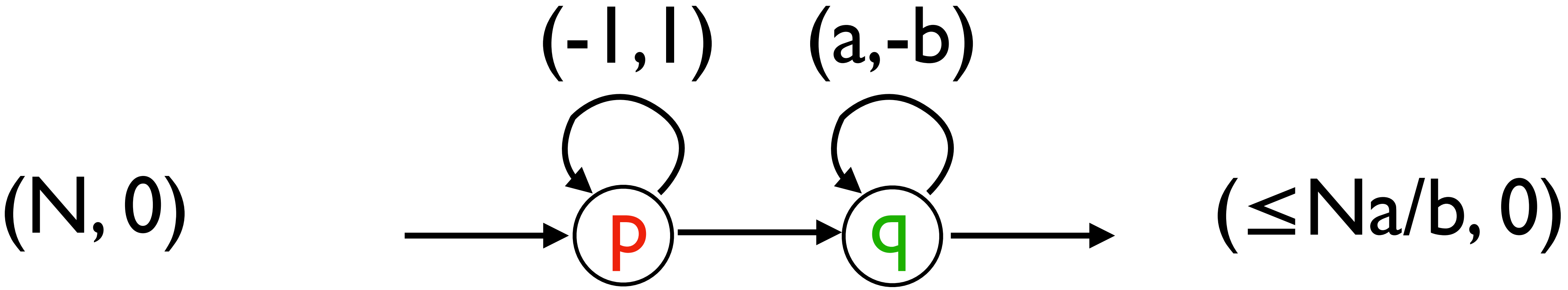
Telescopic equations



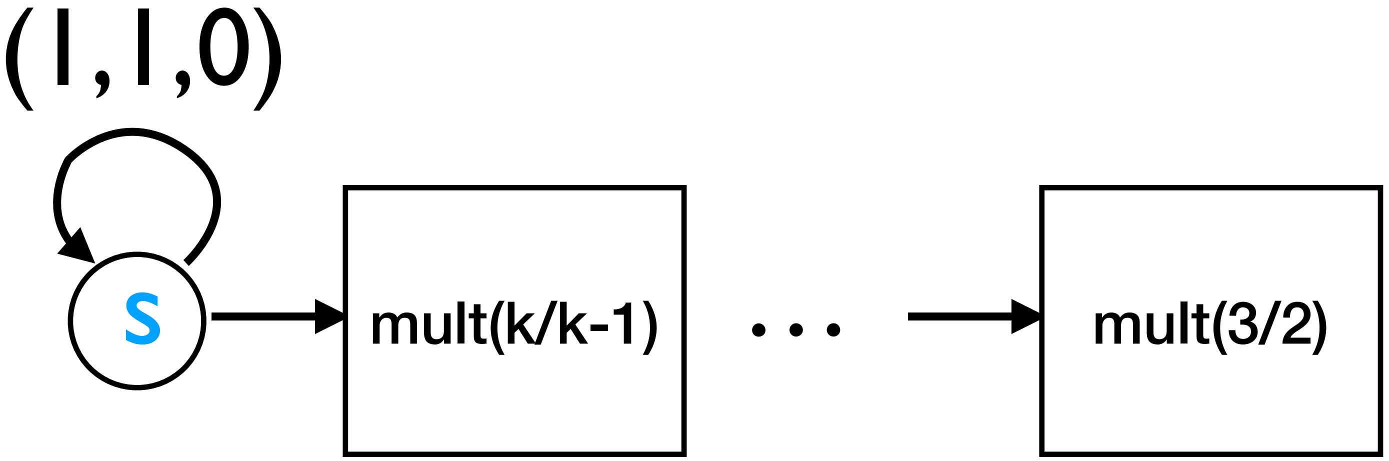
$$k/k-1 \cdot k-1/k-2 \cdot \ldots \cdot 3/2 \cdot 2/1 = k/1$$



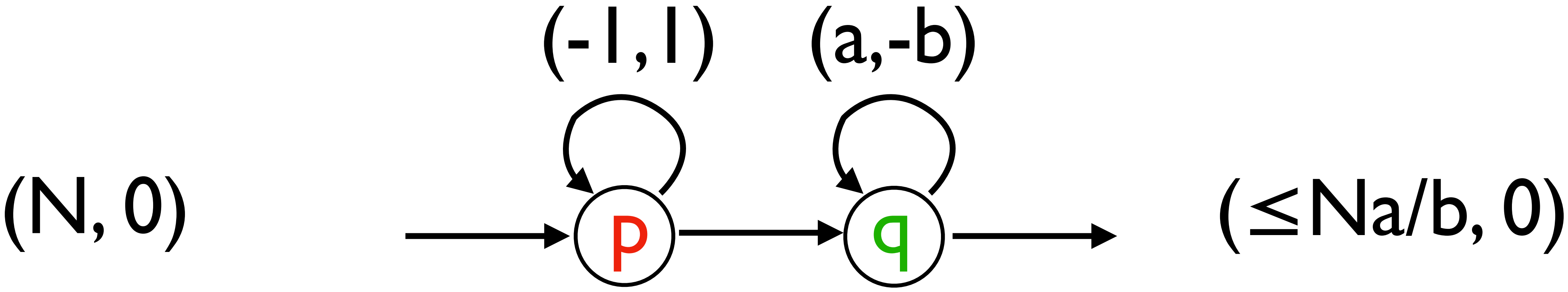
Telescopic equations



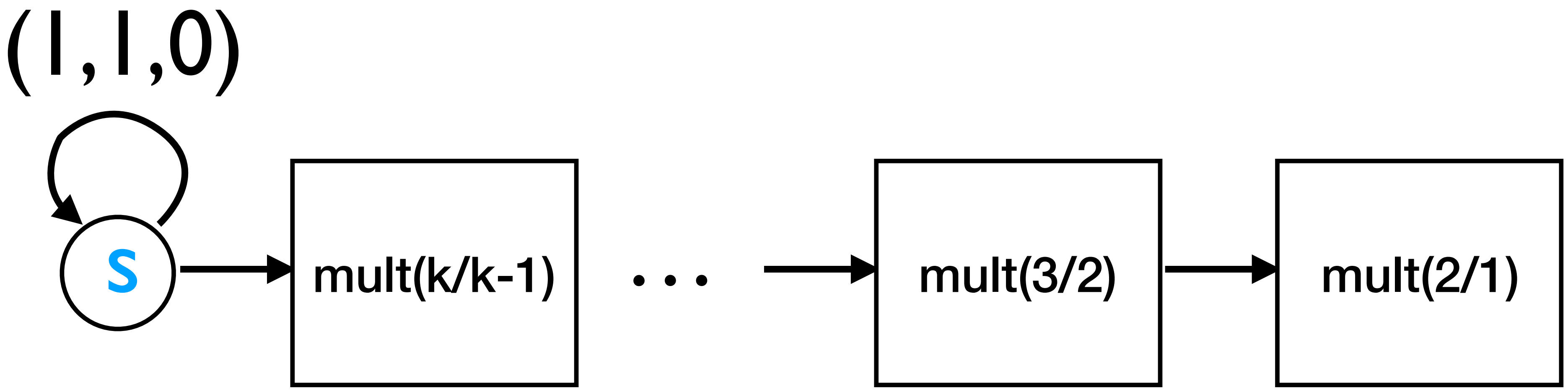
k/k-1 · k-1/k-2 · … · 3/2 · 2/1 = k/1



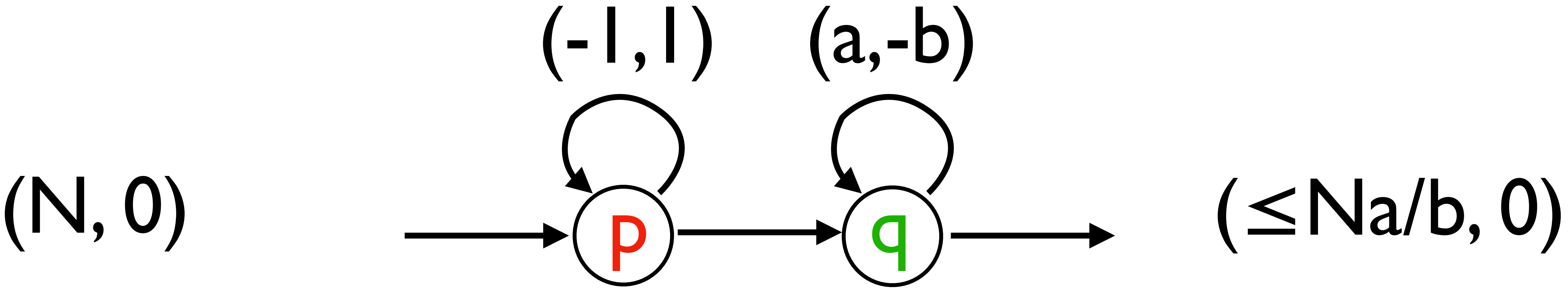
Telescopic equations



$$k/k-1 \cdot k-1/k-2 \cdot \ldots \cdot 3/2 \cdot 2/1 = k/1$$



Telescopic equations



(-1,1) (a,-b)

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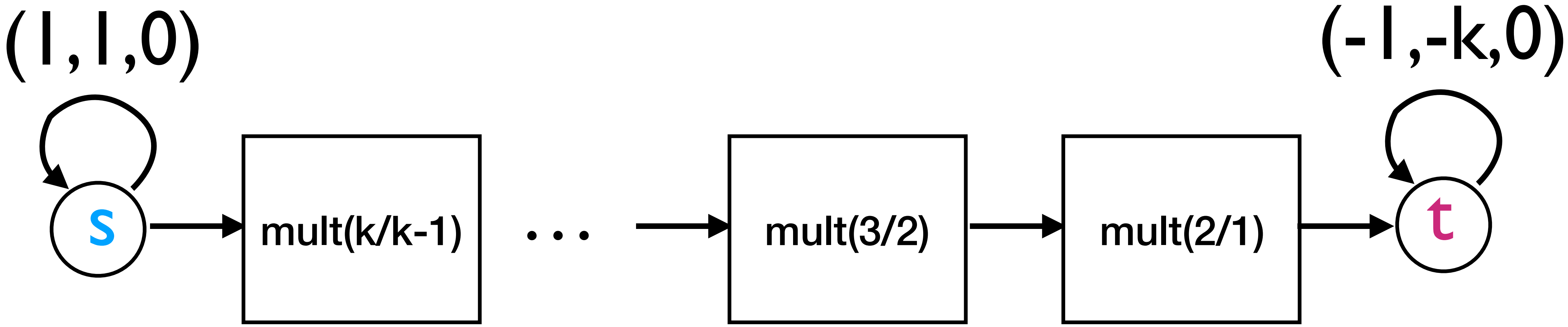
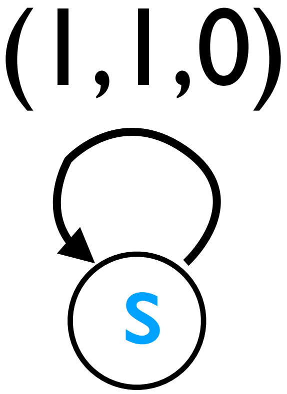


(1,1,0) (-1,-k,0)

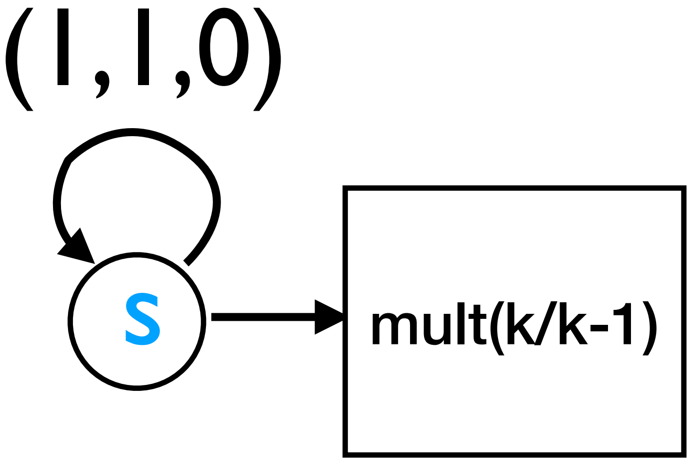
s mult(k/k-1) . . . mult(3/2) mult(2/1) t

Telescopic equations

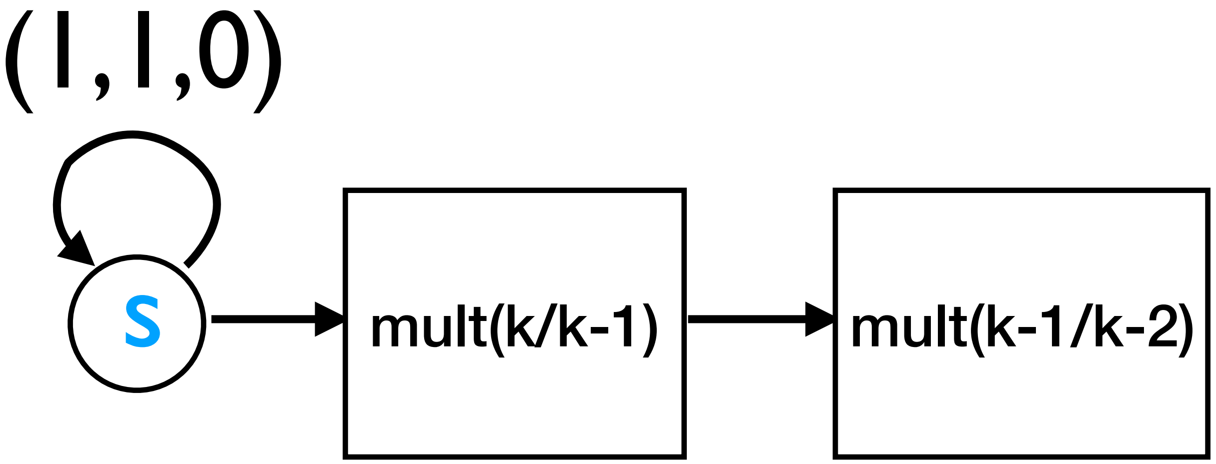
Telescopic equations



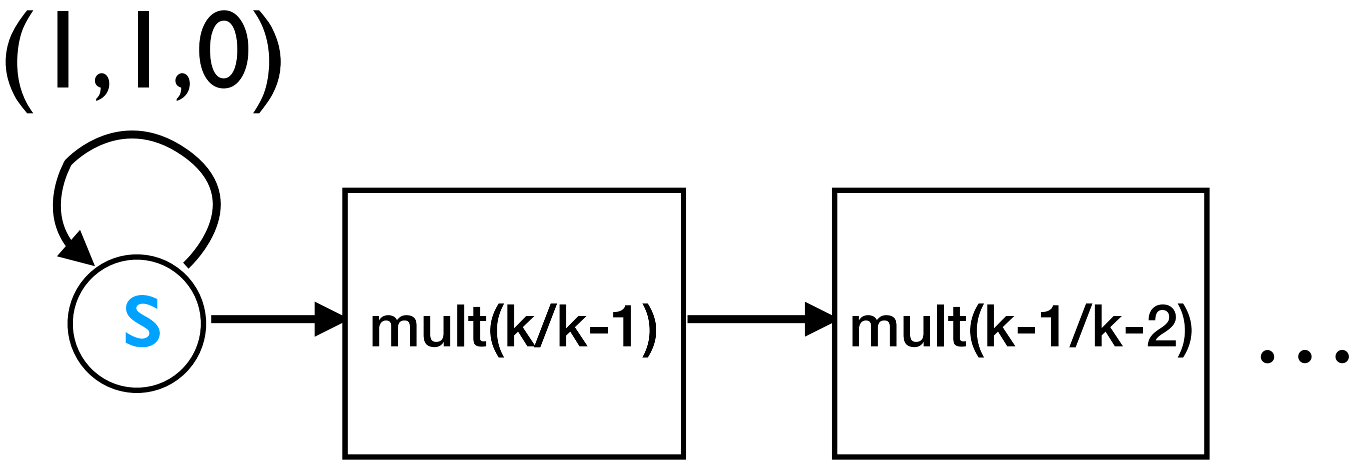
Telescopic equations



Telescopic equations



Telescopic equations



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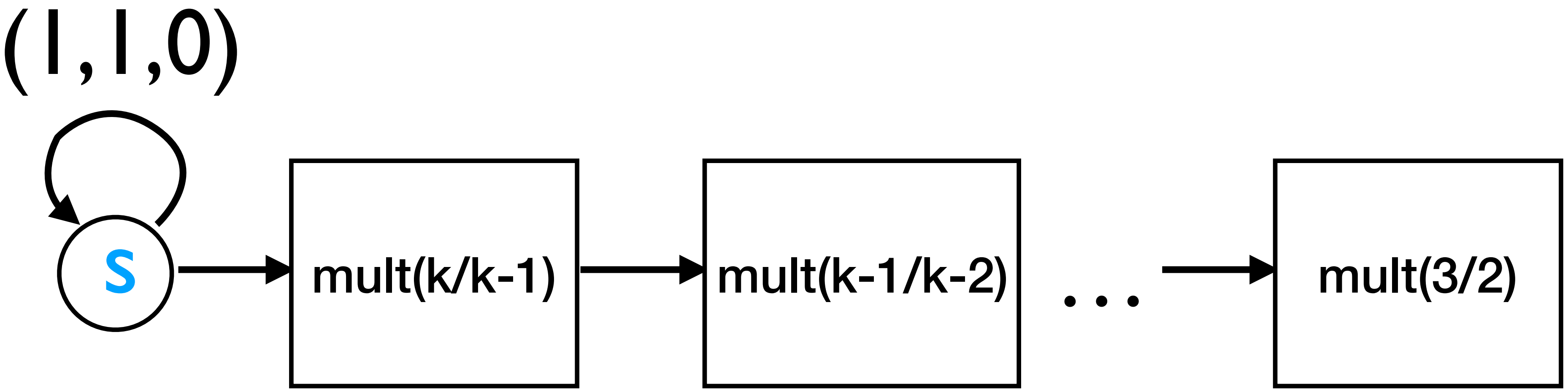
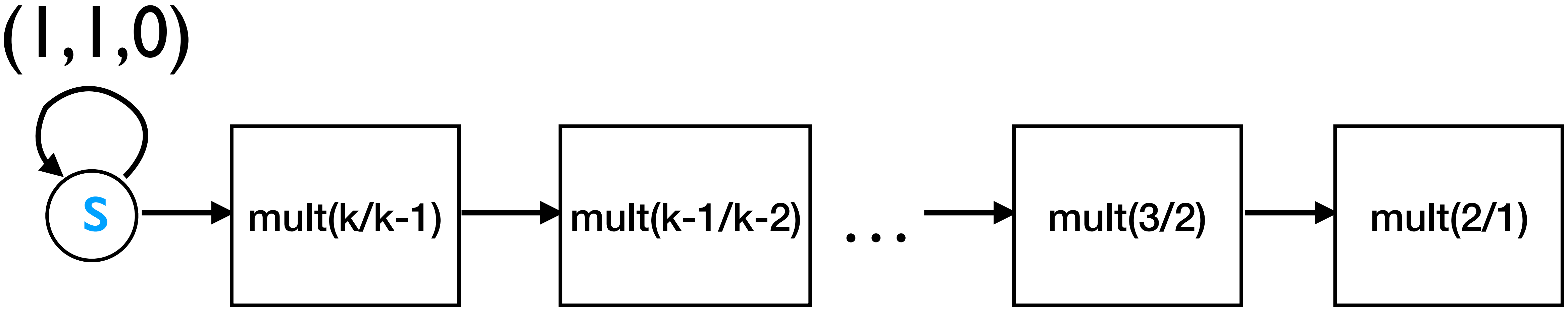
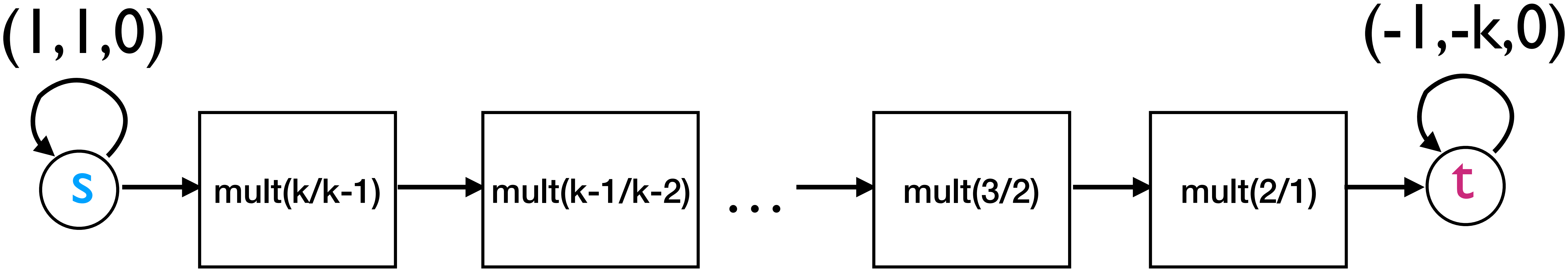


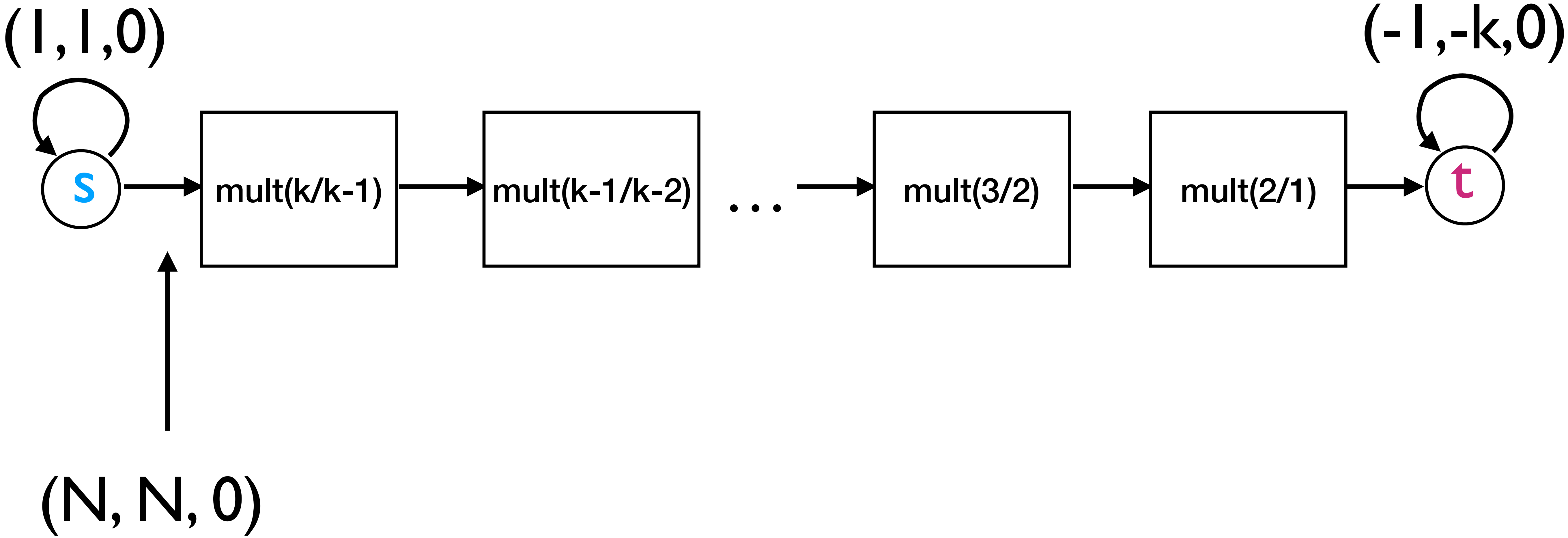
Telescopic equations



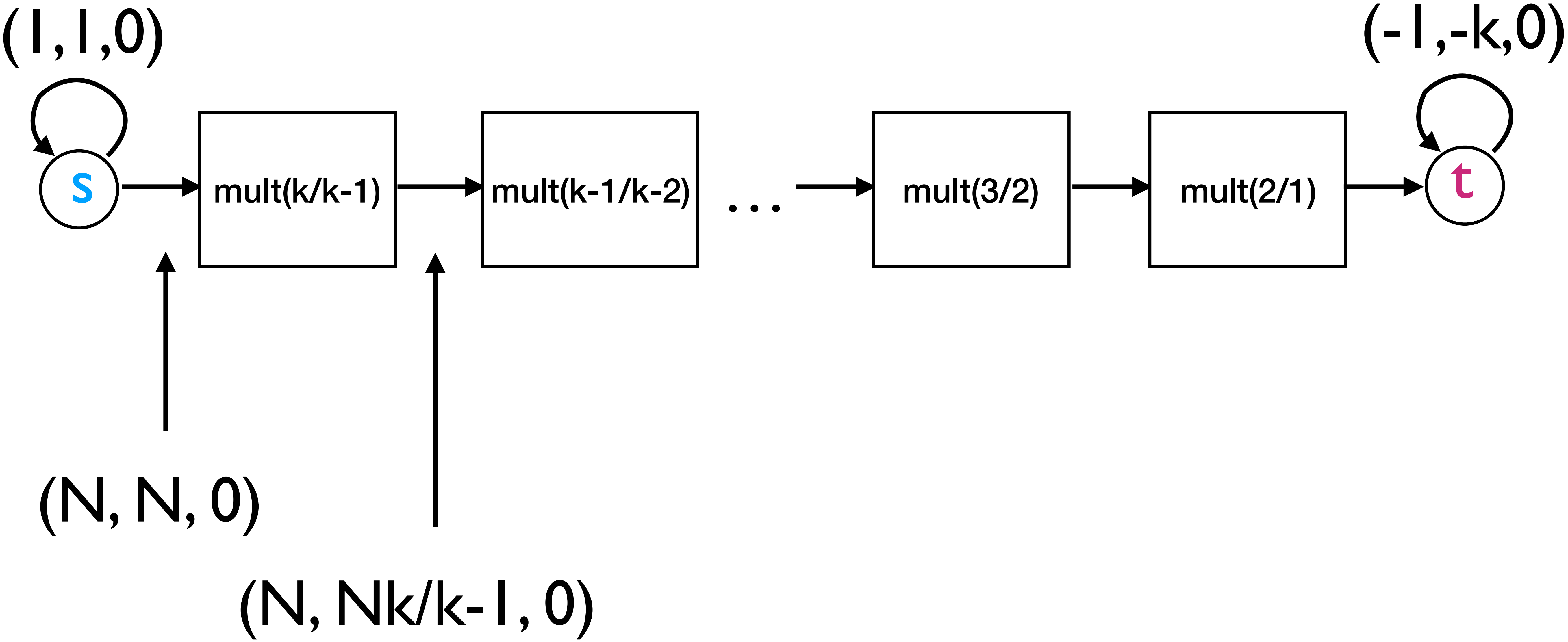
Telescopic equations



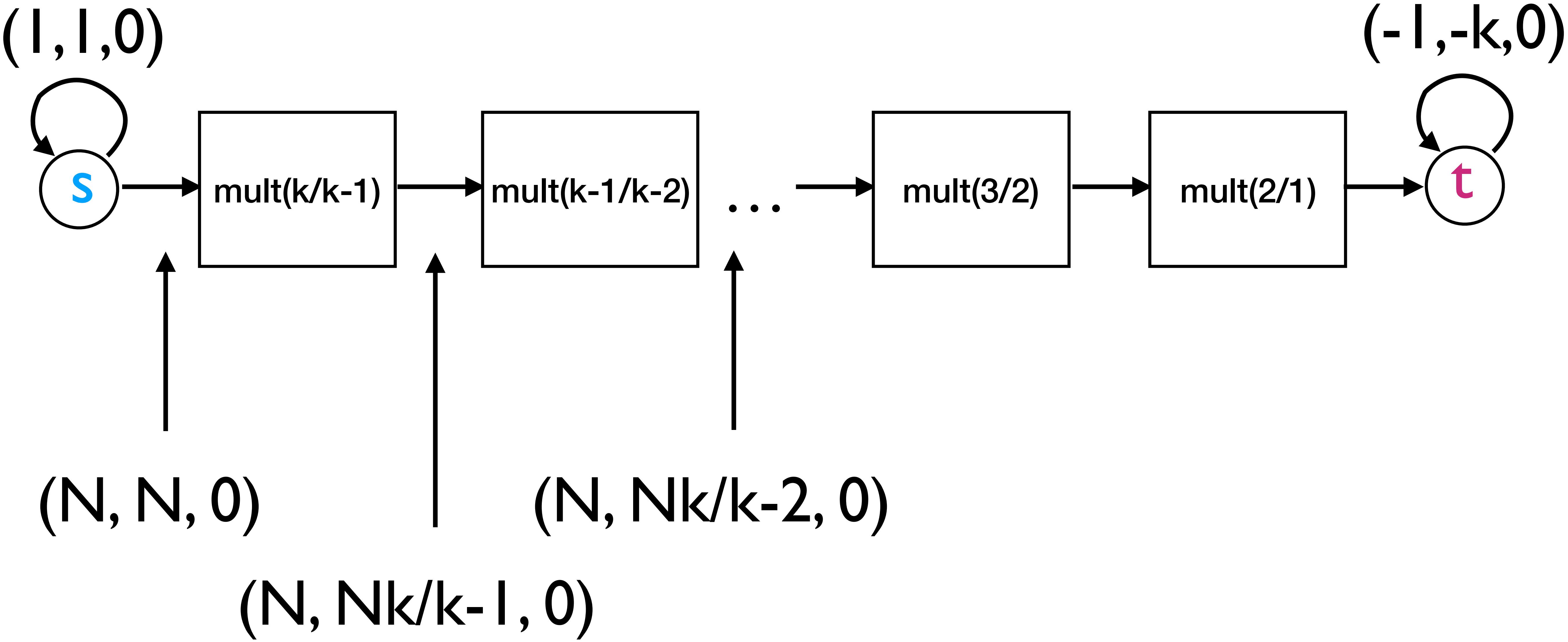
Telescopic equations



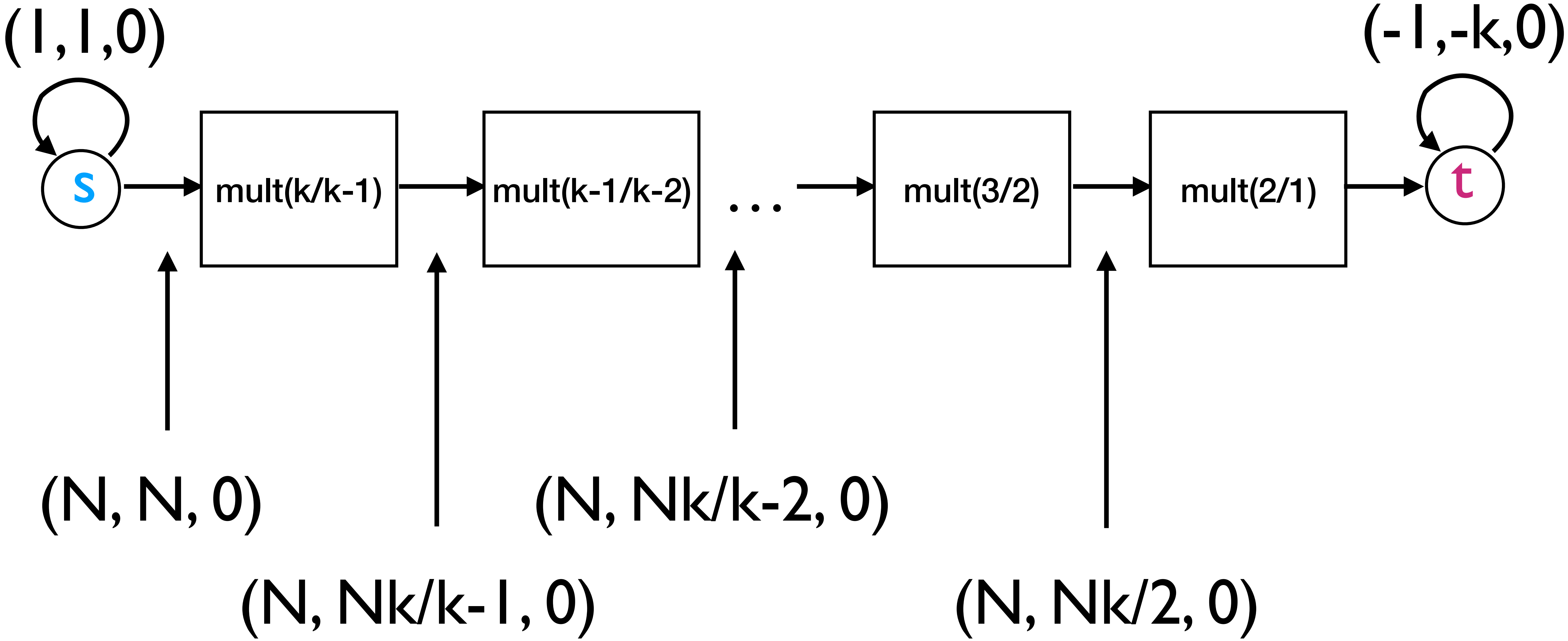
Telescopic equations



Telescopic equations



Telescopic equations



(1,1,0)

(-1,-k,0)

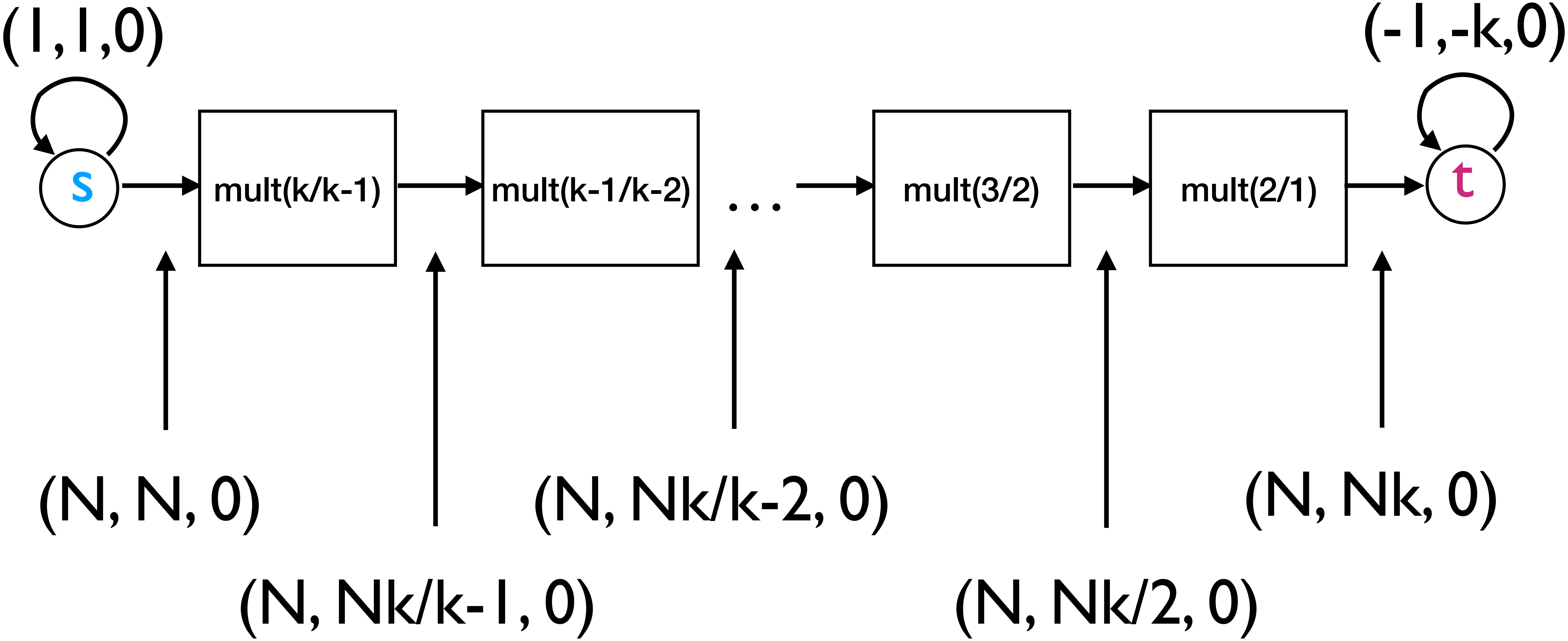
(N, N, 0)

(N, Nk/k-2, 0)

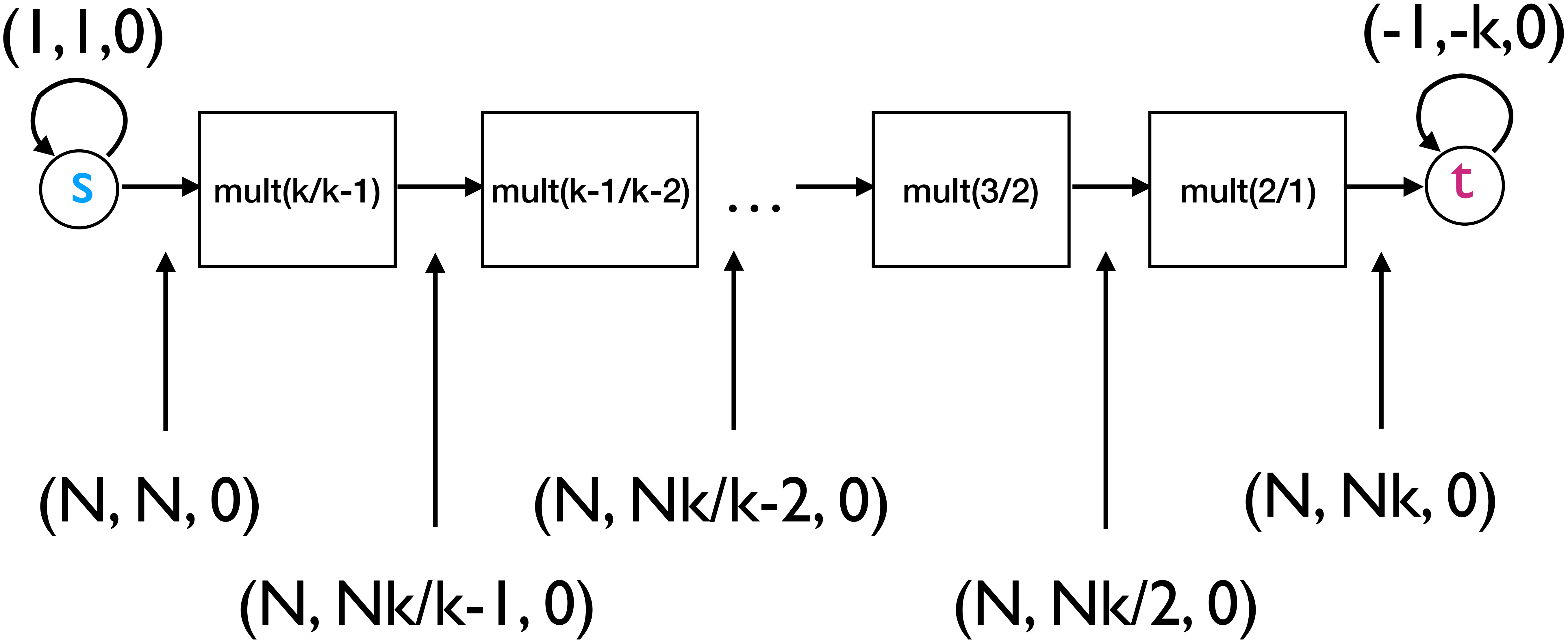
(N, Nk/k-1, 0)

(N, Nk/2, 0)

Telescopic equations



Telescopic equations



(1,1,0)

(-1,-k,0)

s

mult(k/k-1)

mult(k-1/k-2)

. . .

mult(3/2)

mult(2/1)

t

(N, N, 0)

(N, Nk/k-1, 0)

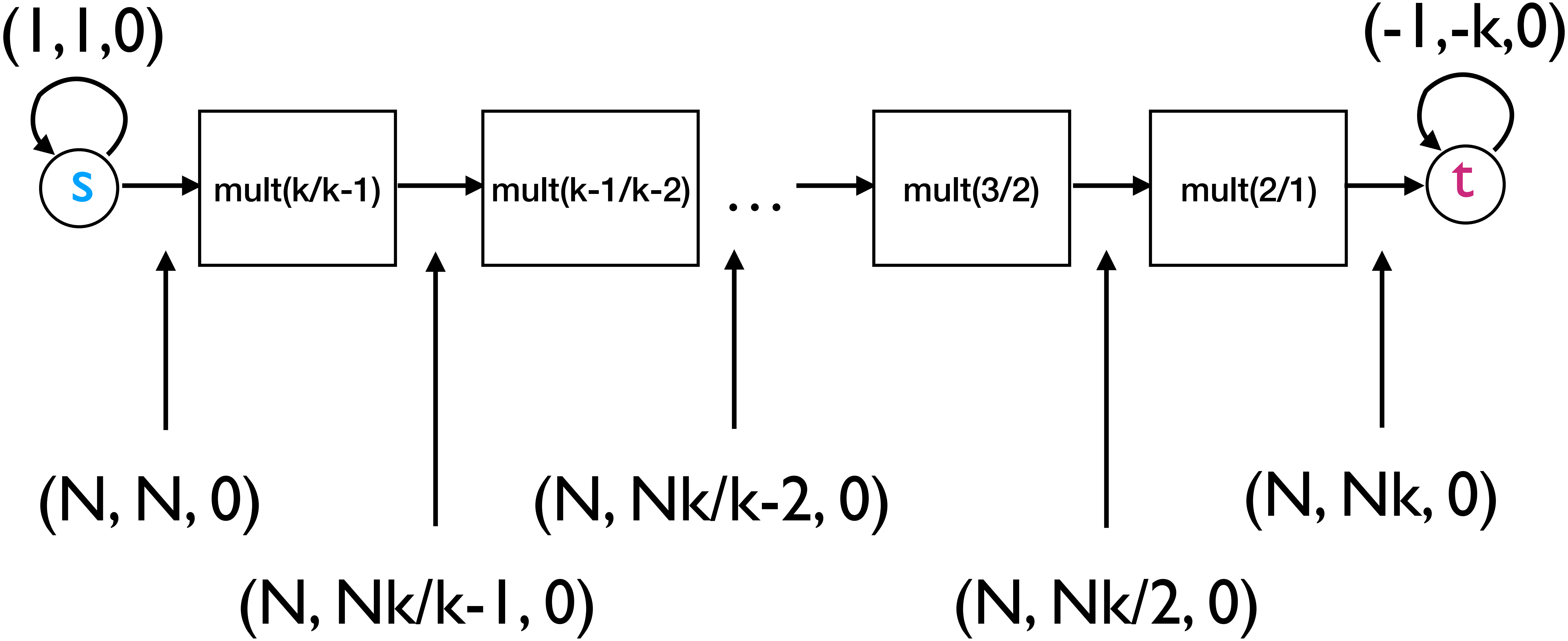
(N, Nk/k-2, 0)

(N, Nk/2, 0)

(N, Nk, 0)

Nk divisible by LCM(2, …, k-1)

Telescopic equations



Nk divisible by LCM(2, …, k-1)

N is exponential

Telescopic equations

Telescopic equations

2-exp run in a 4-VASS

Telescopic equations

2-exp run in a 4-VASS

tower run in Tower-hardness paper

Telescopic equations

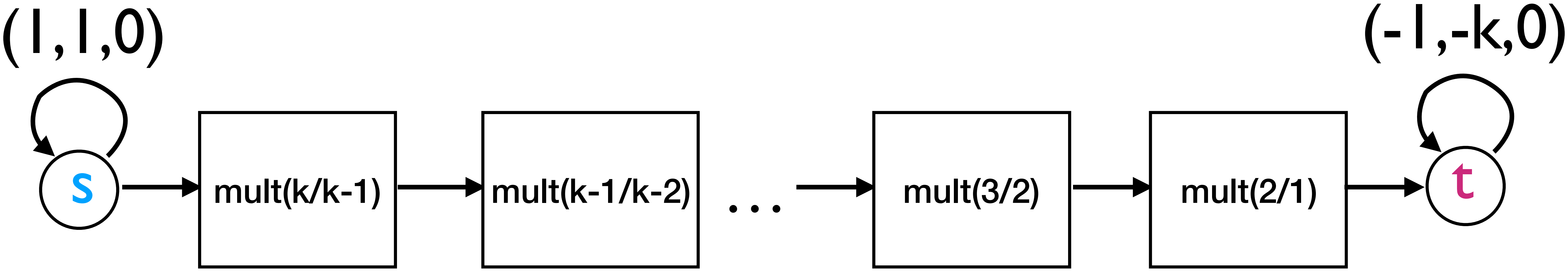
2-exp run in a 4-VASS

tower run in Tower-hardness paper

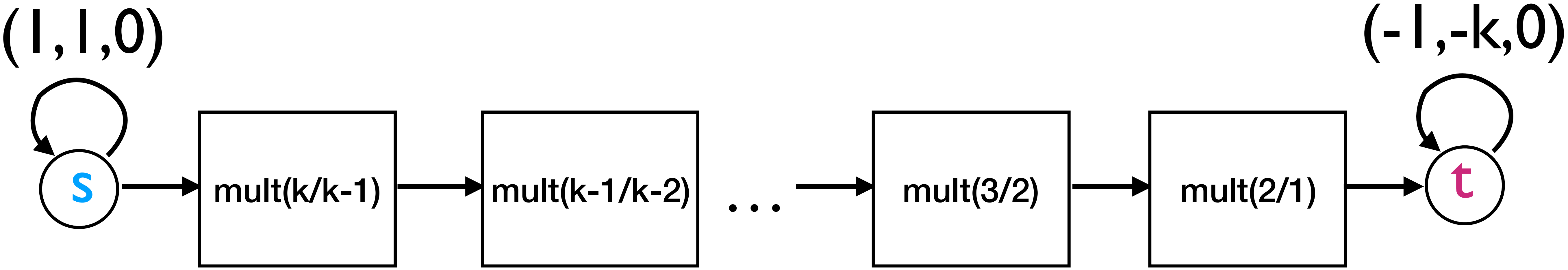
together with multiplication triples

Counter programs

Counter programs

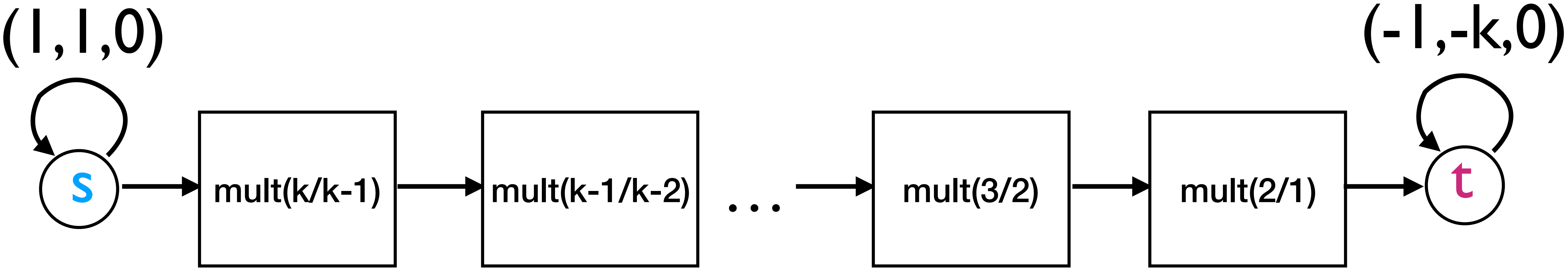


Counter programs



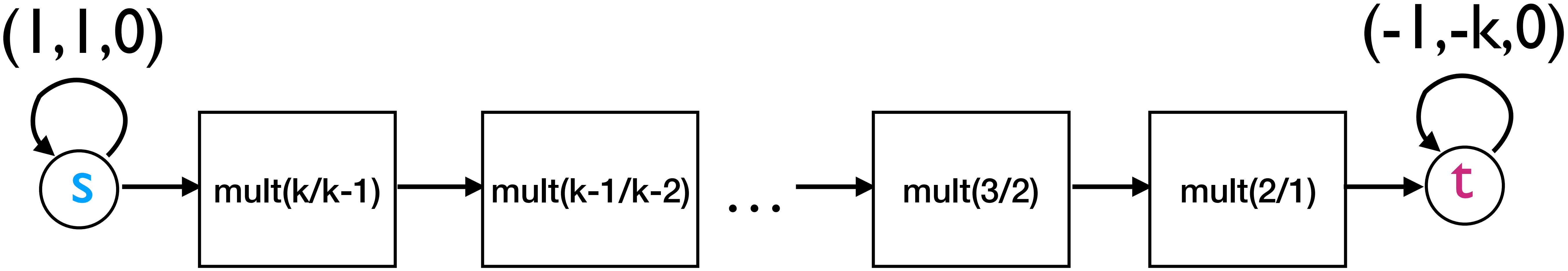
loop

Counter programs



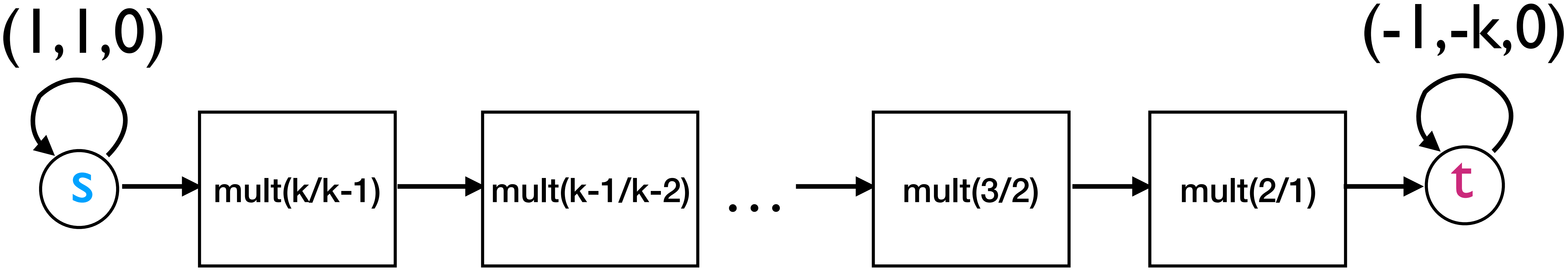
loop x+=1 y+=1

Counter programs



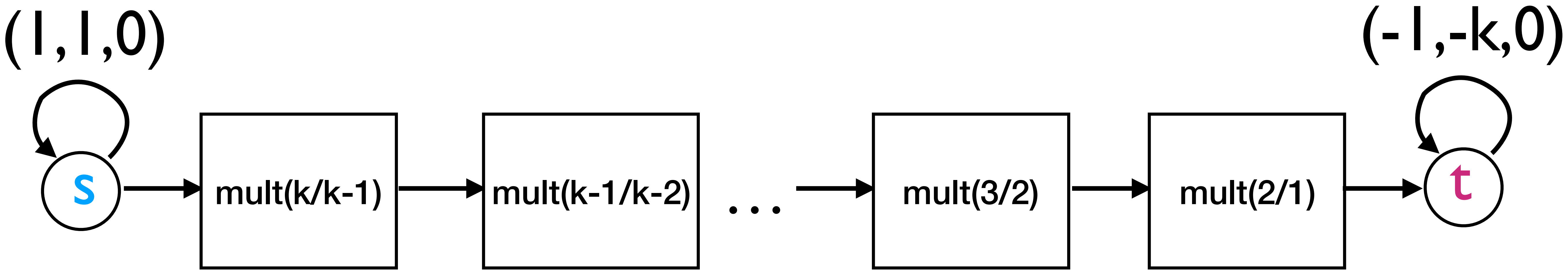
```
loop    x+=1 y+=1
for i=k downto 2 do
```

Counter programs



```
loop    x+=1 y+=1
for i=k downto 2 do
        loop
```

Counter programs



(1,1,0)

s

mult(k/k-1)

mult(k-1/k-2)

. . .

mult(3/2)

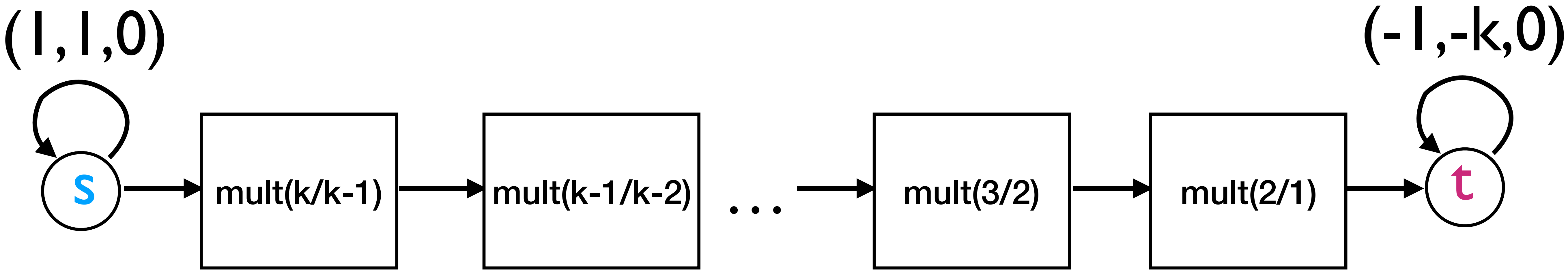
mult(2/1)

t

(-1,-k,0)

```
loop    x+=1 y+=1
for i=k downto 2 do
        loop    y-=1 z+=1
```

Counter programs



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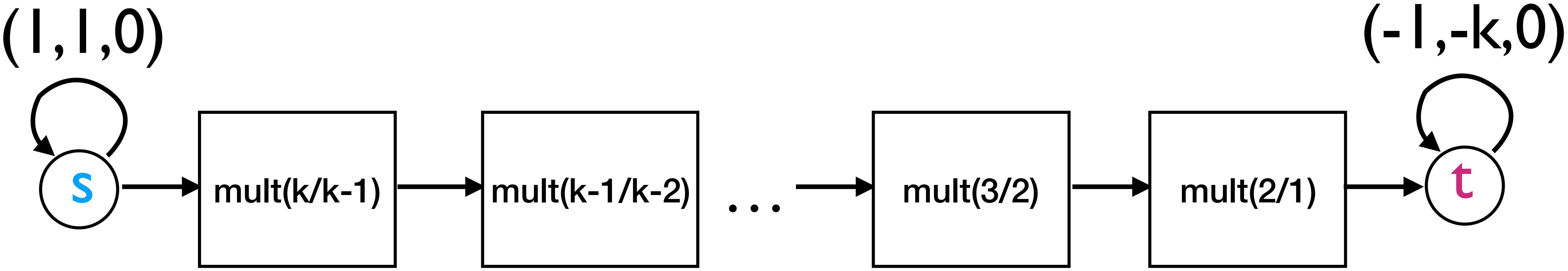
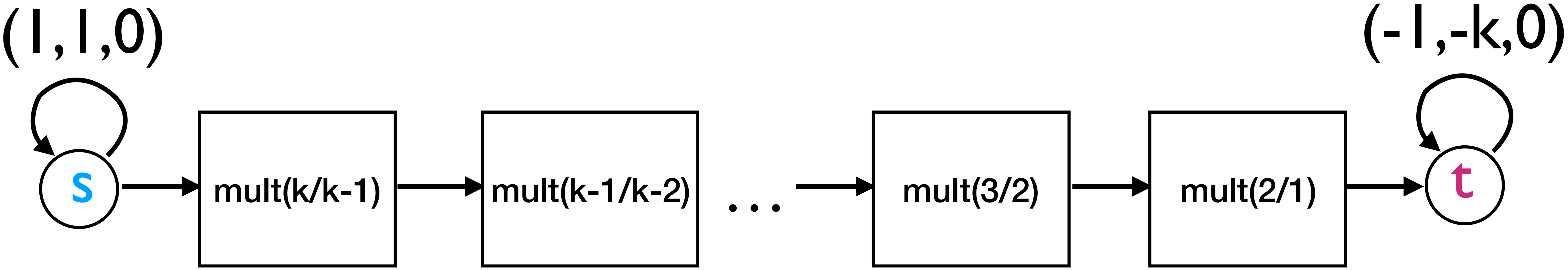


```
loop    x+=1 y+=1
for i=k downto 2 do
        loop    y-=1 z+=1
        loop    y+=i z-=(i-1)
```

Counter programs



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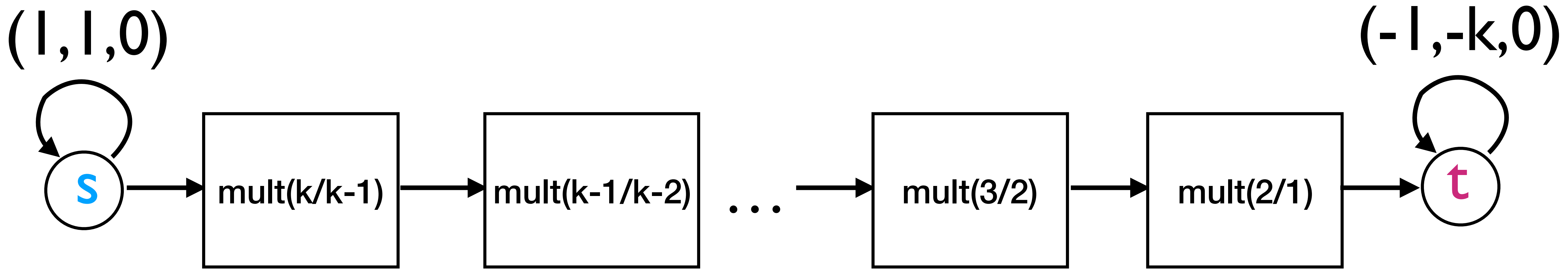
Counter programs



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        loop    y-=1 z+=1
        loop    y+=i z-=(i-1)
loop    x-=1 y-=k
```

Controlling counter

Controlling counter

Goal: zero-test x three times

Controlling counter

Goal: zero-test x three times

Controlling counter

Goal: zero-test x three times

$\longrightarrow c_1$

Controlling counter

Goal: zero-test x three times

$$\longrightarrow \; c_1 \; \longrightarrow$$

Controlling counter

Goal: zero-test x three times

$$\longrightarrow c_1 \longrightarrow c_2$$

Controlling counter

Goal: zero-test x three times

$$\begin{tikzcd}
{} \arrow[r] & c_1 \arrow[r] & c_2 \arrow[r] & {}
\end{tikzcd}$$

Controlling counter

Goal: zero-test x three times

$$\longrightarrow c_1 \longrightarrow c_2 \longrightarrow c_3$$

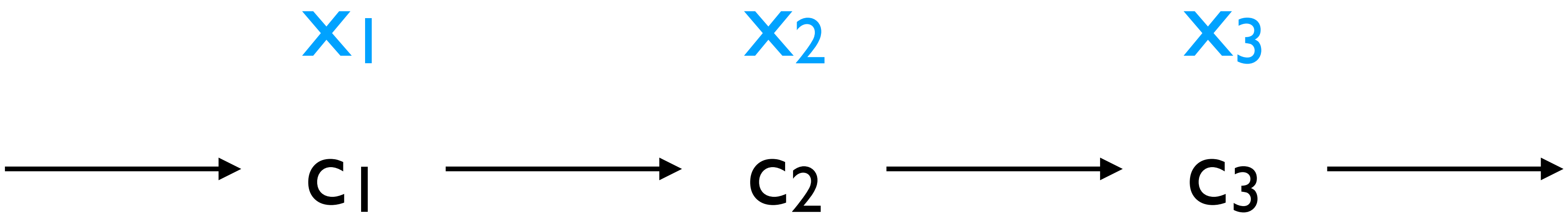
Controlling counter

Goal: zero-test x three times

$$\longrightarrow c_1 \longrightarrow c_2 \longrightarrow c_3 \longrightarrow$$

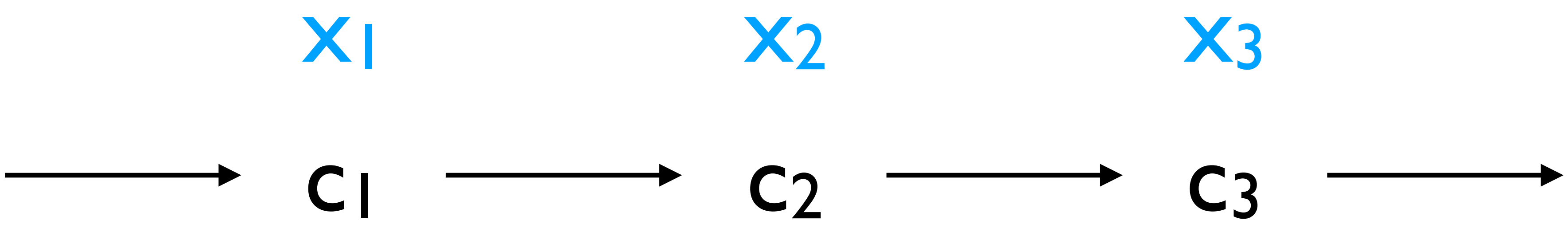
Controlling counter

Goal: zero-test x three times



Controlling counter

Goal: zero-test x three times



Enough to check if $x_1+x_2+x_3 = 0$

Controlling counter

Goal: zero-test x three times

$$\begin{tikzcd} 0 & x_1 & x_2 & x_3 & \\ c_0 \arrow[r] & c_1 \arrow[r] & c_2 \arrow[r] & c_3 \arrow[r] & {} \end{tikzcd}$$

Enough to check if $x_1+x_2+x_3 = 0$

Controlling counter

Goal: zero-test x three times

$$\begin{tikzcd}
0 & x_1 & x_2 & x_3 & {} \\
c_0 \arrow[r, "y_1"] & c_1 \arrow[r] & c_2 \arrow[r] & c_3 \arrow[r] & {}
\end{tikzcd}$$

Enough to check if $x_1+x_2+x_3 = 0$

Controlling counter

Goal: zero-test x three times

$$\begin{tikzcd}
0 & x_1 & x_2 & x_3 & \\
c_0 \arrow[r, "y_1"] & c_1 \arrow[r, "y_2"] & c_2 \arrow[r] & c_3 \arrow[r] & {}
\end{tikzcd}$$

Enough to check if $x_1+x_2+x_3 = 0$

Controlling counter

Goal: zero-test x three times

$$\begin{tikzcd} 0 & x_1 & x_2 & x_3 & \\ c_0 \arrow[r, "y_1"] & c_1 \arrow[r, "y_2"] & c_2 \arrow[r, "y_3"] & c_3 \arrow[r] & {} \end{tikzcd}$$

Enough to check if $x_1+x_2+x_3 = 0$

Controlling counter

Goal: zero-test x three times

$$\begin{tikzcd}
0 & & x_1 & & x_2 & & x_3 & & \\
c_0 \arrow[rr, "y_1"] & & c_1 \arrow[rr, "y_2"] & & c_2 \arrow[rr, "y_3"] & & c_3 \arrow[rr] & & {}
\end{tikzcd}$$

Enough to check if $x_1+x_2+x_3 = 0$

$x_1 = y_1$

Controlling counter

Goal: zero-test x three times

$$\begin{tikzcd} \overset{0}{c_0} \arrow[r, "y_1"] & \overset{x_1}{c_1} \arrow[r, "y_2"] & \overset{x_2}{c_2} \arrow[r, "y_3"] & \overset{x_3}{c_3} \arrow[r] & {} \end{tikzcd}$$

Enough to check if $x_1+x_2+x_3 = 0$

$x_1 = y_1$ $x_2 = y_1+y_2$

Controlling counter

Goal: zero-test x three times

$$\begin{tikzcd}
\overset{0}{c_0} \arrow[r, "y_1"] & \overset{x_1}{c_1} \arrow[r, "y_2"] & \overset{x_2}{c_2} \arrow[r, "y_3"] & \overset{x_3}{c_3} \arrow[r] & {}
\end{tikzcd}$$

Enough to check if $x_1+x_2+x_3 = 0$

$x_1 = y_1$ $\quad$ $x_2 = y_1+y_2$ $\quad$ $x_3 = y_1+y_2+y_3$

Controlling counter

Goal: zero-test x three times

$$\begin{tikzcd}
0 & x_1 & x_2 & x_3 & \\
c_0 \arrow[r, "y_1"] & c_1 \arrow[r, "y_2"] & c_2 \arrow[r, "y_3"] & c_3 \arrow[r] & {}
\end{tikzcd}$$

Enough to check if $x_1+x_2+x_3 = 0$

$x_1 = y_1$ $\qquad$ $x_2 = y_1+y_2$ $\qquad$ $x_3 = y_1+y_2+y_3$

Enough to check if $3y_1+2y_2+y_3 = 0$

Controlling counter

Goal: zero-test x three times

$$\begin{tikzcd} \overset{0}{c_0} \arrow[r, "y_1"] & \overset{x_1}{c_1} \arrow[r, "y_2"] & \overset{x_2}{c_2} \arrow[r, "y_3"] & \overset{x_3}{c_3} \arrow[r] & {} \end{tikzcd}$$

Enough to check if $x_1+x_2+x_3 = 0$

$x_1 = y_1$ $x_2 = y_1+y_2$ $x_3 = y_1+y_2+y_3$

Enough to check if $3y_1+2y_2+y_3 = 0$

New counter c counts $3y_1+2y_2+y_3$

Controlling counter

Controlling counter

Many testing places for many counters

Controlling counter

Many testing places for many counters

One controlling counter c

Controlling counter

Many testing places for many counters

One controlling counter c

Counter c counts sum of all zero-tested counters

Controlling counter

Many testing places for many counters

One controlling counter c

Counter c counts sum of all zero-tested counters

If x waits for k zero-tests
then inc(x) implies c+=k

Controlling counter

Controlling counter

```
for i=1 to k do
```

Controlling counter

```
for i=1 to k do
    loop
```

Controlling counter

```
for i=1 to k do
    loop    x-=1 y+=2
```

Controlling counter

```
for i=1 to k do
    loop     x-=1 y+=2
    loop
```

Controlling counter

```
for i=1 to k do
    loop    x-=1 y+=2
    loop    x+=1 y-=1
```

Controlling counter

(1, 0)

```
for i=1 to k do
    loop    x-=1 y+=2
    loop    x+=1 y-=1
```

Controlling counter

(1, 0) →

```
for i=1 to k do
    loop    x-=1 y+=2
    loop    x+=1 y-=1
```

Controlling counter

(1, 0) →

```
for i=1 to k do
    loop    x-=1 y+=2
    loop    x+=1 y-=1
```

→

Controlling counter

$(1, 0) \longrightarrow$

```
for i=1 to k do
    loop    x-=1 y+=2
    loop    x+=1 y-=1
```

$\longrightarrow (2^k, 0)$

Controlling counter

$(1, 0) \longrightarrow$

```
for i=1 to k do
    loop    x-=1 y+=2
    loop    x+=1 y-=1
```

$\longrightarrow (2^k, 0)$

$\longrightarrow$

Controlling counter

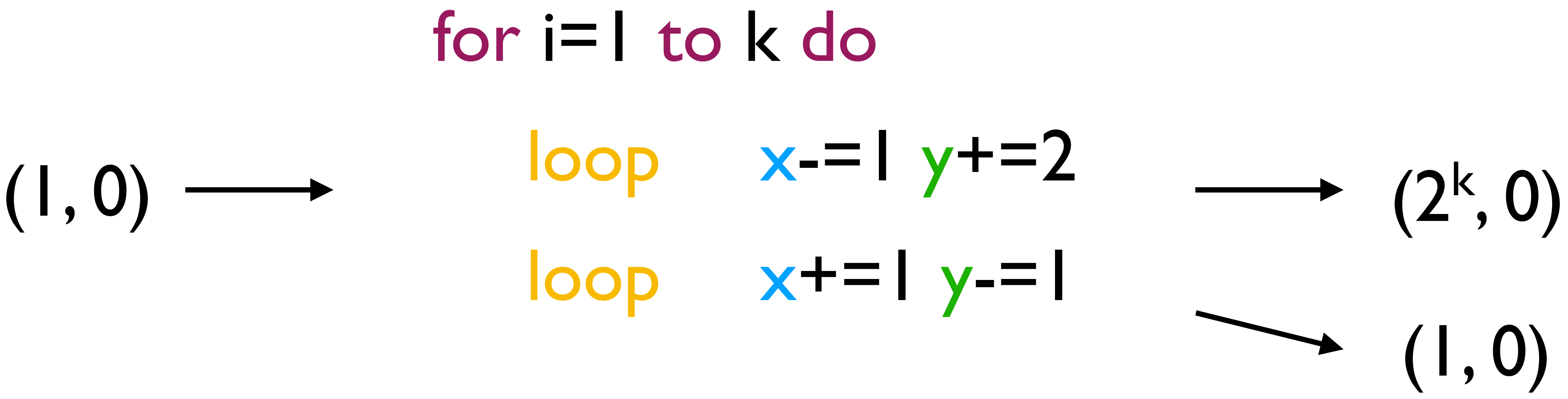
$(1, 0) \longrightarrow$

```
for i=1 to k do
    loop    x-=1 y+=2
    loop    x+=1 y-=1
```

$\longrightarrow (2^k, 0)$

$\longrightarrow (1, 0)$

Controlling counter



$(1, 0)$

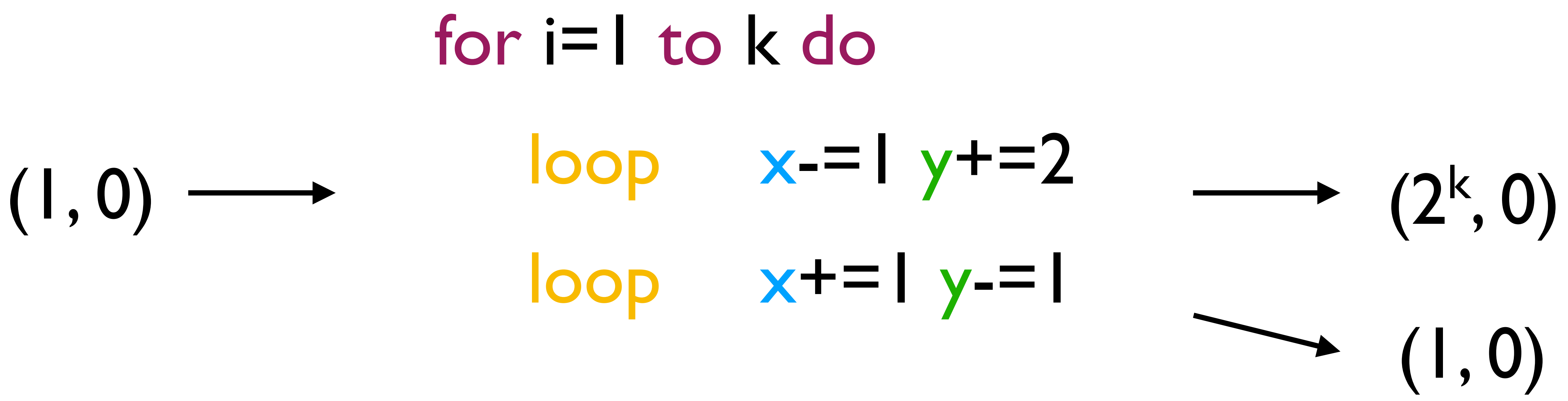
```
for i=1 to k do
    loop    x-=1 y+=2
    loop    x+=1 y-=1
```

$(2^k, 0)$

$(1, 0)$

```
for i=k downto 1 do
```

Controlling counter



```
for i=1 to k do
    loop    x-=1 y+=2
    loop    x+=1 y-=1
```

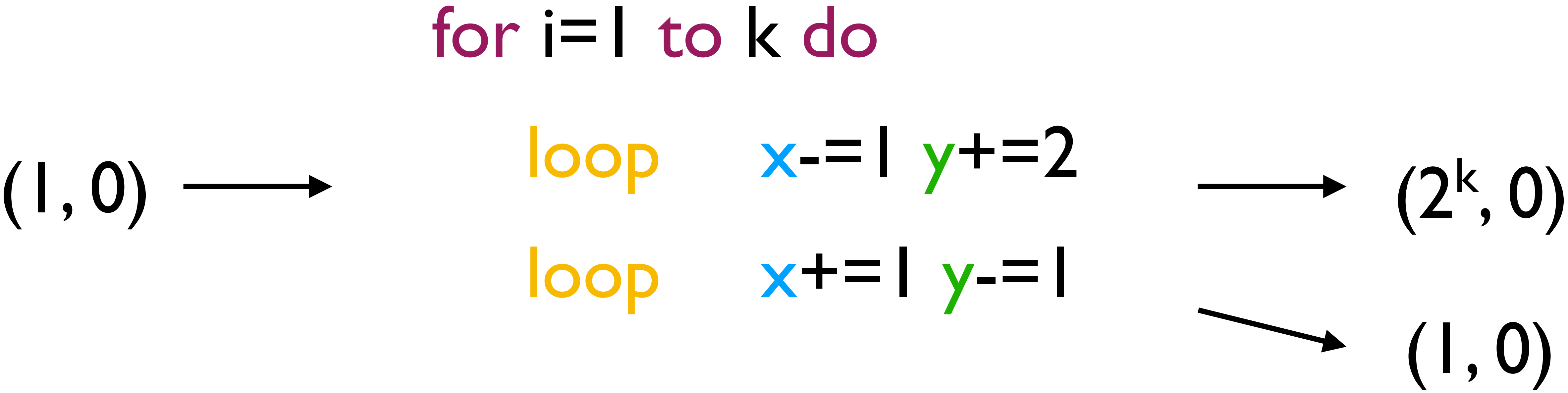
(1, 0) →

→ $(2^k, 0)$

→ (1, 0)

```
for i=k downto 1 do
    loop
```

Controlling counter



(1, 0)

```
for i=1 to k do
    loop    x-=1 y+=2
    loop    x+=1 y-=1
```

$(2^k, 0)$

(1, 0)

```
for i=k downto 1 do
    loop    x-=1 y+=2
```

Controlling counter

$(1, 0) \longrightarrow$

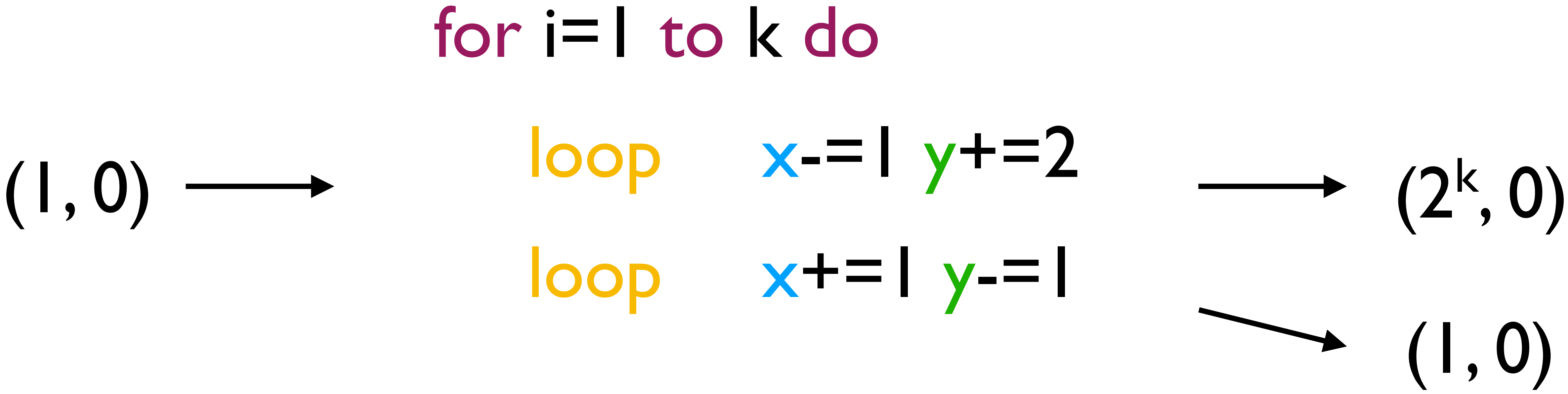
```
for i=1 to k do
    loop    x-=1 y+=2
    loop    x+=1 y-=1
```

$\longrightarrow (2^k, 0)$

$\longrightarrow (1, 0)$

```
for i=k downto 1 do
    loop    x-=1 y+=2
    loop
```

Controlling counter



$(1, 0) \longrightarrow$

```
for i=1 to k do
    loop    x-=1 y+=2
    loop    x+=1 y-=1
```

$\longrightarrow (2^k, 0)$

$\longrightarrow (1, 0)$

```
for i=k downto 1 do
    loop    x-=1 y+=2
    loop    x+=1 y-=1
```

Controlling counter

$(1, 0) \longrightarrow$

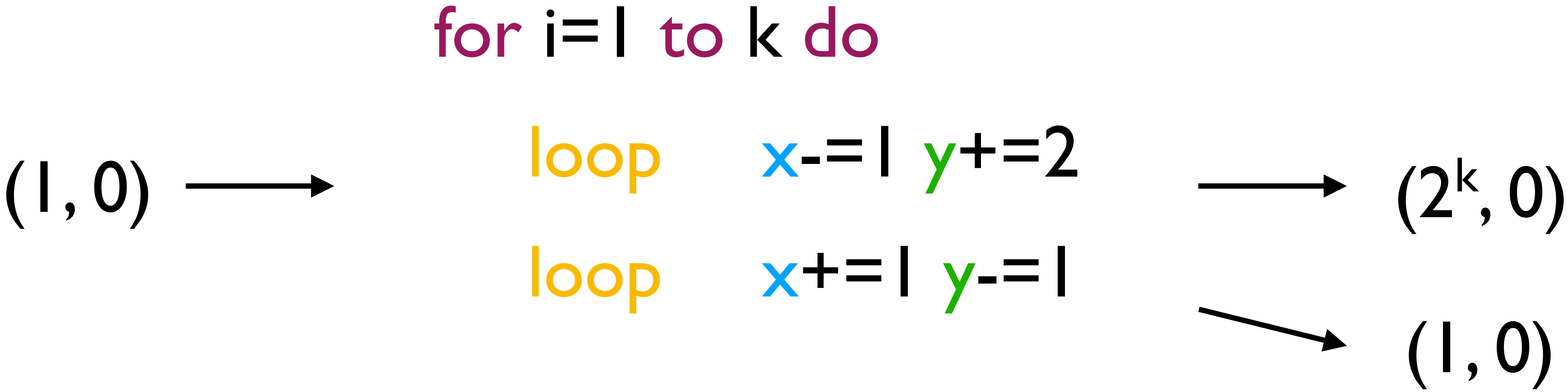
```
for i=1 to k do
    loop    x-=1 y+=2
    loop    x+=1 y-=1
```

$\longrightarrow (2^k, 0)$

$\longrightarrow (1, 0)$

```
for i=k downto 1 do
    loop    x-=1 y+=2    c+=i
    loop    x+=1 y-=1
```

Controlling counter



$(1, 0)$

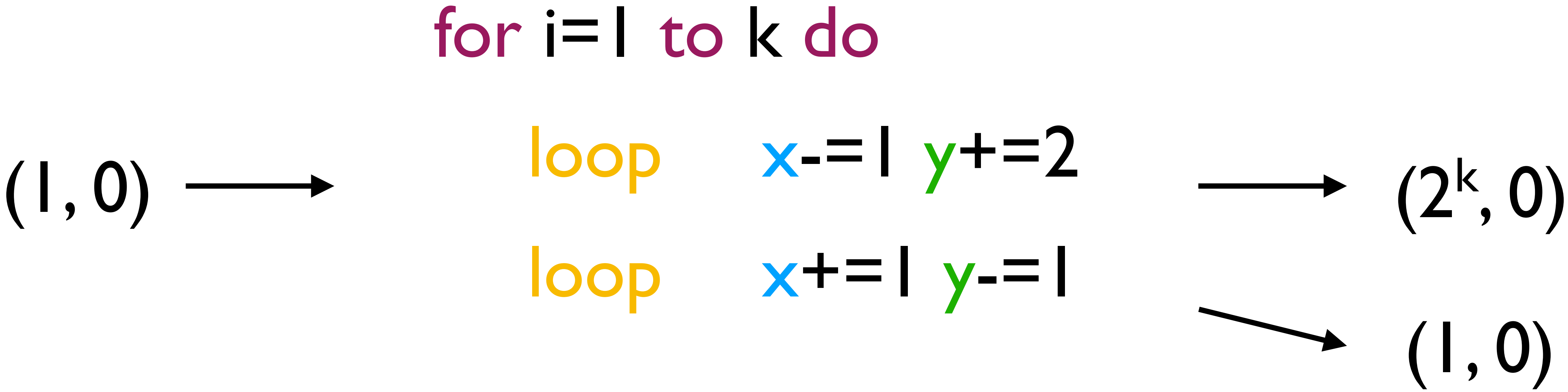
```
for i=1 to k do
    loop    x-=1 y+=2
    loop    x+=1 y-=1
```

$(2^k, 0)$

$(1, 0)$

```
for i=k downto 1 do
    loop    x-=1 y+=2    c+=i
    loop    x+=1 y-=1    c-=1
```

Controlling counter



$(1, 0)$ →

```
for i=1 to k do
    loop    x-=1 y+=2
    loop    x+=1 y-=1
```

→ $(2^k, 0)$

→ $(1, 0)$

$(1, 0, 0)$

```
for i=k downto 1 do
    loop    x-=1 y+=2    c+=i
    loop    x+=1 y-=1    c-=1
```

Controlling counter

$(1, 0) \longrightarrow$

```
for i=1 to k do
    loop    x-=1 y+=2
    loop    x+=1 y-=1
```

$\longrightarrow (2^k, 0)$

$\longrightarrow (1, 0)$

$(1, 0, 0) \longrightarrow$

```
for i=k downto 1 do
    loop    x-=1 y+=2    c+=i
    loop    x+=1 y-=1    c-=1
```

Controlling counter

(1, 0) →

```
for i=1 to k do
    loop    x-=1 y+=2
    loop    x+=1 y-=1
```

→ $(2^k, 0)$

→ (1, 0)

(1, 0, 0) →

```
for i=k downto 1 do
    loop    x-=1 y+=2    c+=i
    loop    x+=1 y-=1    c-=1
```

→

Controlling counter

$(1, 0) \longrightarrow$

```
for i=1 to k do
    loop    x-=1 y+=2
    loop    x+=1 y-=1
```

$\longrightarrow (2^k, 0)$

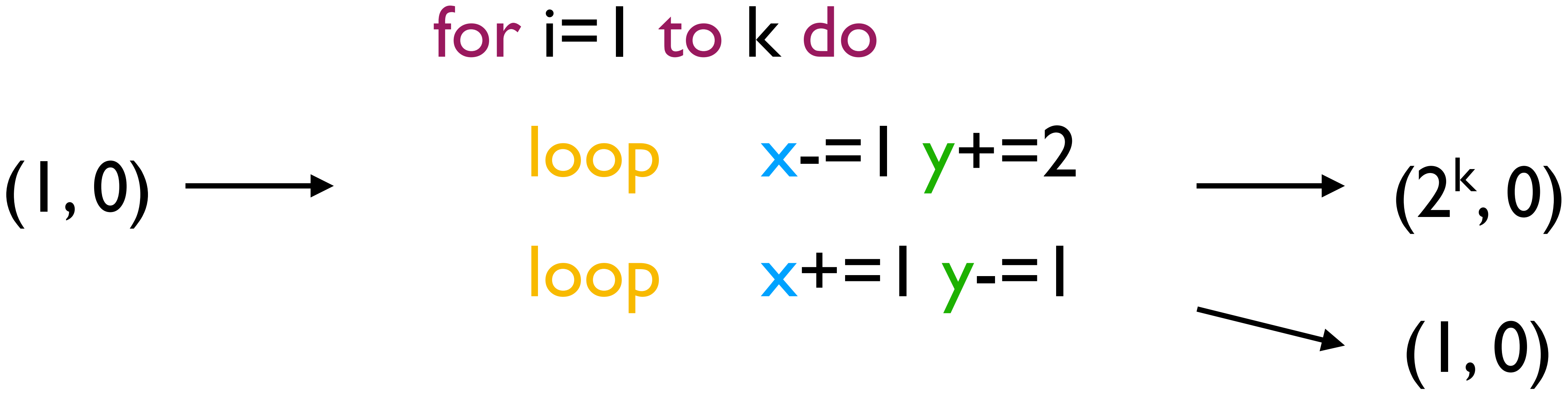
$\longrightarrow (1, 0)$

$(1, 0, 0) \longrightarrow$

```
for i=k downto 1 do
    loop    x-=1 y+=2    c+=i
    loop    x+=1 y-=1    c-=1
```

$\longrightarrow (2^k, 0, 0)$

Controlling counter

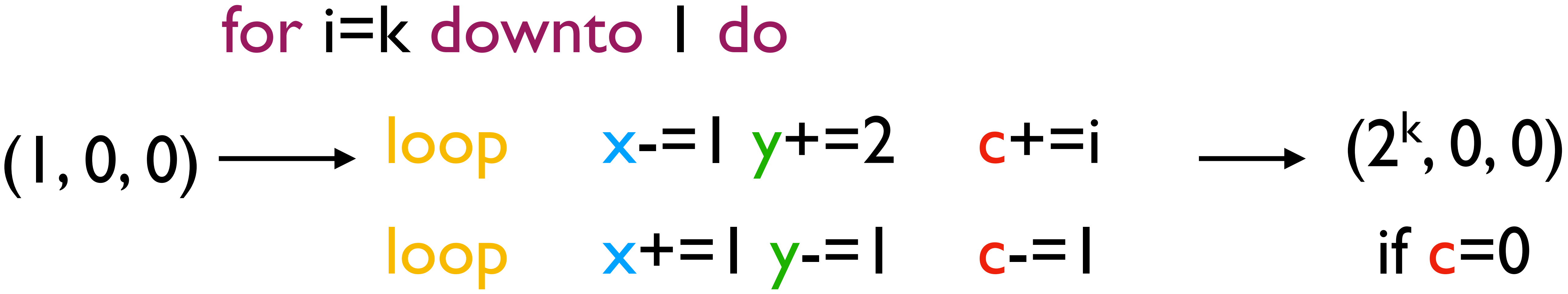


$(1, 0)$

```
for i=1 to k do
    loop x-=1 y+=2
    loop x+=1 y-=1
```

$(2^k, 0)$

$(1, 0)$



$(1, 0, 0)$

```
for i=k downto 1 do
    loop x-=1 y+=2 c+=i
    loop x+=1 y-=1 c-=1
```

$(2^k, 0, 0)$

if c=0

Controlling counter

Controlling counter

used in our Ackermann-hardness

Controlling counter

used in our Ackermann-hardness

many small dimension results

Controlling counter

used in our Ackermann-hardness

many small dimension results

incomparable with multiplication triples

Message

Message

many open problems

Message

many open problems

look at small dimensions

Message

many open problems

look at small dimensions

hardness $\approx$ big reachability set + enforcing

Message

many open problems

look at small dimensions

hardness ≈ big reachability set + enforcing

Thank you!