VAS Coverability in ExpSpace

Fix a d dimensional Vector Addition System (VAS). For two configurations $x, y \in \mathbb{N}^d$ we say that x is *bigger* than y, denoted $x \succeq y$, if x is coordinstewise bigger or equal y. For two configurations source s and target t, both in $\mathbb{N}^d$, a path $s \longrightarrow t'$ is a *covering path from s to t* if $t' \succeq t$. The `Coverability Problem` asks for two given configurations $s, t \in \mathbb{N}^d$ whether there exists a covering path from s to t.

We present here a proof that Coverability Problem is in exponential space, a small modification of the original proof by Charles Rackoff from 1978 (*The Covering and Boundedness Problems for Vector Addition Systems*). It is clearly enough to show that for any two configurations s, t if there is a covering path from s to t then there is such of at most doubly exponential length, which is achieved by the Lemma below. Then algorithm for coverabilitily problem simply guesses such a path and it can verify in exponential space that it is indeed a correct guess.

Let *norm* of a vector be maximal absolute value of any of its entries and *norm* of a VAS be maximal norm of a transition of this VAS. *Norm* of a configuration is the norm of its vector. *Length* of a path is the number of transitions in it.

Lemma 1. *Consider a VAS, which has a norm bounded by M and two its configurations $s, t \in \mathbb{N}^d$ such that norm of t is bounded by M (note that norm of s can be arbitrary). If there is a covering path from s to t then there is such of length at most $(M+1)^{(4d)^{d-1}}$.*

Proof. We proceed by induction on d. For $d = 1$ it is immediate to show that any covering path, which does not repeat a configuration has length at most $M + 1$.

For the induction step assume that ρ is a covering path $s \xrightarrow{\rho} t'$ in a $d+1$ dimensional VAS. Let $B_d = (M+1)^{(4d)^{d-1}}$. We distinguish two cases:

1. norm of every configuration on ρ is bounded by $(M+1) \cdot B_d$;
2. norm of some u on ρ exceeds $(M+1) \cdot B_d$.

Of course we can assume that no configuration on ρ appears more than once, otherwise we unpump ρ. Thus in case 1 length of ρ is bounded by $C = ((M+1) \cdot B_d)^{d+1}$, we estimate C from above later.

In case 2 length of ρ might be long, but we show that we can find another covering path ρ', which is short enough. Let u be the first configuration on ρ with norm exceeding $(M+1) \cdot B_d$. Let $s \xrightarrow{\rho_1} u \xrightarrow{\rho_2} t'$. Clearly length of ρ_1 is bounded by C, similarly as in case 1. Now we aim at modifying ρ_2 to obtain $u \xrightarrow{\rho_3} t''$ for some $t'' \succeq t$ such that ρ_3 is short too. Some coordinate in u is at least $(M+1) \cdot B_d$, assume without loss of generality that it is the last, $d+1$-th coordinate. We ignore for a moment this last coordinate in a whole VAS and by induction assumption we get that there is a path π of length at most B_d such that $u_d \xrightarrow{\pi} t''_d$. Here $u_d, t_d \in \mathbb{N}^d$ are obtained from $u, t \in \mathbb{N}^{d+1}$ by removing the last coordinate and $t''_d \in \mathbb{N}^d$ is some configuration fulfilling $t''_d \succeq t_d$. Then we remind ourselves that coordinate $d+1$ exists, let $u \xrightarrow{\pi} t''$. It is clear by above reasoning that for all coordinates $i \in \{1, \ldots, d\}$ we have $t''[i] \geq t[i]$, but what happens on the last coordinate? By we assumption we have that $u[d+1] \geq (M+1) \cdot B_d$. Every of at most B_d transitions in π can decrease the last coordinate by at most M. So $t''[d+1] \geq (M+1) \cdot B_d - B_d \cdot M = B_d \geq M \geq t[d+1]$. Thus indeed $t'' \succeq t$ and path $s \xrightarrow{\rho_1} u \xrightarrow{\rho_3} t''$ is a covering path from s to t. Length of $\rho' = \rho_1 \rho_3$ is at most $C + B_d$.

In order to finish the argument we have to show that $C + B_d \leq B_{d+1} = (M+1)^{(4(d+1))^d}$. We perform very rough estimations:

$$C+B_d \leq (M+1) \cdot C = (M+1) \cdot \Big((M+1) \cdot B_d\Big)^{d+1} = (M+1) \cdot \Big((M+1) \cdot (M+1)^{(4d)^{d-1}}\Big)^{d+1}$$

$$= (M+1)^{((4d)^{d-1}+1)(d+1)+1} \leq (M+1)^{4 \cdot (4d)^{d-1} \cdot (d+1)} \leq (M+1)^{(4(d+1))^d} = B_{d+1}.$$

This finishes the proof. □