Separability of Reachability Sets of Vector Addition Systems

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General problem

F separability of G

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Given: two sets U and V from family G

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Question: are U and V separable by some set from family F

History of separability

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- regular separability of CFL is undecidable (Szymanski, Williams '76)

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- regular separability of visibly pushdown languages is undecidable (Kopczyński '15)

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- conjecture: decidable for VAS-languages (open)
- recently solved for many subclasses

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- what about recognizable separability of more complicated sets?
- for example VAS reachability sets
- goal now: present technique on a simpler case

Vector Addition System

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initial vector v in N^n

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move: from u to $u+t$
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reachability set: vectors in N^n reachable
from v by a sequence of moves

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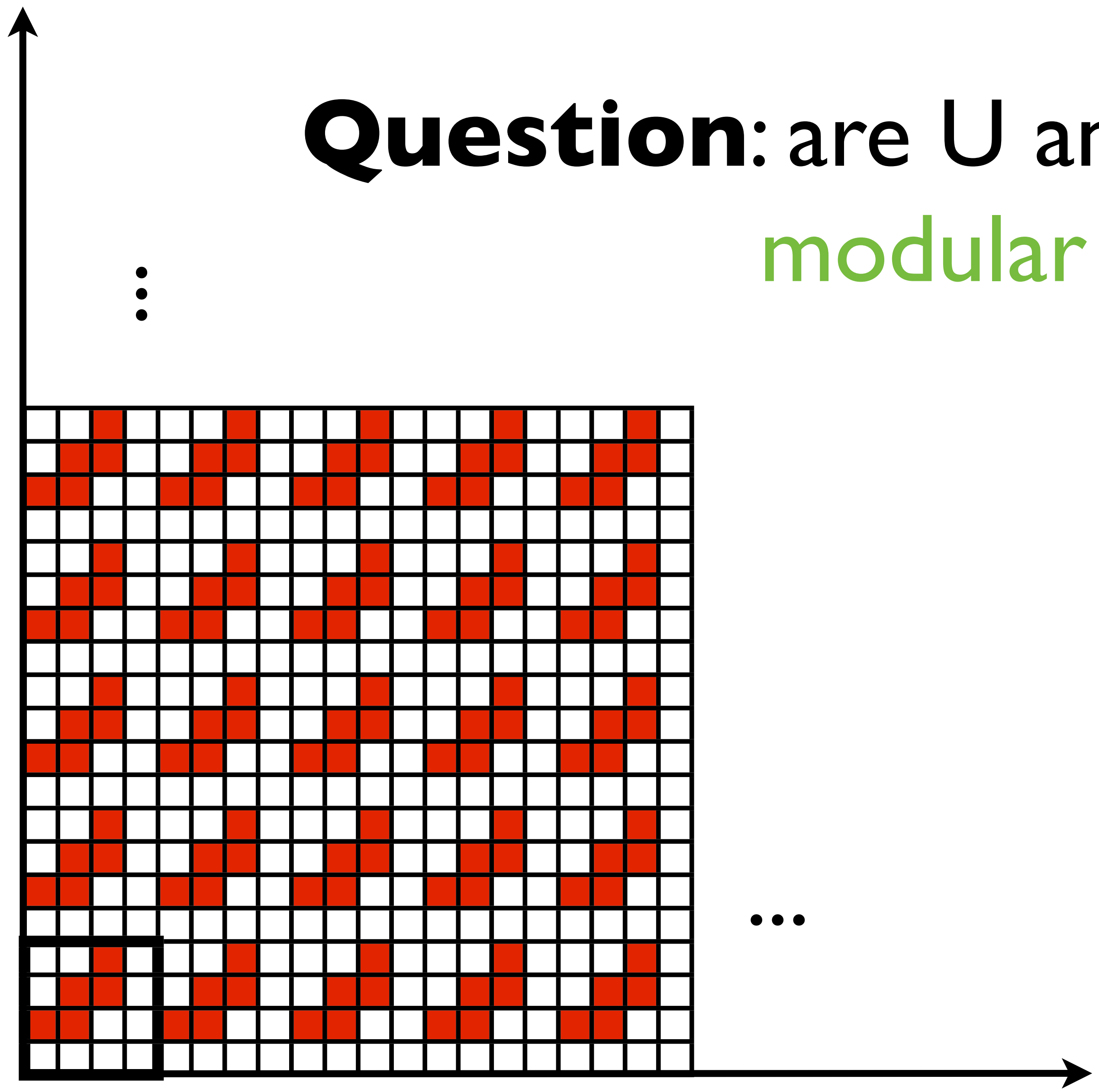

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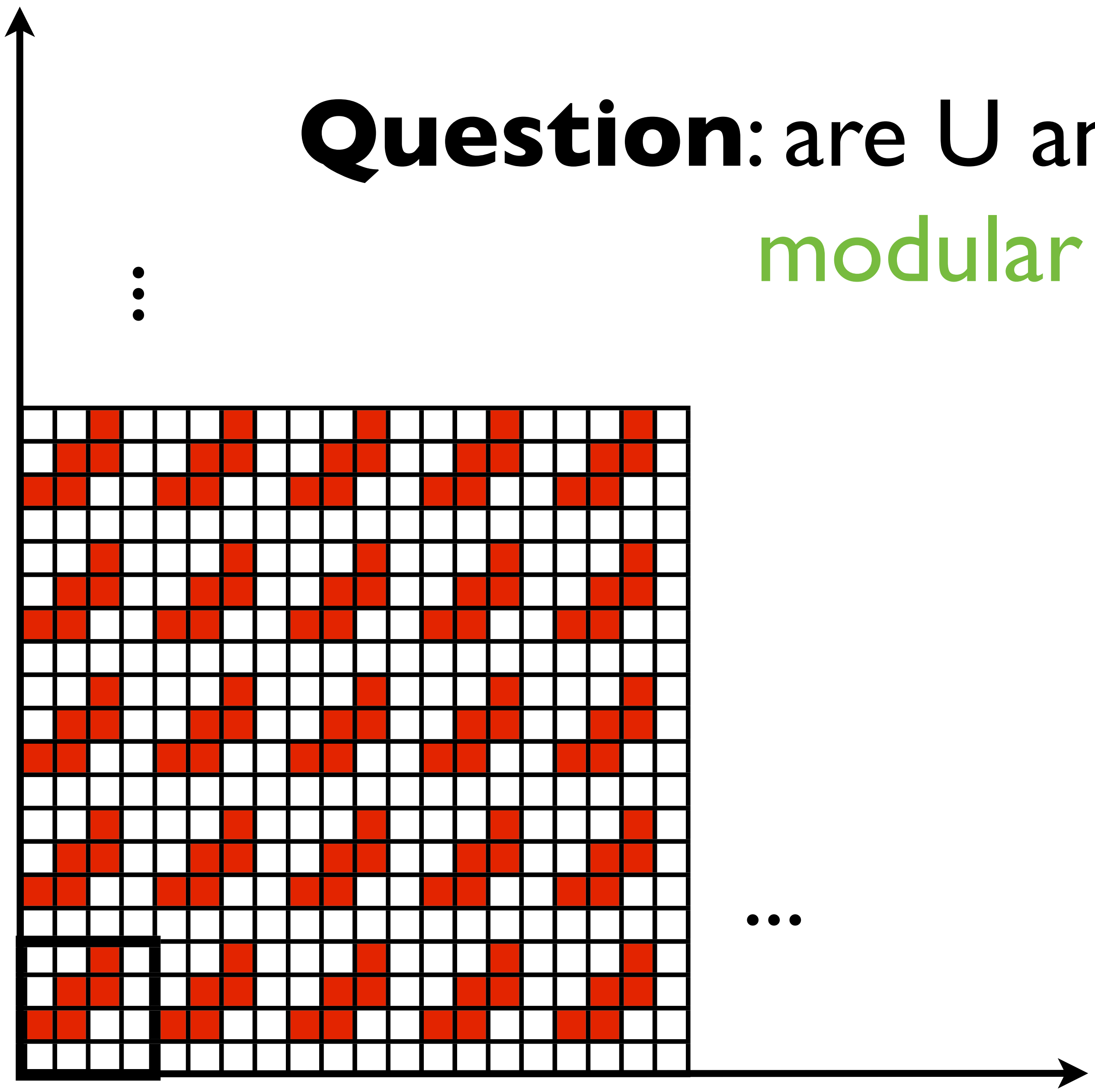

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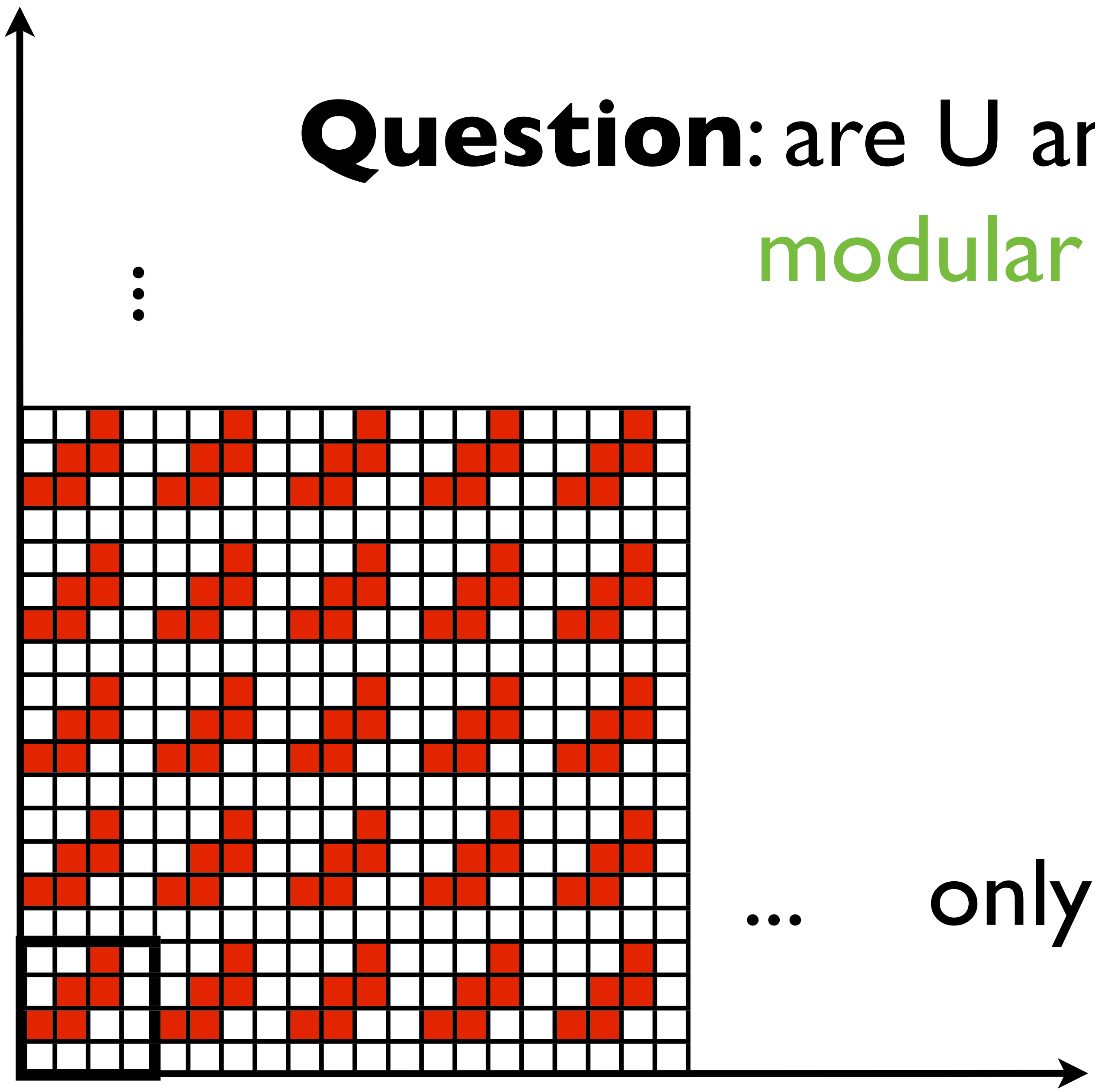

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- Leroux `09: two VAS reachability sets are separable by a semilinear set iff they are disjoint
- so separability by semilinear sets is decidable

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First main result

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Theorem:
Separability of reachability sets of
Vector Addition Systems by modular sets
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3) there exists two special linear sets $L_U \subseteq U$, $L_V \subseteq V$ such that L_U and L_V are not separable by modular sets

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simple by VASS reachability

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simple by linear algebra

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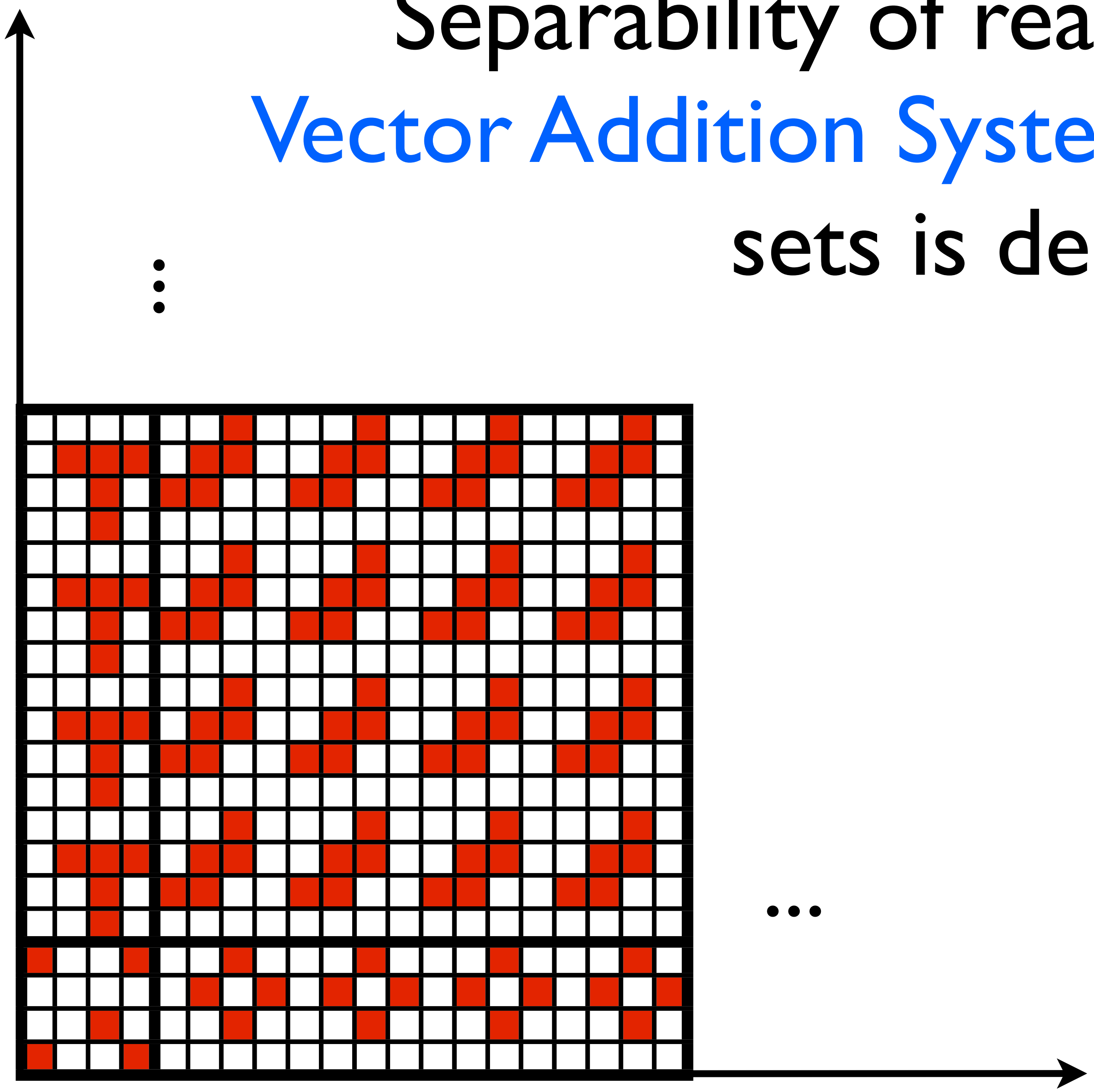

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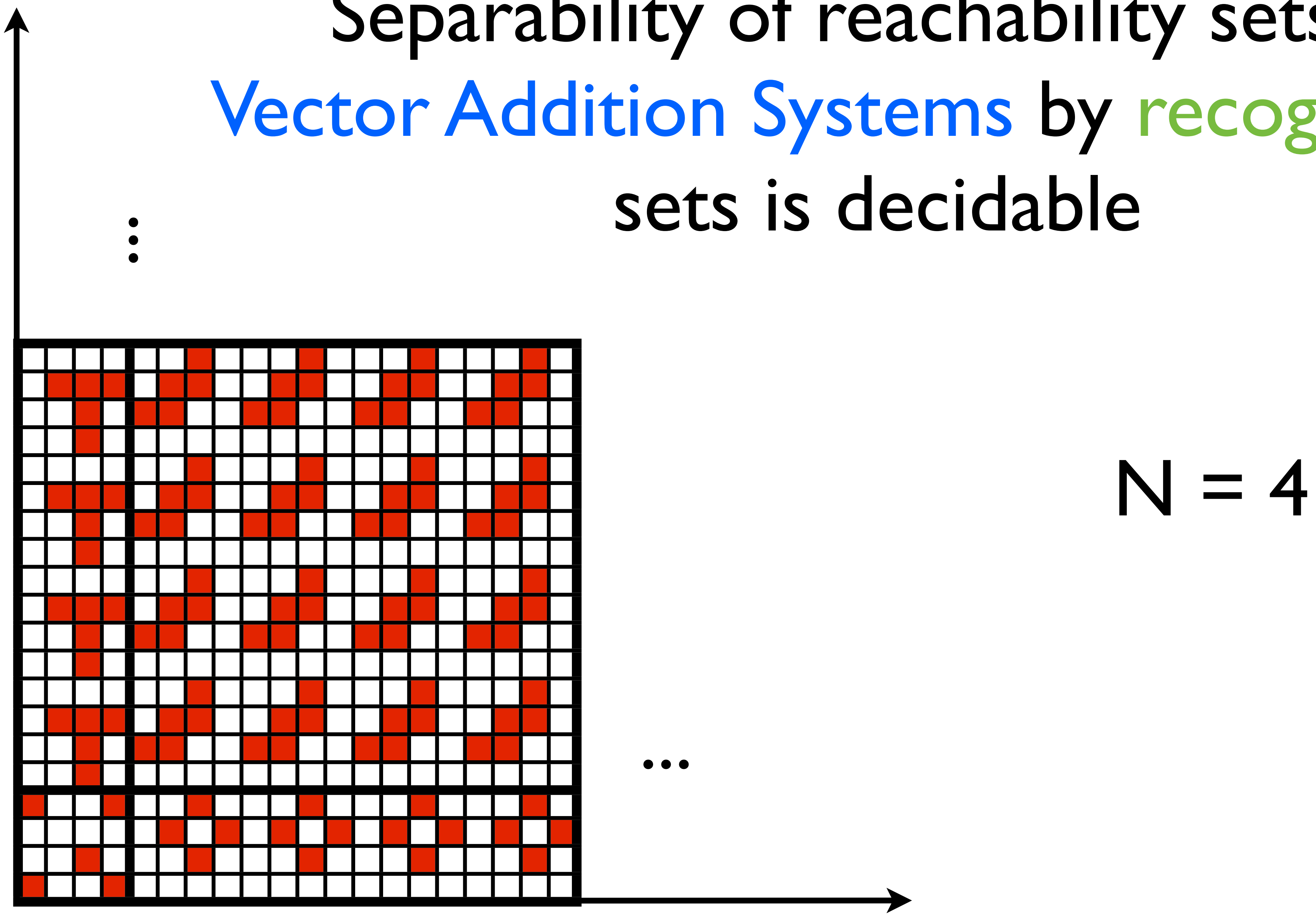

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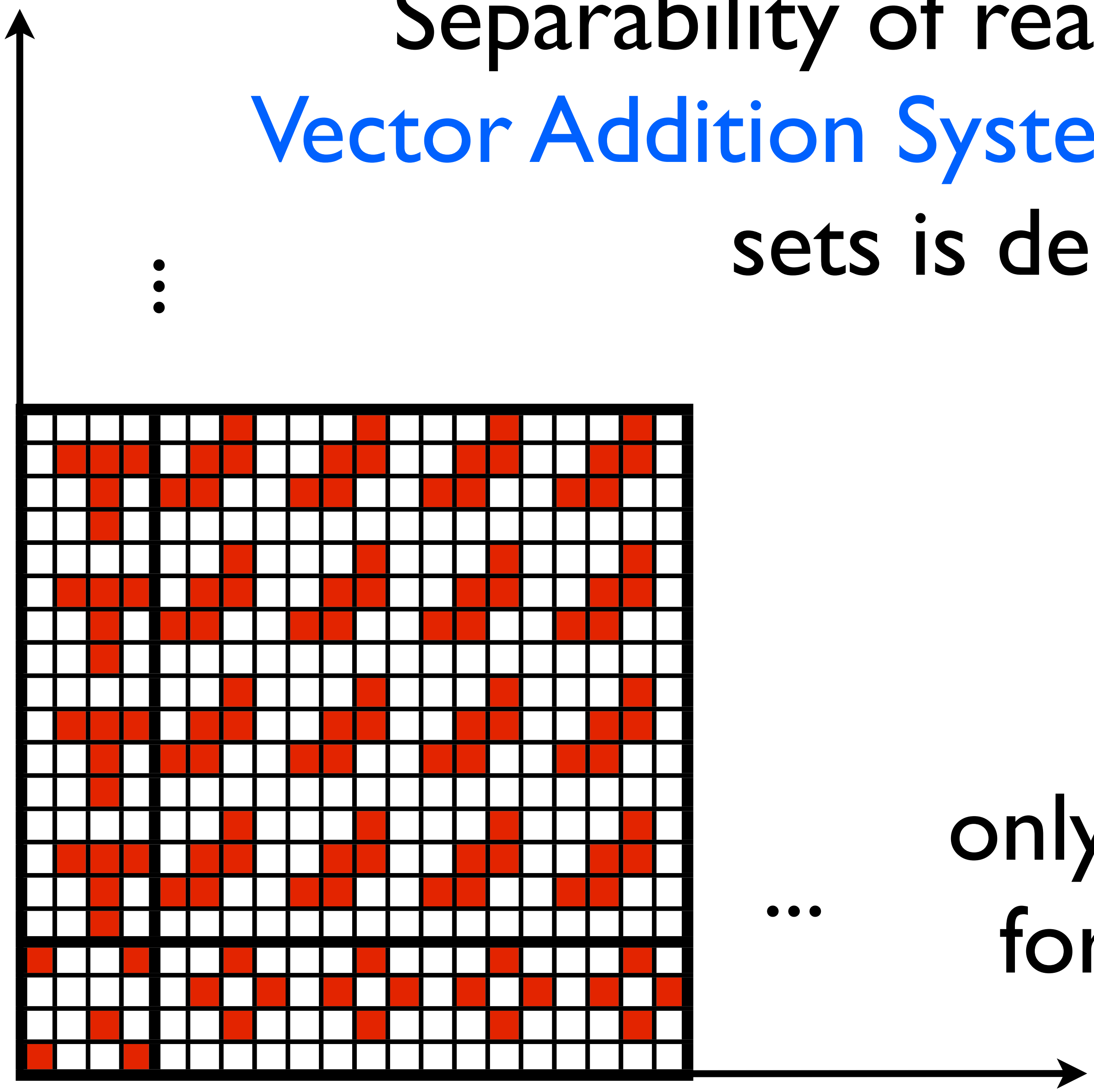

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- start from an infinite witness of modular nonseparability and then fold it to a finite object
- use some linear algebra

WQO on VAS-runs

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- $(X, \leq)$ is a wqo if for every $x_1, x_2, \ldots \in X$ there are $i < j$ such that $x_i \leq x_j$
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- Higman: $a_1 \ldots a_k \leq v$ if $v \in \Sigma^* b_1 \Sigma^* \ldots \Sigma^* b_k \Sigma^*$ for some $a_i \leq_P b_i$, if $\leq_P$ wqo then $\leq$ also wqo

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- our order $\leq$: Higman's order for $\leq_P$ being order on transitions intersected with $\leq_P$ on targets

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Let r, r_1, r_2 be runs from s to t, t_1 and t_2 respectively such that $r \leq r_1$ and $r \leq r_2$.

Then there is a run r' from s to $t + (t_1 - t) + (t_2 - t)$ such that $r_1 \leq r'$ and $r_2 \leq r'$.

Amalgamation for VAS-runs

Let r, r_1, r_2 be runs from s to t, t_1 and t_2 respectively such that $r \preceq r_1$ and $r \preceq r_2$.

Then there is a run r' from s to $t + (t_1 - t) + (t_2 - t)$ such that $r_1 \preceq r'$ and $r_2 \preceq r'$.

Corollary: then there is a run to every $t + n_1(t_1 - t) + \ldots + n_k(t_k - t)$

Special linear set

Special linear set

For a VAS V a V-special set is a set of the form

$$\{t + n_1(t_1 - t) + \dots + n_k(t_k - t) \mid n_i \in N\}$$

for some t, t_i in Reach(V) such that $r \leq r_i$ for some runs to t and to t_i, respectively, for all i

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plan: fold it to a finite object

Working on a sequence

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$\preceq$ on runs is a wqo, so we choose an infinite subsequence, which is non decreasing wrt $\preceq$

An algebraic fact

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For every (possibly infinite) set of vectors $S \subseteq Z^d$ there exist finitely many vectors $v_1, v_2, v_3, \ldots, v_k \in S$, such that $S \subseteq \mathrm{Lin}(v_1, v_2, v_3, \ldots, v_k)$

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There is a finite set

$S_{fin} = \{x_{i1} - y_{i1}, x_{i2} - y_{i2}, \ldots, x_{ik} - y_{ik}\}$

such that for every i

$x_i - y_i = a_{i1}(x_{i1} - y_{i1}) + \ldots + a_{ik}(x_{ik} - y_{ik})$

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$\pitchfork$ $\pitchfork$

L_U L_V

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