Star exercise 2.2
Sketch of the solution

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Problem: short run in $\mathbb{N} \times \mathbb{Z}$-VAS
Consider a system called $\mathbb{N} \times \mathbb{Z}$-VAS defined by a set of transitions $T \subseteq \mathbb{Z}^2$. Configuration in this system is a vector $v \in \mathbb{N} \times \mathbb{Z}$, i.e. first coordinate has to be nonnegative, but second not necessarily. Transitions between configurations are defined as usual in VASes. Let M be the maximal absolute value of a number occurring in a transition. Show that if there is a path from $(0, 0)$ to $(0, 1)$ then there exists such a path of length at most polynomial in M.
Hint: Use the following Steinitz lemma. Let vectors $v_1, \ldots, v_n \in \mathbb{R}^d$ sum up to zero vector and all the coordinates of all v_i have absolute value at most 1 (i.e. their infinity norm is at most 1). Then there exists a permutation $v'_1, \ldots, v'_n$ of the above vectors such that every point reachable by a path starting in zero vector and firing consecutively vectors $v'_1, \ldots, v'_n$ has the infinite norm bounded by d.

Solution sketch
Assume we have some run ρ from $(0, 0)$ to $(0, 1)$ consisting of vectors $v_1, \ldots, v_n$. If we add at the end vector $v_{n+1} = (0, -1)$ then we get a path from $(0, 0)$ back to $(0, 0)$ (the last vector is not a VASS-transition, but we do not care about that). Now we can apply the Steinitz lemma from the hint. If vectors $v_1, \ldots, v_{n+1}$ have norm maximally M then we can rearrange them such that the path fits into a square with side $2Md = 4M$ around $(0, 0)$. This path can go below 0 on first coordinate, but we don't care at the moment about it. We delete the vector $(0, -1)$, which is on that path and get some path $v'_1, \ldots, v'_n$ starting in $(0, 0)$ and finishing in $(0, 1)$ in $\mathbb{Z} \times \mathbb{Z}$. It fits in the square with side $4M + 2$ (it could have moved by 1 as an effect of removing $(0, -1)$). Now we can shorten this path to the length maximally $(4M + 3)^2$ such that it traverses at most once via every point (we unpump all the loops). We now have a short path of polynomial length, the only problem with it is that it fits into $\mathbb{Z} \times \mathbb{Z}$, not into $\mathbb{N} \times \mathbb{Z}$. We will fix it. Let us sort vectors $v'_1, \ldots, v'_n$ so that the first coordinate decreases. We get a sequence $v''_1, \ldots, v''_n$ such that first vectors in the sequence have first coordinate positive, then maybe there are some with first coordinate equal zero and at the end we have vectors with first coordinate negative. It is not hard to see that path via $v''_1, \ldots, v''_n$ goes from $(0, 0)$ to $(0, 1)$ and fits into $\mathbb{N} \times \mathbb{Z}$, it never goes down on the first coordinate. This finishes the proof.