Regularity of Random Processes Using Net-Approximation

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Abstract

We prove some regularity results for processes which are defined on an arbitrary set $T$ equipped with completely bounded metrics $d_i$, $1 \leq i \leq n$, and have bounded increments i.e. we consider $X(t)$, $t \in T$ such that $E \inf_{1 \leq i \leq n} \varphi_i\left(\frac{|X(s) - X(t)|}{d_i(s,t)}\right) \leq 1$, where $\varphi_i$, $1 \leq i \leq n$ are Young functions of exponential growth. In the first part we prove that there exists a family of metrics $\tau_m$ on $T$ (indexed by sequences $m = (m_i)_{i=1}^n$ of probability measures, Borel in respect to $d_i$ topology) such that each process with bounded increments has its samples Lipschitz according to those metrics. The second part concerns the converse issue. Assuming that $d_i$, $1 \leq i \leq n$ are completely bounded on $T$ and there exists a metric $\rho$ such that each process with bounded increments has Lipschitz samples according to $\rho$, then there exists $\tau_m$ such that $\tau_m \leq K\rho$. We also show an application of the introduced technique to prove some already known results.

1 Introduction

Consider an arbitrary set $T$ equipped with separable metrics $d_i$, $1 \leq i \leq n$. We fix normalized Young functions $\varphi_i$, $1 \leq i \leq n$, i.e. convex, increasing, $\varphi_i(0) = 0$, $\varphi(1) = 1$, and say that process $X(t)$, $t \in T$ has bounded increments when it verifies

$$E \inf_{1 \leq i \leq n} \varphi_i\left(\frac{|X(s) - X(t)|}{d_i(s,t)}\right) \leq 1, \text{ for } s, t \in T. \quad (1)$$

If a process with bounded increments has almost surely bounded samples, then the metric $d = \max_{1 \leq i \leq n} d_i$ is completely bounded (cf. Theorem 2.3 in [7]). Consequently

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\(d_i, 1 \leq i \leq n\) are completely bounded and we do assume the property from now on. It follows that there exist compact closures \((\hat{T}_i, \hat{d}_i)\) of \((T, d_i)\) for all \(1 \leq i \leq n\) (by adding Cauchy sequences limits). Clearly each process with bounded increments is \(d\)-continuous in probability, so it has also a \(d\)-separable modification. We thus address to such modification whenever considering a process with bounded increments.

The idea of studying the sample boundedness of processes controlled by several metrics was started in [8] and is nicely expounded in the first chapter of Talagrand’s book [9]. This paper deals with some further questions in the theory, namely we study the continuity modulus of \(X(t), t \in T\) i.e. for each metric \(\rho\) on \(T\) we check the boundedness of \(\sup_{s,t \in T} \frac{|X(s) - X(t)|}{\rho(s,t)}\). Observe that the last variable is measurable if \(X\) is \(d\)-separable and the metric \(\rho\) is Borel as a function on \((T \times T, d \times d)\). In this sense we pursue the approach of Kwapien and Rosinski [5]. Throughout the paper we use the notation \(C(S, \rho)\) for the Banach space of all continuous and bounded functions on \((S, \rho)\).

First we consider the case \(n = 1\) in which the approximation nets can be used to recover some well-known results. Since there is only one completely bounded metric \(d_1 \equiv d\) and one normalized Young function \(\varphi_1 \equiv \varphi\) without a lost of generality we can assume that \((T, d)\) is compact. Define \(D(t, T)\) as the horizon at the point \(t \in T\), i.e. \(D(t, T) = \sup\{d(s, t) : s \in T\}\). Observe that process \(X(t), t \in T\) has bounded increments if there holds

\[
\mathbb{E}\varphi\left(\frac{|X(s) - X(t)|}{d(s, t)}\right) \leq 1, \quad \text{for } s, t \in T. \tag{2}
\]

Let \(m\) be a probability measure on \((T, d)\) such that \(\text{supp}(m) = T\), and \((N_k)_{k \geq 0}\) a sequence of real numbers which verify \(N_0 = 1\) and

\[
c\varphi^{-1}(N_k) \leq \varphi^{-1}(N_{k+1}) \leq C\varphi^{-1}(N_k), \tag{3}
\]

where \(1 < c \leq C < \infty\) (for our purposes we need that \(c > 5\)).

**Remark 1** The condition (3) holds for \(N_k = \varphi(R^k)\) and \(R > 1\). In this case we take \(c = C = R\).

We define

\[r_k(t) = \inf\{\varepsilon > 0 : m(B(t, \varepsilon)) \geq \frac{1}{N_k}\}.
\]

Obviously \(m(B(t, r_k(t))) \geq \frac{1}{N_k}\). In the sequel we prove several properties of functions \(r_k\), among which the following is of particular meaning.

**Lemma 1** Functions \(r_k, k \geq 0\) are 1-Lipschitz and \(r_0(t) = D(t, T)\) for all \(t \in T\).
Proof. Since the support of $m$ is the whole $T$ we have $r_0(t) = D(t, T)$, $t \in T$. Moreover the geometrical argument shows that

$$B(s, r_k(t) + d(s, t)) \supset B(t, r_k(t)),$$

consequently $m(B(s, r_k(t) + d(s, t))) \geq \frac{1}{N_k}$. Hence $r_k(s) \leq r_k(t) + d(s, t)$ and similarly $r_k(t) \leq r_k(s) + d(s, t)$ which implies that $r_k$ is 1-Lipschitz.

There is a close connection between those functions and majorizing measures. Let us define the following quantities

$$\sigma_{m, \varphi}(t) = \int_0^{D(t, T)} \varphi^{-1}(\frac{1}{m(B(t, \varepsilon))})d\varepsilon, \quad M(m, \varphi) = \sup_{t \in T} \sigma_{m, \varphi}(t),$$

$$\bar{M}(m, \varphi) = \int_T \sigma_{m, \varphi}(t)m(dt).$$

Measure $m$ is said to be majorizing if $M(m, \varphi)$ is finite, and weakly majorizing if $\bar{M}(m, \varphi) < \infty$. In the preliminaries the following assertion will be proved

Lemma 2 For each $0 \leq \delta \leq D(t, T)$ there holds

$$\sum_{k=0}^{\infty} \varphi^{-1}(N_k) \min\{r_k(t), \delta\} \leq \frac{c}{c-1} \int_0^{\delta} \varphi^{-1}(\frac{1}{m(B(t, \varepsilon))})d\varepsilon.$$

The main result of the first section is to recover Theorem 1.1 in [1].

Theorem 1 There exists a universal constant $K$ such that for each process $X(t)$, $t \in T$ which verify (2) the inequality holds true

$$E \sup_{s, t \in T} |X(s) - X(t)| \leq K M(m, \varphi).$$

We thus try to find a sufficient condition which imposed on $(T, d)$ would guarantee all processes with bounded increments to have bounded samples. The best characterization is certainly of 'if and only if' type, however it is done only for very few spaces $(T, d)$ (cf. [2] and [7] - section 5). One can even prove some on the regularity of samples in the arbitrary Young function setting (see [3]). However the situation is much simpler when we assume the Young function to be of exponential growth [4].

In the second pursuing the Fernique-Talagrand investigations, we consider the case of general $n$, but under the additional assumption that $\varphi_i, 1 \leq i \leq n$ verify the exponential growth condition, i.e. there exist $K_i \geq 1$ such that for all $x, y \geq 0$.

$$\varphi_i(x) \varphi_i(y) \leq \varphi_i(K_i(x + y)), \quad \text{or equivalently } \varphi_i^{-1}(xy) \leq K_i(\varphi_i^{-1}(x) + \varphi_i^{-1}(y)).$$
Define $B_i(t, \varepsilon) = \{ s \in T : d_i(s, t) \leq \varepsilon \}$ and $D_i(t, T) = \sup \{ d_i(s, t) : s \in T \}$. We also require that there exists an increasing sequence $(N_k)_{k \geq 0}$ of real numbers such that $N_0 = 1$, and
\begin{equation}
 c_i \varphi_i^{-1}(N_k) \leq \varphi_i^{-1}(N_{k+1}) \leq C_i \varphi_i^{-1}(N_k), \quad k \geq 0,
\end{equation}
where $1 < c_i \leq C_i < \infty$ (for our purposes we need that $c_i \geq 2K_i$, which implies $N_{k+1} \geq N_k^2$).

**Remark 2** The above condition is satisfied for $\varphi_i(x) = 2^{x^{p_i}} - 1$, $p_i \geq 1$. Since $\varphi_i^{-1}(x) = (\ln(1+x))^{\frac{1}{p_i}}$, thus taking $N_k = 2^{R^k} - 1$, $R > 1$ we have $\varphi_i^{-1}(N_k) = R^{\frac{1}{p_i}}$ and hence (5) holds with $c_i = C_i = R^{\frac{1}{p_i}}$. Moreover
\[ \varphi_i(x) \varphi_i(y) \leq 2^{x^{p_i}+y^{p_i}} - 1 \leq 2^{(x+y)^{p_i}} - 1 = \varphi_i(x+y), \]
which implies (4) with $K_i = 1$.

Consider sequences $m = (m_i)_{i=1}^n$ such that $m_i$ is a Borel probability measures on $(\hat{T}, \hat{d})$. The goal is to define metrics $\tau_m$ with values in $\hat{R} = \mathbb{R} \cup \{ \infty \}$ which we later use to control the continuity modulus of processes with bounded increments. For all $1 \leq i \leq n$ and $k \geq 0$ we denote
\[ r_k^i(t) := \inf \{ \varepsilon > 0 : \ m_i(B_i(t, \varepsilon)) \geq \frac{1}{N_k} \}, \quad t \in T. \]

**Definition 1** For each $l \in T$ and $t \in T$ we define $f_{l,m}^i : \mathbb{N} \times T \to \mathbb{R} \cup \{ \infty \}$
\[ f_{l,m}^i(p, s) := \max_{1 \leq i \leq n} d_i(s, t) \varphi_i^{-1}(N_p) + \sum_{k=1}^{l} \max_{1 \leq i \leq n} r_k^i(t) \varphi_i^{-1}(N_k), \quad \text{for } 0 \leq p \leq l, \]
and $f_{l,m}^i(p, s) = 0$ for $p > l$. Let $f_{l,m}^i(s) = \min_{0 \leq p \leq l} f_{l,m}^i(s, p)$ and $p_{l,m}^i(s)$ denotes the biggest value $p$ on which the minimum is attained.

**Lemma 3** The following assertions hold true:

1. For $t \in T$, and $s \neq t$ and $l \in \mathbb{N}$ there holds $p_{l,m}^i(s) \leq a_{l,m}(s) + 1$, where $a_{l,m}(s) = \max_{1 \leq i \leq n} d_i^l(s, t)$ and $a^i = a_{l,m}^i(s)$ is such that $r_{a^i+1}^i(t) < d_i(s, t) \leq r_{a^i}^i(t)$.

2. For each $t \in T$ there exist $d$-measurable limits $f_{l,m} = \lim_{l \to \infty} f_{l,m}^i$ and $p_{l,m} = \lim_{l \to \infty} p_{l,m}^i$. The limit $f_{l,m}$ is bounded on $T$ iff $\max_{1 \leq i \leq n} \sigma_{m_i, \phi_i} < \infty$. Moreover
\[ f_{l,m}(s) = \max_{1 \leq i \leq n} d_i(s, t) \varphi_i^{-1}(N_{p_{l,m}(s)}) + \sum_{k=1}^{p_{l,m}(s)} \max_{1 \leq i \leq n} r_k^i(t) \varphi_i^{-1}(N_k). \]
If $f_{t,m}$ is bounded then it additionally verifies

$$f_{t,m}(s) = \inf_{p \geq 0} \max_{1 \leq i \leq n} d_i(s,t)\varphi_i^{-1}(N_p) + \sum_{k=p}^{\infty} \max_{1 \leq i \leq n} r_k^i(t)\varphi_i^{-1}(N_k),$$

and $p_{t,m}(s)$ equals the biggest $p$ on which the infimum is attained.

3. The metric $\tau_m(s,t) := \max\{f_{t,m}(s), f_{s,m}(t)\}$ is well-defined and Borel as a function on $(T \times T, d \times d)$.

**Theorem 2** There exists a universal constant $K$ (depending on $\varphi_1, ..., \varphi_n$ only) such that for each sequence $m = (m_i)_{i=1}^n$ of probability measures on $T$ (resp. Borel in $d_i$-topology) there holds

$$E \inf_{1 \leq i \leq n} \varphi_i(\sup_{s,t \in T} \frac{|X(s) - X(t)|}{K\tau_m(s,t)}) \leq 1,$$

for all processes $X(t), t \in T$ that verify (2).

**Theorem 3** If there exists a metric $\rho$ on $T$ Borel as a function on $(T \times T, d \times d)$ such that for all processes $X(t), t \in T$ which verify (2) there holds

$$\frac{|X(s) - X(t)|}{\rho(s,t)} < \infty, \text{ a.e.}$$

then there exist a constant $K$ and a sequence $m = (m_i)_{i=1}^n$ of probability measures (resp. Borel in $d_i$-topology), such that $\tau_m(s,t) \leq K\rho(s,t)$, for all $s, t \in T$.

**Corollary 1** If all processes $X(t), t \in T$ which verify (2) are sample bounded then there exist $K$ and $m = (m_i)_{i=1}^n$ (as in the above theorem) such that $\tau_m \leq K$.

**Proof.** It is enough to take in Theorem 3 the simplest metric $\rho(s,t) = 1$ if $s \neq t$ and $\rho(s,t) = 0$ if $s = t$ and observe that it is Borel as a function on $T \times T$ with $d \times d$ topology.

**2 Preliminary results**

In the section we prove the announced background results.

**Proof.** of Lemma 2] Let us observe that there exists $k_0$ such that $r_{k_0+1}(t) < \delta \leq r_{k_0}(t)$. Clearly

$$\int_{r_{k+1}(t)}^{r_k(t)} \varphi^{-1}(\frac{1}{m(B(t,\varepsilon))})d\varepsilon \geq (r_k(t) - r_{k+1}(t))\varphi^{-1}(N_k).$$
In the same way we have that
\[
\int_{r_{k_0+1}(t)}^{\delta} \varphi^{-1}(\frac{1}{m(B(t, \varepsilon))})d\varepsilon \geq (\delta - r_{k_0+1}(t))\varphi^{-1}(N_{k_0}).
\]
Thus using (3) we deduce
\[
\int_{0}^{\delta} \varphi^{-1}(\frac{1}{m(B(t, \varepsilon))})d\varepsilon \geq \sum_{k=1}^{\infty} (r_k(t) - r_{k+1}(t))\varphi^{-1}(N_k) + (\delta - r_{k_0+1}(t))\varphi^{-1}(N_{k_0}) \geq
\]
\[
\geq \sum_{k=k_0+1}^{\infty} r_k(t)(\varphi^{-1}(N_k) - \varphi^{-1}(N_{k-1})) + \delta\varphi^{-1}(N_{k_0}) \geq
\]
\[
\geq \sum_{k=k_0+1}^{\infty} r_k(t)(1 - c^{-1})\varphi^{-1}(N_k) + \delta\varphi^{-1}(N_{k_0}).
\]
Since
\[
\sum_{k=0}^{k_0} \varphi^{-1}(N_k) \leq \sum_{k=0}^{k_0} e^{-k} \varphi^{-1}(N_{k_0}) \leq \frac{1}{1 - c^{-1}} \varphi^{-1}(N_{k_0})
\]
we finally obtain
\[
\int_{0}^{\delta} \varphi^{-1}(\frac{1}{m(B(t, \varepsilon))})d\varepsilon \geq (1 - c^{-1}) \sum_{k=0}^{\infty} \varphi^{-1}(N_k) \min\{r_k(t), \delta\}.
\]
\[
\square
\]
**Proof.** [of Lemma 3] 1. Fix \( t \in T, \) and \( s \neq t. \) It is enough to prove the assertion for all \( l > a_{m,t}(s). \) Denoting \( a = a_{m,t}(s) \) we have for each \( 1 \leq k \leq l - a \)
\[
f_{t,m}(a + k, s) - f_{t,m}(a + 1, s) = \max_{1 \leq i \leq n} d_i(s, t)\varphi_i^{-1}(N_{a+k}) - \max_{1 \leq i \leq n} d_i(s, t)\varphi_i^{-1}(N_{a+1}) -
\]
\[
- \sum_{1 \leq i \leq n}^{k-1} \max_{1 \leq i \leq n} r_{a+i}(t)\varphi_i^{-1}(N_{a+i}).
\]
By the definition of \( a \) we have \( r_{a+i}(t) < d_i(s, t), \) for all \( j \geq 1 \) and hence
\[
\sum_{j=1}^{k-1} \max_{1 \leq i \leq n} r_{a+i}(t)\varphi_i^{-1}(N_{a+j}) < \max_{1 \leq i \leq n} d_i(s, t) \sum_{j=1}^{k-1} \varphi_i^{-1}(N_{a+j}).
\]
Due to (5) we obtain \( \varphi_i^{-1}(N_{a+j}) \leq \frac{1}{2} \varphi_i^{-1}(N_{a+j+1}), \) (we use here that \( c_i \geq 2 \),

\[
\max_{1 \leq i \leq n} d_i(s, t)\varphi_i^{-1}(N_{a+1}) + \sum_{l=1}^{k-1} \max_{1 \leq i \leq n} r_{a+i}(t)\varphi_i^{-1}(N_{a+l}) <
\]
\[
< \max_{1 \leq i \leq n} d_i(s, t)(\varphi_i^{-1}(N_{a+1}) + \sum_{j=1}^{k-1} \varphi_i^{-1}(N_{a+j})) \leq \max_{1 \leq i \leq n} d_i(s, t)\varphi_i^{-1}(N_{a+k}).
\]

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Using the above in (6) we thus obtain $f^l_{t,m}(a+k,s) - f^l_{t,m}(a+1,s) > 0$ and consequently $0 \leq p_{t,m}(s) \leq a + 1$ (by Definition 1).

2. Fix $t \in T$. Observe first that $f^l_{t,m}(t) = f^l_{t,m}(l,t) = 0$, $p^l_{t,m}(t) = l$ so $f_{t,m}(t) = 0$ and $p_{t,m}(t) = \infty$. Consider $s \neq t$. Denoting $a = a_{t,m}(s)$ we use the previous assertion to get

$$f^l_{t,m}(s) = \min_{0 \leq p \leq l(s+1)} f^l_{t,m}(p,s).$$

(7)

Hence $f^l_{t,m}(s)$ is increasing for $l > a$, what gives the existence of $f_{t,m}(s) = \lim_{l \to \infty} f^l_{t,m}(s)$. We thus proved that $f_{t,m} = \lim_{l \to \infty} f^l_{t,m}$ is well defined and it is obviously $d$-measurable as the limit of $d$-continuous functions. Due to (7) and Definition 1 we have $p^l_{t,m}(s) = p^{l+1}_{t,m}(s)$, for all $l > a$. Hence there exists $p_{t,m}(s) = \lim_{l \to \infty} p^l_{t,m}(s)$ what together with $p_{t,m}(t) = \infty$ completes the proof of $p_{t,m} = \lim_{l \to \infty} p^l_{t,m}$ existence. One can see that

$$f_{t,m}(s) \leq \lim_{l \to \infty} f^l_{t,m}(0,s) \leq \sum_{i=1}^{n} (D_i(t,T) + \sum_{k=0}^{\infty} r^l_k(t) \phi_i^{-1}(N_k)),$n

thus $f_{t,m}$ is bounded if $\sigma_{m,\phi_i}(t) < \infty$ for all $1 \leq i \leq n$ (by Lemma 2 only then $\sum_{k=0}^{\infty} r^l_k(t)$ is convergent). Conversely if $\sigma_{m,\phi_i}(t) = \infty$ for some $i \in \{1, ..., n\}$ then $\sum_{k=0}^{\infty} \max_{1 \leq i \leq n} r^l_k(t) \phi_i^{-1}(N_k) = \infty$ and consequently by (7) we have

$$f_{t,m}(s) \geq \sum_{k=1}^{\infty} \max_{1 \leq i \leq n} r^l_k(t) \phi_i^{-1}(N_k) = \infty.$$

As we have observed $p^l_{t,m}(s) = p_{t,m}(s)$ for $l > a$, which implies that $f^l_{t,m}(s) = f^l_{t,m}(a,s)$. Thus by Definition 1 we obtain both

$$f_{t,m}(s) = \max_{1 \leq i \leq n} d_i(s,t) \phi_i^{-1}(N_{p_{t,m}(s)}) + \sum_{k=p_{t,m}(s)}^{\infty} \max_{1 \leq i \leq n} r^l_k(t) \phi_i^{-1}(N_k).$$

and in the case when $f_{t,m}$ is bounded

$$f_{t,m}(s) = \inf_{p \geq 0} \left( \max_{1 \leq i \leq n} d_i(s,t) \phi_i^{-1}(N_p) + \sum_{k=p}^{\infty} \max_{1 \leq i \leq n} r^l_k(t) \phi_i^{-1}(N_k) \right).$$

Moreover $p_{t,m}(s)$ equals the biggest value $p$ on which the infimum is attained.

3. By the definition $\tau_m(s,t) = \tau_m(t,s)$ and $\tau(s,t) = 0$ iff $s = t$ it remains to show that $\tau_m$ verifies the triangle inequality i.e. for fixes $s, t, u \in T$ we prove that $\tau_m(s,t) \leq \tau_m(s,u) + \tau_m(u,t)$. Without a lost of generality we can assume that $f_{s,m}, f_{t,m}, f_{t,m}$ are bounded. There are two cases: first when $p_{t,m}(u) \leq p_{u,m}(s)$. Clearly

$$f_{t,m}(s) \leq f_{t,m}(p_{t,m}(u),s) = \max_{1 \leq i \leq n} d_i(s,t) \phi_i^{-1}(N_{p_{t,m}(u)}) + \sum_{k=p_{t,m}(u)}^{\infty} \max_{1 \leq i \leq n} r^l_k(t) \phi_i^{-1}(N_k).$$

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Due to the triangle inequality \( d_i(s, t) \leq d_i(s, u) + d_i(u, t) \) we obtain
\[
 f_{t,m}(s) \leq f_{t,m}(u) + \max_{1 \leq i \leq n} d_i(s, u)\varphi_i^{-1}(N_{p_{i,m}(u)}). 
\]
Applying \( p_{t,m}(u) \leq p_{u,m}(s) \) we have
\[
 f_{t,m}(s) \leq f_{t,m}(u) + f_{u,m}(s) \leq \tau_m(t, u) + \tau_m(u, s).
\]
The second case is when \( p_{u,m}(s) \leq p_{t,m}(u) \). One can see
\[
 f_{t,m}(s) \leq f_{t,m}(p_{u,m}(s), u) = \max_{1 \leq i \leq n} d_i(s, t)\varphi_i^{-1}(N_{p_{i,m}(s)}) + \sum_{k=p_{u,m}(s)}^{\infty} \max_{1 \leq i \leq n} r_k(t)\varphi_i^{-1}(N_k).
\]
Due to Lemma 1 we know that \( r_k(t) \leq r_k(t) + d_i(u, t) \), so
\[
 f_{t,m}(s) \leq \max_{1 \leq i \leq n} d_i(s, t)\varphi_i^{-1}(N_{p_{i,m}(s)}) + \sum_{k=p_{u,m}(s)}^{\infty} \max_{1 \leq i \leq n} d_i(t, u)\varphi_i^{-1}(N_k) + \\
+ \sum_{k=p_{t,m}(u)}^{\infty} \max_{1 \leq i \leq n} r_k(t)\varphi_i^{-1}(N_k) + \sum_{k=p_{u,m}(s)}^{\infty} \max_{1 \leq i \leq n} r_k(u)\varphi_i^{-1}(N_k).
\]
By the triangle inequality \( d_i(s, t) \leq d_i(s, u) + d_i(u, t) \) and since \( p_{u,m}(s) \leq p_{t,m}(u) \) we have
\[
 \max_{1 \leq i \leq n} d_i(s, t)\varphi_i^{-1}(N_{p_{i,m}(s)}) \leq \max_{1 \leq i \leq n} (d_i(s, u) + d_i(t, u))\varphi_i^{-1}(N_{p_{i,m}(s)}) \leq \\
\leq \max_{1 \leq i \leq n} d_i(s, u)\varphi_i^{-1}(N_{p_{u,m}(s)}) + \max_{1 \leq i \leq n} d_i(t, u)\varphi_i^{-1}(N_{p_{t,m}(u)})
\]
From \( \varphi_i^{-1}(N_k) \leq \frac{1}{2}\varphi_i^{-1}(N_{k+1}) \) we deduce
\[
 \max_{1 \leq i \leq n} d_i(t, u)\varphi_i^{-1}(N_{p_{i,m}(s)}) + \sum_{k=p_{t,m}(u)}^{\infty} \max_{1 \leq i \leq n} d_i(t, u)\varphi_i^{-1}(N_k) \leq \\
\leq \max_{1 \leq i \leq n} d_i(t, u)\varphi_i^{-1}(N_{p_{u,m}(s)}) + \sum_{k=p_{u,m}(s)}^{\infty} \varphi_i^{-1}(N_k) \leq \max_{1 \leq i \leq n} d_i(t, u)\varphi_i^{-1}(N_{p_{u,m}(s)}).
\]
Consequently
\[
 f_{t,m}(s) \leq f_{t,m}(u) + \max_{1 \leq i \leq n} d_i(s, u)\varphi_i^{-1}(N_{p_{i,m}(s)}) + \sum_{k=p_{u,m}(s)}^{\infty} r_k(t)\varphi_i^{-1}(N_k) \leq \\
\leq f_{t,m}(u) + f_{u,m}(s) \leq \tau_m(u, t) + \tau_m(s, u).
\]
Similarly we can prove that \( f_{s,m}(t) \leq \tau_m(s, u) + \tau_m(u, t) \), and this completes the proof of \( \tau_m(s, t) \leq \tau_m(s, u) + \tau_m(u, t) \).
3 The case of one metric revisited

We first prove the stronger version of Lemma 2.

**Lemma 4** For each \( D < c, 0 \leq m \leq l \)

\[
\sum_{k=m}^{l} \left( \sum_{i=k}^{l} D^{i-k} r_i(t) \right) \varphi^{-1}(N_k) \leq \frac{c}{c-D} \sum_{k=m}^{\infty} r_k(t) \varphi^{-1}(N_k).
\]

**Proof.** We shall use \( \varphi^{-1}(N_k) \leq c^{k-i} \varphi^{-1}(N_i) \)

\[
\sum_{k=m}^{l} \left( \sum_{i=k}^{l} D^{i-k} r_i(t) \right) \varphi^{-1}(N_k) \leq \sum_{k=m}^{l} \left( \sum_{i=k}^{l} D^{i-k} c^{k-i} r_i(t) \varphi^{-1}(N_i) \right) \leq \sum_{i=m}^{l} r_i(t) \varphi^{-1}(N_i) \left( \sum_{j=0}^{\infty} D^j c^{-j} \right) \leq \frac{c}{c-D} \sum_{i=m}^{l} r_i(t) \varphi^{-1}(N_i).
\]

\[\square\]

Now we skip to the approximation net construction.

**Theorem 4** There exists a sequence of nets \((T_k)_{k \geq 0}, T_k \subset T\) which satisfies the following conditions:

1. \(|T_0| = 1, |T_k| \leq N_k,\)
2. for each \( t \in T \) there exists \( \pi_k(t) \in T_k \) such that \( d(t, \pi_k(t)) \leq 4r_k(t),\)
3. \( r_k(t) \leq 2r_k(x) \) for each \( t \in T_k \) and \( x \in B(t, r_k(t)) \).

**Proof.** For each \( k \geq 0 \) let us define \( t_1 \) as the argument minimum of function \( r_k \). Then we define the following open set in \( T \)

\[ A_1 := \{ s \in T : 2(r_k(s) + r_k(t_1)) > d(t, t_1) \}. \]

Suppose we have constructed points \( t_1, \ldots, t_l \) and open sets \( A_1, \ldots, A_l \). If \( T \setminus \bigcup_{j=1}^{l} A_j \) is non-empty, then we define \( t_{l+1} \) as the argument minimum of \( r_k \) on this set, moreover we put

\[ A_{l+1} := \{ s \in T : 2(r_k(s) + r_k(t_{l+1})) > d(t, t_{l+1}) \}. \]

Observe that the construction necessarily stops after no more then \( N_k \) steps. Indeed let us notice that balls \( B(t_j, r_k(t_j)) \) and \( B(t_l, r_k(t_l)) \) are disjoint whenever \( j \neq l \) \((d(t_j, t_l) \geq 2(r_k(t_j) + r_k(t_l)))\). Consequently

\[ 1 = m(T) \geq \sum_{j=1}^{\left\lfloor \frac{|T_k|}{N_k} \right\rfloor} m(B(t_j, r_k(t_j))) \geq \frac{|T_k|}{N_k}. \]
and thus \(|T_k| \leq N_k\). Clearly \(|T_0| = 1\) and for each \(t \in T\) we can define \(\pi_k(t) = t_{j_0}\), where \(j_0 \in \mathbb{N}\) is the lowest \(j\) such that \(t \in A_j\). By the construction \(2(r_k(t) + r_k(\pi_k(t))) > d(t, \pi_k(t))\) and \(r_k(\pi_k(t)) \leq r_k(t)\), hence \(d(t, \pi_k(t)) < 4r_k(t)\).

To prove the last issue for each \(x \in B(t, r_k(t))\), where \(t \in T_k\). Consider \(\pi_k(x)\), obviously \(t = t_l, \pi_k(x) = t_{j_0}\) for some \(1 \leq j_0, l \leq |T_k|\). If \(l \leq j_0\) then \(r_k(t) \leq r_k(\pi_k(x)) \leq r_k(x)\). If \(j_0 < l\), then \(t \notin \bigcup_{j=1}^{j_0} A_j\) and thus \(d(\pi_k(x), t) \geq 2(r_k(t) + r_k(\pi_k(x)))\), hence by the triangle inequality

\[
2(r_k(t) + r_k(\pi_k(x))) \leq d(\pi_k(x), t) \leq d(x, \pi_k(x)) + d(x, t) \leq d(x, \pi_k(x)) + r_k(t),
\]

where the last is because \(x \in B(t, r_k(t))\). On the other hand \(x \in A_{j_0}\), so

\[
d(x, \pi_k(x)) < 2(r_k(x) + r_k(\pi_k(x))).
\]

It proves that

\[
2(r_k(t) + r_k(\pi_k(x))) \leq d(x, \pi_k(x)) + r_k(t) \leq 2(r_k(x) + r_k(\pi_k(x))) + r_k(t),
\]

hence \(r_k(t) \leq 2r_k(x)\), whenever \(x \in B(t, r_k(t))\). \(\blacksquare\)

Observe that \(\pi_0(t)\) does not depend on \(t\) (since \(|T_0| = 1\)). We prove now the majorizing measure theorem (cf. Theorem 1.2. [1]).

**Theorem 5** There exists a probability measure \(\nu\) on \((T \times T, d \times d)\), for each \(d\)-continuous function \(f\) on \(T\) there holds

\[
\sup_{s, t \in T} |f(s) - f(t)| \leq A(\sigma_{m, \varphi}(t) + \sigma_{m, \varphi}(s)) + B\mathcal{M}(m, \varphi) \int_{T \times T} \varphi\left(\frac{|f(u) - f(v)|}{d(u, v)}\right) \nu(du, dv),
\]

where \(A, B > 0\) are universal constants.

**Proof.** For the proof we need that \(c > 5\). Fix \(l \geq 0, t \in T\). We define \(t_l = \pi_l(t)\) and by the reverse induction \(t_k = \pi_k(t_{k+1})\). Due to Theorem 4 we have \(d(t_k, t_{k+1}) \leq 4r_k(t_{k+1})\). We give some estimation on \(d(t, t_k)\). Observe that

\[
d(t, t_k) \leq d(t, t_{k+1}) + d(t_{k+1}, t_k) \leq d(t, t_{k+1}) + 4r_k(t_{k+1}).
\]

But \(r_k\) is 1-Lipschitz (Lemma 1), so \(r_{k+1}(t_{k+1}) \leq r_k(t) + d(t, t_{k+1})\) and therefore

\[
d(t, t_k) \leq 5d(t, t_{k+1}) + 4(1 + \varepsilon)r_k(t).
\]

By the induction \(d(t, t_{k+1}) \leq \sum_{i=k}^{l} 5^{i+1-k} r_i(t)\). Thus by Lemma 2 and 4 we deduce

\[
\sum_{k=0}^{l} d(t, t_k)\varphi^{-1}(N_k) \leq \frac{c}{c - 5} \sum_{k=0}^{l} r_k(t)\varphi^{-1}(N_k) \leq \frac{c^2}{(c - 1)(c - 5)} \sigma_{m, \varphi}(t).
\]
and by the triangle inequality
\[
\sum_{k=0}^{l-1} d(t_k, t_{k+1}) \varphi^{-1}(N_k) \leq \frac{2\epsilon^2}{(c - 1)(c - 5)} \sigma(m, \varphi)(t). \tag{8}
\]

Since \( t_0 = \pi_0(t) \) the chain argument implies
\[
|f(t_i) - f(t_0)| \leq \sum_{k=0}^{l-1} |f(t_k) - f(t_{k+1})|. \tag{9}
\]

For all Young functions \( \varphi \) the following inequality holds true
\[
\frac{x}{y} \leq 1 + \frac{\varphi(x)}{\varphi(y)}, \ x, y > 0.
\]

Taking \( x = \frac{|f(t_k) - f(t_{k+1})|}{d(t_k, t_{k+1})} \), \( y = \varphi^{-1}(N_{k+1}) \) we obtain
\[
\frac{|f(t_k) - f(t_{k+1})|}{d(t_k, t_{k+1})\varphi^{-1}(N_{k+1})} \leq 1 + \frac{1}{N_{k+1}} \varphi(\frac{|f(t_k) - f(t_{k+1})|}{d(t_k, t_{k+1})}).
\]

Due to (3) we have \( \varphi^{-1}(N_{k+1}) \leq C \varphi^{-1}(N_k) \), thus
\[
|f(t_k) - f(t_{k+1})| \leq Cd(t_k, t_{k+1})\varphi^{-1}(N_k)(1 + \frac{1}{N_{k+1}} \varphi(\frac{|f(t_k) - f(t_{k+1})|}{d(t_k, t_{k+1})})�).
\]

Together with (8) and (9) it proves that
\[
|f(t_i) - f(t_0)| \leq \sum_{k=0}^{l-1} Cd(t_k, t_{k+1})\varphi^{-1}(N_k)(1 + \frac{1}{N_{k+1}} \varphi(\frac{|f(t_k) - f(t_{k+1})|}{d(t_k, t_{k+1})}) ≤
\]
\[
\leq A \sigma_{m, \varphi}(t) + C \sum_{k=0}^{l-1} \frac{d(t_k, t_{k+1})\varphi^{-1}(N_k)}{N_{k+1}} \varphi(\frac{|f(t_k) - f(t_{k+1})|}{d(t_k, t_{k+1})}) ≤
\]
\[
\leq A \sigma_{m, \varphi}(t) + C \sum_{k=0}^{\infty} \sum_{s \in T_{k+1}} d(s, \pi_k(s))\varphi^{-1}(N_k) \varphi(\frac{|f(s) - f(\pi_k(s))|}{d(s, \pi_k(s))}),
\]
\[
\text{where } A = \frac{2\epsilon^2 C}{(c - 1)(c - 5)}. \text{ Finally define measure } \nu \text{ on } T \times T
\]
\[
\nu := C \sum_{k=0}^{\infty} \sum_{s \in T_{k+1}} \frac{d(s, \pi_k(s))\varphi^{-1}(N_k)}{N_{k+1}} \delta_{(s, \pi_k(s))}.
\]

Consequently
\[
|f(t_i) - f(t_0)| ≤ A \sigma_{m, \varphi}(t) + \int_{T \times T} \varphi(\frac{|f(u) - f(v)|}{d(u, v)}) \nu(du, dv).
\]
It remains to show that \( \nu \) is bounded. Observe firstly that \( m(B(s, r_{k+1}(s))) \geq \frac{1}{N_{k+1}} \), hence

\[
\nu(1) \leq C \sum_{k=0}^{\infty} \sum_{s \in T_{k+1}} d(s, \pi_k(s)) \varphi^{-1}(N_k)m(B(s, r_{k+1}(s))) = \\
= C \sum_{k=0}^{\infty} \sum_{s \in T_{k+1}} d(s, \pi_k(s)) \varphi^{-1}(N_k) \int_{B(s, r_{k+1}(s))} 1m(dx).
\]

By Theorem 4 we have \( d(s, \pi_k(s)) \leq 4r_k(s) \), moreover \( r_k \) is 1-Lipschitz so \( r_k(s) \leq r_k(x) + r_{k+1}(s) \) for each \( x \in B(s, r_{k+1}(s)) \). Again by Theorem 4 we have \( r_{k+1}(s) \leq 2r_{k+1}(x) \) for \( x \in B(s, r_{k+1}(s)) \), \( s \in T_{k+1} \), hence for all \( x \in B(s, r_{k+1}(s)) \)

\[
d(s, \pi_k(s)) \leq 4r_k(s) \leq 4(r_k(x) + r_{k+1}(s)) \leq 4(r_k(x) + 2r_{k+1}(x)).
\]

It gives that \( \nu(1) \leq 4C \sum_{k=0}^{\infty} \sum_{s \in T_{k+1}} \int_{B(s, r_{k+1}(s))} (r_k(x) + 2r_{k+1}(x)) \varphi^{-1}(N_k)m(dx) \). Due to Theorem 4 balls \( B(s, r_{k+1}(s)) \), \( s \in T_{k+1} \) are disjoint so

\[
\nu(1) \leq 4C \sum_{k=0}^{\infty} \sum_{s \in T_{k+1}} \int_{B(s, r_{k+1}(s))} (r_k(x) + 2r_{k+1}(x)) \varphi^{-1}(N_k)m(dx) \leq \\
\leq 4C \sum_{k=0}^{\infty} \int_T (r_k(x) + 2r_{k+1}(x)) \varphi^{-1}(N_k)m(dx) \leq 12C \int_T \sum_{k=0}^{\infty} r_k(x) \varphi^{-1}(N_k)m(dx).
\]

Using Lemma 2 we thus obtain

\[
\nu(1) \leq \frac{12C}{c-1} \int_T \sigma_{m, \varphi}(t)m(dt) = \frac{1}{2}B\tilde{\lambda}(m, \varphi),
\]

where \( B = \frac{2kcC}{c-1} \). Tending with \( l \) to infinity we see

\[
|f(t) - f(\pi_0(t))| \leq A \sigma_{m, \varphi}(t) + \frac{1}{2}B\tilde{\lambda}(m, \varphi) \int_{T \times T} \varphi\left(\frac{|f(u) - f(v)|}{d(u, v)}\right)\nu(du, dv).
\]

Using the triangle inequality we end the proof of Theorem 1

\[
|f(s) - f(t)| \leq A(\sigma_{m, \varphi}(t) + \sigma_{m, \varphi}(s)) + B\tilde{\lambda}(m, \varphi) \int_{T \times T} \varphi\left(\frac{|f(u) - f(v)|}{d(u, v)}\right)\nu(du, dv).
\]

A simple consequence of the above result is Theorem 1.

**Proof.**[of Theorem 1] We use the fact that

\[
E \sup_{s, t \in T} |X(s) - X(t)| = \sup_{F \subset T} E \sup_{s, t \in F} |X(s) - X(t)|,
\]

12
where the supremum is taken over all finite subsets of $T$. Consequently we need to prove Theorem 1 for each finite subset $F \subset T$. Following the proof of Theorem 11.9 from [6] we find that there exists a probability measure $m^F$ supported on $F$ such that $m(B(t, \varepsilon)) \leq m^F(B(t, 2\varepsilon))$. Therefore $\mathcal{M}(m^F, \varphi) \leq 2\mathcal{M}(m, \varphi)$, $\bar{\mathcal{M}}(m^F, \varphi) \leq 2\bar{\mathcal{M}}(m, \varphi)$. Applying Theorem 5 and the Fubini Theorem we obtain

\[
\mathbf{E} \sup_{s,t \in F} |X(s) - X(t)| \leq A\mathcal{M}(m^F, \varphi) + B\bar{\mathcal{M}}(m^F, \varphi) \mathbf{E} \int_{T \times T} \varphi\left(\frac{|X(u) - X(v)|}{d(u, v)}\right) \nu(du, dv) \leq 2(A\mathcal{M}(m, \varphi) + B\bar{\mathcal{M}}(m, \varphi))
\]

\[\blacksquare\]

4 The estimation from above

We start from the approximation net construction adapted to the general case.

**Theorem 6** Fix $\varepsilon > 0$. There exists a sequence of nets $(T_k)_{k \geq 0}$, $T_k \subset T$ which verify the following conditions:

1. $|T_0| = 1$, $|T_k| \leq N^n_k$,
2. for each $t \in T$ there exists $\pi_k(t) \in T_k$ such that

\[
\max_{1 \leq i \leq n} \varphi^{-1}_i(N_k)d_i(t, \pi_k(t)) \leq (2 + \varepsilon) \max_{1 \leq i \leq n} \varphi^{-1}_i(N_k)r^i_k(t).
\]

**Proof.** Fix $k \geq 0$. Due to Lemma 1 we know that $r^i_k$ is $d_i$-continuous on $T$ for all $1 \leq i \leq n$, and hence $g_k(t) := \max_{1 \leq i \leq n} r^i_k(t)\varphi^{-1}_i(N_k)$ is $d$-continuous. We define $t_1 \in T$ as an $\varepsilon$-argument minimum of $g_k$ on $T$, i.e. a point which verifies the inequality $g_k(t_1) \leq (1 + \varepsilon) \inf_{t \in T} g_k(t)$. Then we define the following $d$-closed subset in $T$

\[A_1 := \{t \in T : r^i_k(t) + r^i_k(t_1) \geq d_i(t_1, t), \text{ for all } 1 \leq i \leq n\}\]

Suppose we have constructed $t_1, ..., t_l$ and $d$-closed sets $A_1, ..., A_l$. Since $T \setminus \bigcup_{j=1}^{l} A_j$ is $d$-open we can define $t_{l+1} \in T \setminus \bigcup_{j=1}^{l} A_j$ as an $\varepsilon$-argument minimum of $g_k$ on $T \setminus \bigcup_{j=1}^{l} A_j$, i.e. a point which verifies $g_k(t_{l+1}) \leq (1 + \varepsilon) \inf_{t \in T \setminus \bigcup_{j=1}^{l} A_j} g_k(t)$. We put also

\[A_{l+1} := \{t \in T : r^i_k(t) + r^i_k(t_{l+1}) \geq d_i(t, t_{l+1}), \text{ for all } 1 \leq i \leq n\}\]

We show why the procedure ends after no more then $N^n_k$ steps. The reason is of volume type, for each $t \in T_k$ we define the following subset

\[B_k(t) = B_1(t, r^1_k(t)) \times B_2(t, r^2_k(t)) \times ... \times B_n(t, r^n_k(t)) \subset T^n.\]
By the definition if \( t_j, t_i \in T_k \) and \( j \neq l \) then there exists at least one value one value (e.g. \( 1 \leq i_0 \leq n \)) such that

\[
r_k^{i_0}(t_j) + r_k^{i_0}(t_i) < d_{i_0}(t_j, t_i)
\]

Consequently balls \( B_{i_0}(t_j, r_k^{i_0}(t_j)) \) and \( B_{i_0}(t_i, r_k^{i_0}(t_i)) \) are disjoint and as a result \( B_k(t_j) \) and \( B_k(t_i) \) are disjoint. If we consider measure \( m^n = m_1 \otimes m_2 \otimes \ldots \otimes m_n \) on \( \hat{T}_1 \times \ldots \times \hat{T}_n \) then by the definition of \( r_k^n \)

\[
m^n(B_k(t)) = \prod_{i=1}^{n} m_i(B_i(t, r_k^i(t))) \geq \prod_{i=1}^{n} \frac{1}{N_k} = \frac{1}{N_k^n}, \quad \text{for } t \in T.
\]

Thus

\[
1 = m^n(\hat{T}_1 \times \ldots \hat{T}_n) \geq \sum_{t \in T_k} m^n(B_k(t)) \geq \frac{|T_k|}{N_k^n},
\]

which gives \( |T_k| \leq N_k^n \). Clearly \( |T_0| = 1 \) and for each \( t \in T \) we can define \( \pi_k(t) := t_{j_0} \), where \( j_0 = \min\{j \in \mathbb{N} : t \in A_j\} \). By the construction

\[
\max_{1 \leq i \leq n} r_k^i(\pi_k(t)) \varphi_i^{-1}(N_k) \leq (1 + \varepsilon) \max_{1 \leq i \leq n} r_k^i(t) \varphi_i^{-1}(N_k)
\]

and for all \( i \) we have \( d_i(t, \pi_k(t)) \leq r_k^i(t) + r_k^i(\pi_k(t)) \), thus

\[
\max_{1 \leq i \leq n} d_i(t, \pi_k(t)) \varphi_i^{-1}(N_k) \leq (2 + \varepsilon) \max_{1 \leq i \leq n} r_k^i(t) \varphi_i^{-1}(N_k).
\]

\[\blacksquare\]

**Theorem 7** There exists a probability measure \( \nu \) on \((T \times T, d \times d)\) such that for each \( d\)-continuous function \( f \) on \( T \) the following inequality holds

\[
\sup_{s, t \in T} \frac{|f(s) - f(t)|}{K \tau_{m}(s, t)} \leq 1 + \int_{T \times T} \inf_{1 \leq i \leq n} \varphi_i(\frac{|f(u) - f(v)|}{d_i(u, v)}) \nu(du, dv).
\]

where \( K > 0 \) depends on \( \varphi_1, ..., \varphi_n \) only.

**Proof.** We apply Theorem 6 for fixed \( \varepsilon > 0 \), thus there exists an approximating net \((T_k)_{k \geq 0}\) which satisfies \( |T_0| = 1, \ |T_k| \leq N_k^n \) and such that for each \( x \in T \) there exists \( \pi_k(x) \in T_k \)

\[
\max_{1 \leq i \leq n} \varphi_i^{-1}(N_k)d_i(x, \pi_k(x)) \leq (2 + \varepsilon) \max_{1 \leq i \leq n} \varphi_i^{-1}(N_k)r_k^i(x).
\]

Denote for simplicity \( x_k = \pi_k(x) \) for all \( x \in T \). Observe that due to (10) and \( \text{supp}(m_i) = \hat{T}_i \) we have that \( \lim_{k \to \infty} \pi_i(t) = t \) in \( d\)-topology. We fix \( s, t \in T \) and consider the chaining

\[
|f(s) - f(t)| \leq \sum_{k=p}^{\infty} |f(t_k) - f(t_{k+1})| + \sum_{k=p}^{\infty} |f(s_k) - f(s_{k+1})| + |f(s_p) - f(t_p)|,
\]

\[\text{for } s, t \in T, d \geq 0, p \in \mathbb{N} \text{ and } s = (s_p, s_{p-1}, \ldots, s_1), t = (t_p, t_{p-1}, \ldots, t_1).\]
where \( p = \max\{p_{t,m}(s), p_{s,m}(t)\} \). Observe that taking \( x = \frac{|f(t_k) - f(t_{k+1})|}{d_i(t_k, t_{k+1})} \), \( y = \varphi_i^{-1}(N_{k+1}^n) \)

in the inequality

\[
\frac{x}{y} \leq 1 + \frac{\varphi_i(x)}{\varphi_i(y)}, \quad x, y > 0,
\]

we obtain

\[
\frac{|f(t_k) - f(t_{k+1})|}{d_i(t_k, t_{k+1})\varphi_i^{-1}(N_{k+1}^n)} \leq 1 + \frac{1}{N_{k+1}^{2n}} \varphi_i(\frac{|f(t_k) - f(t_{k+1})|}{d_i(t_k, t_{k+1})})
\]

Since \( i \) is arbitrary it implies that

\[
|f(t_k) - f(t_{k+1})| \leq (\max_{1 \leq i \leq n} d_i(t_k, t_{k+1})\varphi_i^{-1}(N_{k+1}^n))(1 + \frac{1}{N_{k+1}^{2n}} \inf_{1 \leq i \leq n} \varphi_i(\frac{|f(t_k) - f(t_{k+1})|}{d_i(t_k, t_{k+1})})).
\]

Consequently

\[
\sum_{k=p}^{\infty} |f(t_k) - f(t_{k+1})| \leq (\sum_{k=p}^{\infty} \max_{1 \leq i \leq n} d_i(t_k, t_{k+1})\varphi_i^{-1}(N_{k+1}^n))(1 + \frac{1}{N_{k+1}^{2n}} \sum_{1 \leq i \leq n} \inf \varphi_i(\frac{|f(u) - f(v)|}{d_i(u, v)})).
\]

In the same way we get

\[
\sum_{k=p}^{\infty} |f(s_k) - f(s_{k+1})| \leq (\sum_{k=p}^{\infty} \max_{1 \leq i \leq n} d_i(s_k, s_{k+1})\varphi_i^{-1}(N_{p+1}^n))(1 + \frac{1}{N_{p+1}^{2n}} \sum_{1 \leq i \leq n} \inf \varphi_i(\frac{|f(u) - f(v)|}{d_i(u, v)}))
\]

and

\[
|f(t_p) - f(s_p)| \leq \max_{1 \leq i \leq n} d_i(s_p, t_p)\varphi_i^{-1}(N_{p+1}^n)(1 + \frac{1}{N_{p+1}^{2n}} \sum_{1 \leq i \leq n} \inf \varphi_i(\frac{|f(u) - f(v)|}{d_i(u, v)})).
\]

Applying (11) and following triangle inequalities

\[
d_i(t_k, t_{k+1}) \leq d_i(t_k, t) + d_i(t, t_{k+1}), \quad d_i(s_k, s_{k+1}) \leq d_i(s_k, s) + d_i(s, s_{k+1}),
\]

\[
d_i(s_p, t_p) \leq d_i(s_p, s) + d_i(s, t) + d_i(t, t_p)
\]

we obtain

\[
|f(s) - f(t)| \leq (\max_{1 \leq i \leq n} (d_i(s, t)\varphi_i^{-1}(N_{p}^n))) + 2 \sum_{x \in \{s, t\}} \sum_{k=p}^{\infty} \max_{1 \leq i \leq n} d_i(x, x^k)\varphi_i^{-1}(N_{k+1}^{2n}))(1 + \int_{T \times T} \sum_{1 \leq i \leq n} \inf \varphi_i(\frac{|f(u) - f(v)|}{d_i(u, v)})\hat{\nu}(du, dv)),
\]

15
where
\[ \hat{\nu}(du, dv) = \sum_{k=0}^{\infty} \frac{1}{N_{2k+1}^n} \sum_{u \in T_k} \sum_{v \in T_k} \delta_{(u,v)} + \sum_{k=0}^{\infty} \sum_{u \in T_k} \sum_{v \in T_{k+1}} \frac{1}{N_{2k+1}^n} \delta_{(u,v)}. \]

Let us observe that due to (4) and (5) we have
\[ \varphi_i^{-1}(N_{k+1}^{2n}) \leq K_i' \varphi_i^{-1}(N_{k+1}) \leq C_i K_i' \varphi_i^{-1}(N_k), \]
where \( K_i' \) depends on \( \varphi_i \) and \( n \) only. Denoting \( D := (2 + \varepsilon) \max_{1 \leq i \leq n} C_i K_i' \) we deduce from (10) that
\[ \max_{1 \leq i \leq n} d_i(x, x_k) \varphi_i^{-1}(N_{k+1}^n) \leq D \max_{1 \leq i \leq n} r_i^1(x) \varphi_i^{-1}(N_k), \quad x \in \{s, t\} \]
\[ \max_{1 \leq i \leq n} d_i(s, t) \varphi_i^{-1}(N_{p+1}^n) \leq D \max_{1 \leq i \leq n} d_i(s, t) \varphi_i^{-1}(N_p). \]

By Lemma 3 and the definition \( p = \max\{p_{i,m}(s), p_{s,m}(t)\} \) we obtain
\[ (\max_{1 \leq i \leq n} (d_i(s, t) \varphi_i^{-1}(N_p)) + 2 \sum_{x \in \{s, t\}} \sum_{k=p}^{\infty} \max_{1 \leq i \leq n} r_i^1(x) \varphi_i^{-1}(N_k)) \leq 2\tau_m(s, t). \]

Hence
\[ |f(s) - f(t)| \leq 2D\tau_m(s, t)(1 + \int_{T \times T} \inf_{1 \leq i \leq n} \varphi_i(\frac{|f(u) - f(v)|}{d_i(u, v)})\hat{\nu}(du, dv). \]

It remains to show that \( \hat{\nu}(1) < \infty \). We have
\[ \hat{\nu}(1) \leq \sum_{k=0}^{\infty} \frac{|T_k|^2}{N_{2k+1}^{2n}} + \sum_{k=0}^{\infty} \frac{|T_k||T_{k+1}|}{N_{2k+1}^{2n+1}}. \]

Due to Theorem 6 \( |T_k| \leq N_k^n \), furthermore \( N_0 = 1, N_{k+1} \geq N_k^2, N_{k+1} \geq 2N_k \) (by (4) and (5)), thus
\[ \hat{\nu}(1) \leq 2 \sum_{k=0}^{\infty} \frac{1}{N_k} \leq 2 \sum_{k=0}^{\infty} 2^{-k} = 4. \]

Consequently
\[ \frac{|f(s) - f(t)|}{8D\tau_m(s, t)} \leq 1 + \int_{T \times T} \inf_{1 \leq i \leq n} \varphi_i(\frac{|f(u) - f(v)|}{d_i(u, v)})\nu(du, dv), \nu(du, dv), \]
where \( \nu = \hat{\nu}/\hat{\nu}(1) \). It ends the proof of Theorem 7 (with \( K = 8D \)).

**Corollary 2** There exist a universal constant \( K' \) and probability measure \( \nu \) on \( (T \times T, d \times d) \) such that for all \( f \in C(T, d) \) we have
\[ \inf_{1 \leq i \leq n} \varphi_i(\sup_{s, t \in T} \frac{|f(s) - f(t)|}{K'\tau_m(s, t)}) \leq 1 + \int_{T \times T} \inf_{1 \leq i \leq n} \varphi_i(\frac{|f(u) - f(v)|}{d_i(u, v)})\nu(du, dv). \]
Proof. We use Theorem 7 and (4). First assume that
\[ b := \int_{T \times T} \inf_{1 \leq i \leq n} \varphi_i(\frac{|f(u) - f(v)|}{d_i(u, v)}) \nu(du, dv) \leq 1. \]
Then it follows by Theorem 7 that \( \sup_{s, t \in T} \frac{|f(s) - f(t)|}{2K_T(s, t)} \leq 1 \). Now assume that \( b > 1 \). We define \( a_i = K_i \varphi_i^{-1}(b) \) and \( a = \max_{1 \leq i \leq n} a_i \). By the Theorem 7 we have
\[ \sup_{s, t \in T} \frac{|f(s) - f(t)|}{4K_T(s, t)} \leq \frac{1}{2} + \frac{1}{2} \int_{T \times T} \inf_{1 \leq i \leq n} \varphi_i(\frac{|f(u) - f(v)|}{2a_i d_i(u, v)}) \nu(du, dv). \]
Due to the convexity (and since \( \varphi_i(1) = 1 \))
\[ \varphi_i(\frac{|f(u) - f(v)|}{a_i d_i(u, v)}) \leq \frac{1}{2} + \frac{1}{2} \varphi_i(\frac{|f(u) - f(v)|}{a_i d_i(u, v)}) - 1. \]
By (4) and \( b > 1 \) we obtain
\[ \varphi_i(\frac{|f(u) - f(v)|}{a_i d_i(u, v)}) \leq \varphi_i(\frac{|f(u) - f(v)|}{K_i d_i(u, v) - \varphi_i^{-1}(b)}) \leq \frac{1}{b} \varphi_i(\frac{|f(u) - f(v)|}{d_i(u, v)}). \]
Thus
\[ \int_{T \times T} \inf_{1 \leq i \leq n} \varphi_i(\frac{|f(u) - f(v)|}{2a_i d_i(u, v)}) \nu(du, dv) \leq \frac{1}{2} + \frac{1}{2} = 1 \]
and finally
\[ \sup_{s, t \in T} \frac{|f(s) - f(t)|}{4K_T(s, t)} \leq a \]
what implies
\[ \inf_{1 \leq i \leq n} \varphi_i(\sup_{s, t \in T} \frac{|f(s) - f(t)|}{4K_T(s, t)}) \leq b. \]
The result easily follows with \( K' = 4K \max_{1 \leq i \leq n} K_i \).
\[ \blacksquare \]

Proof. [of Theorem 2] As in the case of one metric we use that for any metric \( \rho \)Borel as function on \( (T \times T, d \times d) \) we have
\[ E \inf_{1 \leq i \leq n} \varphi_i(\sup_{s, t \in T} \frac{|X(s) - X(t)|}{\rho(s, t)}) = \sup_{F \subset T} E \inf_{1 \leq i \leq n} \varphi_i(\sup_{s, t \in F} \frac{|X(s) - X(t)|}{\rho(s, t)}), \]
where the supremum is taken over all finite subsets of \( T \). Hence we need to prove the result for each finite \( F \subset T \). Following the proof of Theorem 11.9 from [6] there exist a sequence \( m^F = (m^F_i)_{i=1}^n \) of probability measures supported on \( F \) and such that \( m(B(t, \varepsilon)) \leq m^F(B(t, \varepsilon)) \). Using the convexity of \( \varphi_i \), Corollary 2 and the Fubini Theorem we obtain
\[ E \inf_{1 \leq i \leq n} \varphi_i(\sup_{s, t \in F} \frac{|X(s) - X(t)|}{4K'_T(m^F)(s, t)}) \leq \frac{1}{2} E \inf_{1 \leq i \leq n} \varphi_i(\sup_{s, t \in F} \frac{|X(s) - X(t)|}{K'_T(m^F)(s, t)}) \leq \frac{1}{2}(1 + E \int_{F \times F} \inf_{1 \leq i \leq n} \varphi_i(\frac{|X(u) - X(v)|}{d_i(u, v)}) \nu(du, dv) \leq 1. \]
\[ \blacksquare \]
5 The minimality

In this section we prove the converse statement, namely that for each continuity modulus \( \rho \) (Borel as a function on \((T \times T, d \times d)\)) which is suitable for all processes with bounded increments, there exist a metric \( \tau_m \) which is up to a constant better than \( \rho \) (i.e. \( \tau_m \leq K \rho \)).

**Proof.** [of Theorem 3] Using the idea described in the Talagrand’s paper [7] Theorem 2.3 we obtain that there exist an absolute constant \( C > 0 \) and a positive linear functional \( \Lambda \) on \( C(T \times T \setminus \triangle, d \times d) \), where \( \triangle = \{(t, t) : t \in T\} \) such that \( \Lambda(1) = 1 \) and for each \( d \)-continuous \( f \) on \( T \) there holds

\[
\sup_{s, t \in T} \frac{|f(s) - f(t)|}{\rho(s, t)} \leq C(1 + \Lambda(\inf_{1 \leq i \leq n} \varphi_i(\frac{|f(u) - f(v)|}{d_i(u, v)}))).
\]

Observe that if \( g \in C(T, d_i) \) then \( g|T \in C(T, d_i) \) and consequently \( g|T \in C(T, d) \). Define a probability measure \( m_i \) on \( T_i \) by the formula

\[
\int_{T_i} g(t)m_i(dt) = \frac{1}{2}\Lambda(g|T(u) + g|T(v)),
\]

We thus obtain a sequence \( m = (m_i)_{i=1}^n \) such that \( m_i \) is a probability measure on \((T_i, d_i)\).

Fix \( t \in T \) and \( l \in \mathbb{N} \). We define \( f \in C(T, d) \) by \( f(s) = D^{-1}f_{t,m}(s) \), where the constant \( D \) we choose later. We prove that

\[
\inf_{1 \leq i \leq n} \varphi_i(\frac{|f(u) - f(v)|}{d_i(u, v)}) \leq \sum_{i=1}^n \frac{1}{(m_i(B_i(t, d_i(t, u))))^\frac{1}{2}} + \frac{1}{(m_i(B_i(t, d_i(t, v))))^\frac{1}{2}}. \tag{13}
\]

One can see that

\[
|f(u) - f(v)| = D^{-1}(\min_{0 \leq p \leq l} f_{t,m}^l(p, u) - \min_{0 \leq p \leq l} f_{t,m}^l(p, v)) \leq
\]

\[
\leq D^{-1} \max_{x \in \{u, v\}} (|f_{t,m}^l(p_{t,m}(x), u) - f_{t,m}^l(p_{t,m}(x), v)|).
\]

and

\[
\max_{x \in \{u, v\}} |f_{t,m}^l(p_{t,m}(x), u) - f_{t,m}^l(p_{t,m}(x), v)| =
\]

\[
= \max_{x \in \{u, v\}} \max_{1 \leq i \leq n} d_i(t, u)\varphi_i^{-1}(N_{p_{t,m}(x)}) - \max_{1 \leq i \leq n} d_i(t, v)\varphi_i^{-1}(N_{p_{t,m}(x)}) \leq
\]

\[
\leq \max_{1 \leq i \leq n} |d_i(t, u) - d_i(t, v)|\varphi_i^{-1}(N_{p_{t,m}(x)}) \leq \sum_{x \in \{u, v\}} \max_{1 \leq i \leq n} d_i(u, v)\varphi_i^{-1}(N_{p_{t,m}(x)}).
\]

Consequently

\[
\inf_{1 \leq i \leq n} \varphi_i(\frac{|f(u) - f(v)|}{d_i(u, v)}) \leq \sum_{x \in \{u, v\}} \max_{1 \leq i \leq n} \varphi_i(D^{-1}\varphi_i^{-1}(N_{p_{t,m}(x)})).
\]

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We can choose $D = 2 \max_{1 \leq i \leq n} C_i K_i$, and hence by (4) and (5)

$$\varphi_i(D^{-1} \varphi_i^{-1}(N_{i}(x))) \leq \varphi_i^{-1}((2K_i)^{-1} \varphi_i^{-1}(N_{i}(x))) \leq (N_{i}(x)-1)^{1/2}.$$ 

It follows that

$$\inf_{1 \leq i \leq n} \varphi_i(\frac{|f(u) - f(v)|}{d_i(u, v)}) \leq \sum_{x \in \{u,v\}} (N_{i}(x)-1)^{1/2}. \tag{14}$$

Due to Lemma 3 there exits at least one $i_0$ such that $d_{i_0}(t, x) \leq r_{i_0}(t, x)$, thus

$$\frac{1}{N_{i_0}(x)-1} \leq m_{i_0}(B_{i_0}(t, r_{i_0}(x)-1)(t)) \leq \frac{1}{m_{i_0}(B_{i_0}(t, d_{i_0}(x), r_{i_0}(x)-1)(t))).$$

Using this fact in (14) we complete the proof of (13). It remains to integrate (13)

$$\Lambda\left(\inf_{1 \leq i \leq n} \varphi_i(\frac{|f(u) - f(v)|}{d_i(u, v)})\right) \leq \Lambda\left(\sum_{i=1}^{n} \frac{1}{m_i(B_i(t, d_i(t, u)))^{1/2}} + \frac{1}{m_i(B_i(t, d_i(t, v)))^{1/2}}\right) \leq 2 \int_{T} \sum_{i=1}^{n} \frac{1}{m_i(B_i(t, d_i(t, u)))^{1/2}} m_i(du) \leq 4n,$$

where the last inequality comes from $\int_{T} \frac{1}{m_i(B_i(t, d_i(t, u)))^{1/2}} m_i(du) \leq \int_{0}^{1} \frac{1}{\sqrt{x}} dx = 2$. By (12) we obtain

$$\frac{D^{-1}|f_{l,m}(s)|}{\rho(s,t)} \leq \frac{|f(s) - f(t)|}{\rho(s,t)} \leq C(1 + 4n)$$

and tending with $l$ to infinity, $|f_{l,m}(s)| \leq CD(1 + 4n)\rho(s,t)$. In the same way we can prove $|f_{s,m}(t)| \leq CD(1 + 4n)\rho(s,t)$. Thus $\tau_{m}(s,t) \leq K\rho(s,t)$, where $K = CD(1 + 4n)$.

\section*{References}


