Improved induced matchings in sparse graphs

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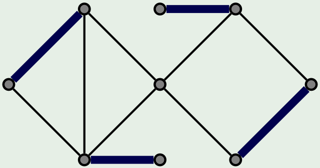
Definitions

Matching

For given undirected graph $G = (V, E)$, the **matching** is a subset of edges $M \subseteq E$, such that no two edges from M share a common endpoint vertex.

Example

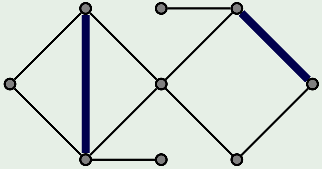


Definitions

Induced Matching

For given undirected graph $G = (V, E)$, the **induced matching** is a subset of edges $M \subseteq E$, such that the M is a matching, and for the graph induced by vertices of M, the edge set $E' = M$.

Example



Problem definition

Induced Matching Problem

Given undirected graph G, find the maximum cardinality induced matching $M \subseteq E(G)$.

- we also consider decision version, with additional paramter k, we have to decide whatever the maximum cardinality ind. matching has size $\geq k$,
- the problem has been introduced by Stockmeyer and Vazirani as "risk-free" marriage problem.

Definitions

Twinless graphs

The undirected graph $G = (V, E)$ is twinless if for any two vertices $x, y \in V$, $x \neq y$, the set of adjacent vertices of x and y are different $(adj(x) \neq adj(y))$.

Observation

Among two vertices with the same neighborhood (called *twins*) at most one is matched in any induced matching, and if one of them is matched then there is another matching of the same size that matches the other vertex. So we can reduce any graph G to a twinless representation G', with the same size of maximal cardinality induced matching.

Previous results

Bounds

Planar twinless graphs:

	lower bound	upper bound
Our result	$\frac{n}{28} + O(1)$	$\frac{n}{28} + O(1)$
Kanj et al. STACS, 08	$\frac{n}{27} + O(1)$	$\frac{n}{40}$

Algorithms

Planar twinless graphs:

	parametrized complexity
Our result	$O(2^{26\sqrt{k}} + n)$
Kanj et al. STACS, 08	$O(2^{159\sqrt{k}} + n)$

Coloring lemma

Kanj, 2008

Let $\mathcal{G}$ be a minor-closed family of graphs and let c be a constant such that any graph in G is c-colorable. Moreover, let G be a graph from $\mathcal{G}$ and let M be a matching in G. Then G contains an induced matching of size at least $|M|/c$.

Simple consequence

For planar graphs, G contains an induced matching of size at least $|M|/4$.

Lower bounds, large minimum deg.

Kanj, 2008

A planar graph of minim degree 3 contains a matching of size $\frac{n+8}{5}$, and induced matching of size at least $\frac{n+8}{20}$.

Lower bounds, large minimum deg.

Nishiezeki and Baybars, 1979

Let G be an n-vertex planar graph of minimum degree δ and let M be a maximum cardinality matching in G. Then,

(*i*) if $\delta = 3$ and $n \geq 10$, then $|M| \geq \frac{n+2}{3}$,

(*ii*) if $\delta = 4$ and $n \geq 16$, then $|M| \geq \frac{2n+3}{5}$,

(*iii*) if $\delta = 5$ and $n \geq 34$, then $|M| \geq \frac{5n+6}{11}$.

Lower bounds, large minimum deg.

Conclusion

Let G be an n-vertex planar graph of minimum degree δ and let M be a maximum cardinality induced matching in G. Then,

(*i*) if $\delta = 3$ and $n \geq 10$, then $|M| \geq \frac{n+2}{12}$,

(*ii*) if $\delta = 4$ and $n \geq 16$, then $|M| \geq \frac{2n+3}{20}$,

(*iii*) if $\delta = 5$ and $n \geq 34$, then $|M| \geq \frac{5n+6}{44}$. □

Proof outline

Contract edges of maximum cardinality matching, apply Four Color Theorem, choose the largest color set.

Lower Bounds

Theorem

Every n-vertex twinless graph G of genus g contains a matching of size $\frac{n+10(1-g)-1}{7}$.

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Every n-vertex twinless graph of genus g contains an induced matching of size $(2n + 20(1 - g) - 2)/(49 + 7\sqrt{1 + 48g})$.

Lower Bounds

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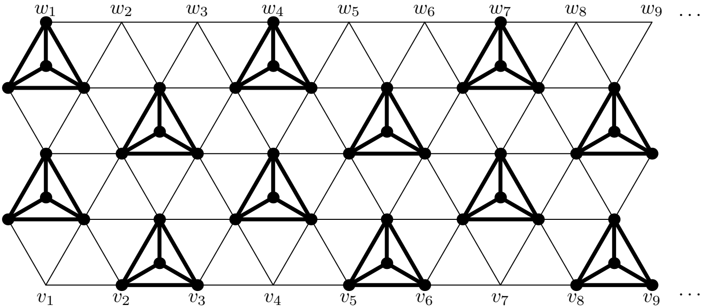
Theorem

Every n-vertex twinless graph of genus g contains an induced matching of size $(2n + 20(1 - g) - 2)/(49 + 7\sqrt{1 + 48g})$.

Lower Bound Theorem

Every n-vertex twinless planar graph contains an induced matching of size $\frac{n+9}{28}$.

Upper bound



We also identify vertices v_i with w_i (for $i = 1, 2, \ldots$).
A graph T_k, contains k copies of K_4, $|V| = 4k$,
$|E| = 12k + O(1)$, $8k + O(1)$ faces.

Upper bound

- the T_k has $4k$ vertices,
- for each face of T_k add vertex of degree 3, (additional $8k$ vertices),
- for each edge of T_k add vertex of degree 2, (additional $12k$ vertices),
- for each vertex of T_k add vertex of degree 1, (additional $4k$ vertices).

In total the constructed graph has $28k + O(1)$ vertices. Each edge of constructed graph is adjacent to some K_4, so the any induced matching has size at most k.

Simple algorithm

Algorithm outline

Given an input planar graph G:

- remove twins,
- find a maximum matching M,
- remove the unmatched vertices,
- contract the edges from M,
- color the resulting graph and choose the subset of M which corresponds to the biggest color class.

Simple algorithm

Algorithm complexity

Let n' is the number of vertices after removing twins. If we use maximal cardinality matching and 5-coloring algorithm, then the algorithm will compute matching of size $\frac{n'}{35}$ (35 since we use 5 colors, instead of 4) in $O(n^{1.5})$ time. If we want $O(n)$ time, then we can use maximal matching algorithm, and achieve ind. matching of size $\frac{n'}{70}$.

Few additional lemmas

Lemma

In any graph without isolated vertices if D is a minimum dominating set and M is a maximum cardinality matching, then $|D| \leq |M|$.

Few additional lemmas

Lemma

In any graph without isolated vertices if D is a minimum dominating set and M is a maximum cardinality matching, then $|D| \leq |M|$.

Fomin et al., 06

For any planar graph G with dominating set D,

$$bw(G) \leq 3\sqrt{4.5 \cdot |D|}.$$

Few additional lemmas

Bw-Lemma

For any planar graph G with maximum induced matching I,

$$bw(G) \leq 3\sqrt{18 \cdot |I|} \cong 12.7 \cdot \sqrt{|I|}.$$

Proof

Simple consequence of two previous lemmas.

Outline of exact algorithm

Outline of decision algorithm *IndMatchExists*(G, k)

Step 1. Remove twins from the graph,

Step 2. If $n' > 28k$ return *True*, (consequence of lower bound theorem)

Step 3. Compute the optimal branch-decomposition of graph H. ($O(k^4)$ time),

Step 4. If $bw(G) \geq 12.7\sqrt{k}$ return *True* (consequence of Bw-Lemma),

Step 5. Use the dynamic programming approach for finding a maximum cardinality induced matching in graph G. ($O(m \cdot 4^l) = O(2^{25.5\sqrt{k}})$ time)

From decision algorithm to solution

Any $f(n, m, k)$ decision algorithm for induced matching problem can be used for computing the solution for the problem. The algorithm, requires m steps, for each edge of graph, it decides, whatever the edge is included in the solution or not. The total time complexity of such solution is $O(m \cdot f(n, m, k))$.

Outline

for each $(u, v) \in E$ **do**

$\quad G' = G - \{(u, x), (v, x) : x \in V(G)\}$

$\quad$ **if** $\mathit{IndMatchExists}(G', k - 1)$ **then**

$\quad\quad I = I \cup \{(u, v)\}; \ G = G'; \ k = k - 1$

Thank you, for your attention!