The following is a sketch of proof of:

**Theorem 1** *First-order intuitionistic logic with* $\forall$ *and* $\to$ *is undecidable.*

Consider a finite set of tile types $T$ with an initial tile $t_0 \in T$ and with two relations $v, h \subseteq T^2$ of vertical and horizontal neighbourhood. A function $f : \mathbb{N}^2 \to T$ is a *tiling* when $f(0,0) = t_0$ and for every $i, j \in \mathbb{N}$ it holds that:

$$\langle\, f(i,j), f(i+1,j)\,\rangle \in h \text{ and } \langle\, f(i,j), f(i,j+1)\,\rangle \in v.$$

It is undecidable whether such a tiling exists. In what follows we fix $T$, $t_0$, $h$, $v$, and we write aa formula $\Phi$ such that $\Phi$ **is provable** if and only if **there is no tiling**. Let $T = \{t_0, ..., t_n\}$.

The vocabulary of $\Phi$ consists of:

– nullary relation symbols *start*, *loop*;
– unary relation symbols $X, Y,$ *set*;
– binary relation symbols $H, V$;
– unary relation symbols $t$, for every $t \in T$.

Let $\Gamma$ consist of the following formulas:

1. $\forall x(X(x) \to Y(x) \to t_0(x) \to \mathit{loop}) \to \mathit{start}$;
2. $\forall x(X(x) \to \forall y(H(x,y) \to X(y) \to \mathit{set}(y)) \to \mathit{loop})$;
3. $\forall x(Y(x) \to \forall y(V(x,y) \to Y(y) \to \mathit{set}(y)) \to \mathit{loop})$;
4. $\forall xyz(H(x,y) \to V(x,z) \to \forall u(H(z,u) \to V(y,u) \to \mathit{set}(u)) \to \mathit{loop})$;
5. $\forall x((t_0(x) \to \mathit{loop}) \to \cdots \to (t_n(x) \to \mathit{loop}) \to \mathit{set}(x))$;
6. $\forall xy(V(x,y) \to t(x) \to t'(y) \to \mathit{loop})$, for all pairs $\langle\, t, t'\,\rangle \notin v$;
7. $\forall xy(H(x,y) \to t(x) \to t'(y) \to \mathit{loop})$, for all pairs $\langle\, t, t'\,\rangle \notin h$.

Define $\Phi = \gamma_1 \to \cdots \to \gamma_d \to \mathit{start}$, where $\gamma_1, \ldots, \gamma_d$ are all formulas in $\Gamma$. We must show that $\Gamma \vdash \mathit{start}$ holds if and only if there is no tiling.

**Part 1:** *No tiling yields a proof.*

Consider an environment $\Delta$ consisting solely of atoms, and let $u$ be a variable in $\Delta$. We define a *position* of $u$ in $\Delta$ as follows:

- If $X(u), Y(u) \in \Delta$ then $\langle\, 0, 0\,\rangle$ is a position of $u$;
- If $X(u), H(y,u) \in \Delta$, and $\langle\, i, 0\,\rangle$ is a position of $y$, then $\langle\, i+1, 0\,\rangle$ is a position of $u$;
- If $Y(u), V(y,u) \in \Delta$, and $\langle\, 0, i\,\rangle$ is a position of $y$, then $\langle\, 0, i+1\,\rangle$ is a position of $u$;
- If $H(x,y), V(x,z), H(z,u), V(y,u) \in \Delta$, and $\langle\, i, j\,\rangle$ is a position of $x$, then $\langle\, i+1, j+1\,\rangle$ is a position of $u$.

We say that an environment $\Delta$ is *proper* when $\Delta$ consists solely of atoms and

- every variable in $\Delta$ has exactly one position;[1]
- the set of positions of variables in $\Delta$ is componentwise downward closed;
- for every variable $u$ in $\Delta$, there is exactly one tile $t \in T$ with $t(u) \in \Delta$ .

A proper environment is *good* if no two variables have the same position. The intention is of course that a good environment describes a partial tiling. Observe that $H(x,y) \in \Delta$ if and only if $\langle\, i, j\,\rangle$ is a position of $x$, and $\langle\, i+1, j\,\rangle$ is a position of $y$, for some $i, j$, and that a similar statement holds for vertical neighbourhood. (Proof by simultaneous induction).

[1]But there can be multiple variables with the same position. $\exists$ros' strategy will however avoid this.

Assume that no tiling exists. We define a strategy for ∃ros. Initally, he has little choice: he must use the assumption 1 and this yields the proof obligation $\Gamma, X(x), Y(x), t_0(x) \vdash \mathit{loop}$. The environment $\{X(x), Y(x), t_0(x)\}$ is good.

Assume that ∃ros wants to prove $\Gamma, \Delta \vdash \mathit{loop}$, with some good $\Delta$. If he can discover an inconsistency in the tiling, either vertical or horizontal, then he wins instantly by applying either formula 6 or 7, respectively. Assume therefore that there is no inconsistency, i.e., the partial tiling is correct so far. Then ∃ros expands it by adding one more position. He chooses a pair $\langle\, i, j \,\rangle$ which is not a position for $\Delta$ and is minimal with respect to the componentwise order. (This ensures that he will systematically cover all positions.)

Case 1: $i = 0, j = j' + 1$. Then $\langle\, 0, j \,\rangle$ is a position of a variable $x$ with $Y(x) \in \Delta$. ∃ros applies axiom 3, and obtains the judgment $\Gamma, \Delta, V(x, y), Y(y) \vdash \mathit{set}(y)$, where $y$ is a fresh variable. Then he must play axiom 5 with $x := y$. This splits the proof into $n$ branches of the form $\Gamma, \Delta, H(x, y), Y(y), t(y) \vdash \mathit{loop}$, for all $t \in T$. Put it differently, ∀phrodite chooses a tile $t$ to be placed at position $\langle\, 0, j \,\rangle$. Observe that the environment $\Delta, H(x, y), Y(y), t(y)$ is good.

Case 2: $i = i' + 1, j = 0$ is similar.

Case 3: $i = i' + 1, j' + 1$. The minimality and downward-closedness implies that pairs $\langle\, i', j' \,\rangle$, $\langle\, i'+1, j' \,\rangle$, $\langle\, i', j'+1 \,\rangle$ are positions of some variables $x, y, z$. Because $\Delta$ is good, it follows that $H(x, y), V(x, z) \in \Delta$. This enables ∃ros to play axiom 4. As a result, a fresh $u$ is introduced and the next judgment is $\Gamma, \Delta, H(z, u), V(y, u) \vdash \mathit{set}(u)$. As in Case 1, the next step is a universal split and we end up with the new judgment $\Gamma, \Delta, H(z, u), V(y, u), t(u) \vdash \mathit{loop}$, where the environment $\Delta, H(z, u), V(y, u), t(u)$ is again good.

We know that there is no tiling, i.e., every function $f : \mathbb{N}^2 \to T$, satisfying $f(0, 0) = t_0$, must exhibit an inconsistency either of the form $\langle\, f(i, j), f(i + 1, j) \,\rangle \notin h$ or of the form $\langle\, f(i, j), f(i, j + 1) \,\rangle \notin v$. Every branch of the proof corresponds to such a function, the value of $f(i, j)$ being determined by an application of axiom 5, when a variable at position $\langle\, i, j \,\rangle$ is introduced. In addition, ∃ros' strategy is such that every pair $\langle\, i, j \,\rangle$ must eventually become a position at every branch. This enables ∃ros to exploit the inconsistency.

**Part 1:** *Tiling yields a refutation.*

If there exists a tiling $f$ then ∀phrodite has a winning strategy (refutation) based on $f$. Then every proof attempt can never be completed and must be infinite. Any proof must necessarily start with axiom 1, and recurrently address judgments of the form $\Gamma, \Delta \vdash \mathit{loop}$, where $\Delta$ consists solely of atoms. The environment $\Delta$ is proper, although now there can be multiple variables with the same position. In principle, such variables could be assigned different tiles, but ∀phrodite's strategy prevents it. Formally, we show the following statement:

*Let $\Delta$ be proper. Assume that, for every variable $u$ at position $\langle\, i, j \,\rangle$ in $\Delta$ and every $t \in T$, if $f(i, j) = t$ then $t(u) \in \Delta$. Then $\Gamma, \Delta \nvdash \mathit{loop}$.*

Assume the contrary and consider the shortest counterexample proof. The proof term begins with an application of an axiom. Since $f$ is a tiling, axioms 6 or 7 cannot be used. Indeed, to apply e.g., 6 one needs variables $x, y$ with $V(x, y)$. But then positions of $x$, $y$ are some $\langle\, i, j \,\rangle, \langle\, i, j + 1 \,\rangle$ which must be tiled with $\langle\, f(i, j), f(i, j + 1) \,\rangle \in h$. Applicable axioms are therefore 2–4, and each introduces a new variable $u$ at a certain position $\langle\, i, j \,\rangle$. The next proof goal is $\mathit{set}(u)$ and the proof splits into branches. In one of them, the assumption $t(u)$ is added, where $t = f(i, j)$ (∀phrodite chooses the tile according to $f$). This leads to a proof of $\Gamma, \Delta' \vdash \mathit{loop}$, where $\Delta'$ is proper. As the obtained proof is shorter, we get a contradiction.