

# An Axiomatization of the Eigenvector and Katz Centralities

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## Abstract

Feedback centralities are one of the key classes of centrality measures. They assess the importance of a vertex recursively, based on the importance of its neighbours. Feedback centralities includes the Eigenvector Centrality, as well as its variants, such as the Katz Centrality or the PageRank, and are used in various AI applications, such as ranking the importance of websites on the Internet and most influential users in the Twitter social network. In this paper, we study the theoretical underpinning of the feedback centralities. Specifically, we propose a novel axiomatization of the Eigenvector Centrality and the Katz Centrality based on six simple requirements. Our approach highlights the similarities and differences between both centralities which may help in choosing the right centrality for a specific application.

## Introduction

*Feedback centralities* are one of the key classes of centrality measures (Koschützki et al. 2005). They assess the importance of a vertex recursively, based on the importance of its neighbours. In the AI literature, such measures have been popularized by PageRank (Page et al. 1999), which is used by Google to rank the importance of websites on the Internet. Since then, feedback centralities have been widely used in a number of AI applications, such as finding the most influential users in the Twitter social network (Weng et al. 2010) and balancing energy consumption in wireless sensor networks (Jain and Reddy 2015). The goal of this paper is to study the theoretical underpinnings of feedback centralities.

Arguably, the most well-known feedback centrality is *the Eigenvector Centrality* (Bonacich 1972). It is a natural extension of the Degree Centrality which simply counts the number of ties that a vertex has. In result, in the Degree Centrality, each neighbour is treated equally regardless of its importance. The idea behind the Eigenvector Centrality is that the important vertex has connections to other vertices that are themselves important. In result, the Eigenvector Centrality of a vertex is proportional not to a number of neighbours, but to the sum of their Eigenvector Centralities.

In our work we study weighted directed networks, where the Degree and Eigenvector Centralities account for both the directions of edges and their weights. In particular, only

the vertices with edges incoming to a vertex in question are taken into account, and this proportionally to the weights of such an edge. For instance, consider journal citation network where vertices represent journals and weighted and directed ties represent citations. Now, the Degree Centrality simply counts the number of citations, i.e., sums the weights of incoming ties. On the other hand, the Eigenvector Centrality counts each citation with the centrality of the journal which this citation comes from. In result, one citation from a good journal might be more important than a couple of citations from a weak one.

Over the years, many extensions of the Eigenvector Centrality have been proposed. Two most prominent ones are the Katz Centrality (Katz 1953) and the PageRank (Page et al. 1999). In this paper, beside the original Eigenvector Centrality, we study the former one. The Katz Centrality is defined as the sum of the predecessors' importance shifted by a constant value,  $b$ . While seemingly insignificant, this change means that the importance of a vertex depends more than previously on the number of neighbours and less on the importance of the neighbours. Thus, the Katz Centrality can be considered as a middle point between the Degree Centrality and the Eigenvector Centrality.

Despite their popularity, there is a striking lack of the theoretical analysis that captures and highlights the differences between the feedback centralities. As noted by Brandes and Erlebach (2005), “*there are several approaches concerning axiomatization, but up to now there is a lack of structure and generality*”. While there have been several attempts to axiomatize various feedback centralities, many problems still remain (see the Related Work section for details).

To address this issue, in this paper we propose a novel axiomatization of both the Eigenvector Centrality and the Katz Centrality. We begin with the axiomatization of the Eigenvector Centrality based on four requirements. The first one, namely *Source Dependency*, specify centrality in a simple graph with only two vertices and two edges. The remaining requirements – *Endpoint Removal*, *Weak Set Locality* and *Convex Combination* – define several graph operations and specify how they affect the centrality in the graph. Next, we analyze the Katz Centrality and prove that replacing Source Dependency and Weak Set Locality with two axioms – *Compound Dependency* and *Set Locality* uniquely characterize the Katz Centrality.

## Preliminaries

In this section, we introduce basic definitions and notations used in this paper.

**Graphs:** In our work we consider directed, weighted graphs with possible loops. Such a graph is an ordered triplet  $G = (V, E, \omega)$ , where  $V$  is the set of  $n = |V|$  vertices,  $E \subseteq V \times V$  is the set of edges and  $\omega : E \rightarrow \mathbb{R}_+$  is a *weight function* reflecting how strong is the connection between a pair of vertices. Sometimes, it is convenient to use generalised weight function  $\omega^* : V \times V \rightarrow \mathbb{R}_+ \cup \{0\}$  defined as follows:

$$\omega^*(u, v) = \omega(u, v) \text{ if } (u, v) \in E, \quad \omega^*(u, v) = 0 \text{ otherwise.}$$

If the function  $\omega$  is constant, i.e., there exists  $\alpha \in \mathbb{R}_+$  such that  $\omega(e) = \alpha$ , for every  $e \in E$ , we will simply write  $\omega(\cdot) = \alpha$  to define it. For a subset of vertices,  $U \subseteq V$ , a subgraph induced by  $U$  is denoted  $G|_U$ . Formally,  $G|_U = (U, E|_U, \omega|_U)$ , where  $E|_U = \{(u, v) \in E : u, v \in U\}$  and  $\omega|_U(e) = \omega(e)$  for every  $e \in E|_U$ . The set of all possible graphs is denoted by  $\mathcal{G}$ .

Every edge  $(u, v)$  is an *outgoing* edge for the vertex  $u$  which is the start of this edge and an *incoming* edge for the vertex  $v$  – its end. If  $u \neq v$ , we say the edge is *proper*. Otherwise, it is called a *loop*. The set of starts of all incoming edges for any vertex  $v$  is called a set of *predecessors* and is denoted by  $\mathcal{N}_G^-(v)$  for a graph  $G$ .

An ordered set of pairwise different vertices,  $p = (v_1, v_2, \dots, v_k)$ , such that every but the last vertex is a predecessor of the next one, i.e.,  $(v_i, v_{i+1}) \in E$  for every  $i \in \{1, \dots, k-1\}$ , is called a *path*. If for every pair of vertices,  $u, v \in V$ , there exists a path from  $v$  to  $u$  and from  $u$  to  $v$ , then the graph is *strongly connected*.

A *cycle* is a path in which the last vertex is a predecessor of the first one, i.e.,  $(v_k, v_1) \in E$ . The set of all cycles in  $G$  will be denoted by  $C(G)$ . For a cycle,  $c = (v_1, v_2, \dots, v_k)$ , the *length of a cycle*, denoted by  $|c|$ , is the number of vertices in  $c$ . Furthermore, the *weight of a cycle*, denoted by  $\omega(c)$ , is the product of weights of all edges that constitute cycle  $c$ :

$$\omega(c) = \omega(v_k, v_1) \cdot \prod_{i=1}^{k-1} \omega(v_i, v_{i+1}). \quad (\text{weight of a cycle})$$

If for two graphs  $G_1 = (V_1, E_1, \omega_1)$  and  $G_2 = (V_2, E_2, \omega_2)$  there exists a bijection  $f : V_1 \rightarrow V_2$  such that  $(u, v) \in E_1 \Leftrightarrow (f(u), f(v)) \in E_2$  and  $\omega_1(u, v) = \omega_2(f(u), f(v))$ , then these graphs are *isomorphic*, which is denoted by  $G_1 \simeq G_2$ .

For a graph,  $G = (V, E, \omega)$ , multiplying it by a scalar  $a > 0$  is an operation producing a graph

$$a \cdot G = (V, E, a \cdot \omega).$$

Similarly, for two graphs  $G_1 = (V_1, E_1, \omega_1)$  and  $G_2 = (V_2, E_2, \omega_2)$  the sum  $G_1 + G_2$  is a graph defined as follows:

$$G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2, \omega_1 + \omega_2)$$

where  $\omega_1 + \omega_2(e) = \omega_1^*(e) + \omega_2^*(e)$  for every  $e \in E_1 \cup E_2$ .

**Centralities:** For a graph,  $G = (V, E, \omega)$ , a *centrality* is a positive vector  $\mathbf{x} \in \mathbb{R}_+^n$  with each coordinate corresponding

to a specified vertex of  $G$  which reflects its importance. A centrality of particular vertex  $v$  is denoted by  $x_v$  and  $\mathbf{x}_{-v}$  is a notation for a vector  $\mathbf{y} \in \mathbb{R}^{n-1}$  such that  $y_u = x_u$  for every  $u \in V \setminus \{v\}$ .

Following the definition of Eigenvector Centrality, in this paper we define the centrality function as a function that assigns a set, and not a single centrality for each graph.<sup>1</sup>

**Definition 1. (Centrality Function)** A centrality function  $F$  is a function that for every graph,  $G \in \mathcal{G}$ , returns a set of centralities:  $F(G) \subseteq \mathbb{R}_+^n$ .

*Eigenvector Centrality Function* (Bonacich 1972), denoted by  $EV$ , is a centrality function such that  $\mathbf{x} \in EV(G)$  for a graph  $G = (V, E, \omega)$  if and only if there exists  $\lambda \in \mathbb{R}$  that satisfies

$$\lambda x_v = \sum_{u \in \mathcal{N}_G^-(v)} \omega(u, v) \cdot x_u, \quad (1)$$

for every vertex  $v \in V$ . In other words, for the *adjacency matrix*  $A$  of graph  $G$  (i.e.,  $a_{ij} = \omega^*(j, i)$ ), every  $\mathbf{x} \in EV(G)$  is an eigenvector of the matrix  $A$  and  $\lambda$  would be the corresponding eigenvalue. Formally:

$$EV(G) = \{\mathbf{x} \in \mathbb{R}_+^n : A\mathbf{x} = \lambda\mathbf{x}\}.$$

In general, each matrix can have up to  $n$  different eigenvalues, where  $n$  is its dimension. However, from Perron-Frobenius Theorem (Perron 1907) we know that in every strongly connected graph  $G$  every positive eigenvector  $\mathbf{x}$  corresponds to the same eigenvalue  $\lambda$  – the maximal eigenvalue of the adjacency matrix. Moreover, the ratio of centralities of any two vertices is always the same.

In this paper, we will not restrict ourselves to strongly connected graphs. If graph  $G$  is not strongly connected, then it is possible that the set of centralities  $EV(G)$  loses its proportionality property or is empty. However, what is important is that since we allow only positive values of centralities, all centralities for a specific graph always refer to the same, nonnegative eigenvalue  $\lambda$ .

*Katz centrality* is a similar concept (Katz 1953). We say that centrality function  $F$  is a *Katz Centrality function* if there exists  $a, b > 0$  such that

$$x_v = a \sum_{u \in \mathcal{N}_G^-(v)} \omega(u, v) \cdot x_u + b, \quad (2)$$

for every graph  $G$  and  $\mathbf{x} \in F(G)$ . As before it has its algebraic interpretation using adjacency matrix

$$K_{a,b}(G) = \{\mathbf{x} \in \mathbb{R}_+^n : aA\mathbf{x} + \mathbf{b} = \mathbf{x}\},$$

where  $\mathbf{b}$  is  $n$ -dimensional vector of  $bs$ . Notice that contrary to Eigenvector Centrality, because of fixed  $a$  and  $b$ , there is at most one solution for every graph and there are no solutions if  $a$  is too big.<sup>2</sup>

<sup>1</sup>Similar concept, under the name *centrality correspondence*, was proposed by Dequiedt and Zenou (2014).

<sup>2</sup>More precisely, the set of solutions is empty if and only if  $a$  is greater or equal  $\frac{1}{\lambda}$ , where  $\lambda$  is the maximal eigenvalue of  $A$ .

## Axiomatization of the Eigenvector Centrality

In this section, we introduce our axiomatization of the Eigenvector Centrality. Our axiomatization consists of four axioms – *Source Dependency*, *Endpoint Removal*, *Weak Set Locality* and *Convex Combination*. In a description of axioms, we will often refer to the journal citation interpretation from the introduction.

Our first axiom, called *Source Dependency*, specifies a centrality in a simple borderline case. Assume there are only two journals, the second one does not cite anyone, and the first one cite the other journal  $k$  times more often than itself. Then, Source Dependency states that the second journal is  $k$  times more important.

**Source Dependency:** For every two vertices  $u, v$  and function  $\omega$ :

$$F(\{u, v\}, \{(u, u), (u, v)\}, \omega) = \left\{ \mathbf{x} \in \mathbb{R}_+^2 : \frac{x_u}{x_v} = \frac{\omega(u, u)}{\omega(u, v)} \right\}.$$

Our next three axioms describe several graph operations and specify how these operations affect the centrality of vertices. For our second axiom, called *Endpoint Removal*, assume that there exists a journal that does not cite anyone else. Endpoint Removal states that removing this journal does not affect the relative importance of the others. The intuition behind this axiom is that an assessment of an importance of a journal should be made based on how often it is cited (i.e., incoming edges) and not how often it cites others (i.e., outgoing edges).

**Endpoint Removal:** For every  $G = (V, E, \omega)$  and  $v \in V$  such that  $(v, u) \notin E$  for every  $u \neq v$ :

$$\mathbf{x} \in F(G) \Rightarrow \mathbf{x}_{-v} \in F(G|_{V \setminus \{v\}}).$$

Our next axiom, called *Weak Set Locality*, considers a graph that consists of several independent parts. Assume there exists a number of exactly identical and independent "worlds", i.e., copies of the same journal citation network. Since the worlds are independent, a centrality in one world should not affect the centrality in other worlds. According to this intuition, *Weak Set Locality* states that every combination of centralities in each world/copy is a valid centrality of the whole system.

**Weak Set Locality:** For  $G_1 = (V_1, E_1, \omega_1), \dots, G_k = (V_k, E_k, \omega_k) \in \mathcal{G}$  such that  $G_1 \simeq \dots \simeq G_k$  and  $V_i \cap V_j = \emptyset$ , for every  $i \neq j$ :

$$F(G_1 + \dots + G_k) = F(G_1) \times \dots \times F(G_k).$$

Eigenvector Centrality depends on the structure of the entire network. Because of that, identical structure of the 'worlds' is crucial and the stronger version of this axiom – *Set Locality* (introduced in the next section) is not satisfied.

Finally, our last axiom imposes a certain consistency condition on centrality function. Assume there are two separate citation networks considering the same journals, e.g., citations limited to theory papers and citations limited to application papers. Now, *Convex Combination* states that if both network have the same centrality, then the (convex)

combination of both networks has this centrality. To put it differently, if some assessment of importance of journal is true based on both networks, then both can be combined and the assessment will remain the same. Moreover, if one of the networks and a convex combination have the same centrality, then the second network has this centrality, too.

**Convex Combination:** For every  $G_1 = (V, E_1, \omega_1)$ ,  $G_2 = (V, E_2, \omega_2)$  and  $t \in (0, 1)$ :

$$\mathbf{x} \in F(G_1) \Rightarrow \left( \mathbf{x} \in F(G_2) \Leftrightarrow \mathbf{x} \in F(t \cdot G_1 + (1-t) \cdot G_2) \right).$$

In Theorem 7 we will prove that these four axioms characterize the Eigenvector Centrality, i.e., it is the only centrality function that satisfies them. Now, we begin with the theorem stating that the Eigenvector Centrality indeed satisfies these axioms.

**Theorem 1.** *The Eigenvector Centrality satisfies Source Dependency, Endpoint Removal, Weak Set Locality and Convex Combination.*

The proofs of our further theorems rely on the notion of *unity graphs*.

**Definition 2.** (*Unity Graphs*) A graph  $G = (V, E, \omega) \in \mathcal{G}$  is called a *unity graph* if and only if every vertex of  $G$  has at most one incoming edge and there exists at least one edge. The set of all possible unity graphs is denoted by  $\mathcal{UG}$ :

$$\mathcal{UG} = \left\{ G \in \mathcal{G} : \forall v \in V |\mathcal{N}_G^-(v)| \leq 1, E \neq \emptyset \right\}.$$

If additionally, every vertex has exactly one incoming edge and there exists  $\alpha$  such that the weight of every cycle,  $c$ , equals  $\alpha^{|c|}$ , then we say that the graph is *regular*. For  $\alpha \in \mathbb{R}_+$ , the set of all possible regular unity graphs is denoted by  $\mathcal{RUG}^\alpha$ :

$$\mathcal{RUG}^\alpha = \left\{ G \in \mathcal{G} : \forall v \in V |\mathcal{N}_G^-(v)| = 1, \forall c \in G \omega(c) = \alpha^{|c|} \right\}.$$

Unity graphs are of our interest because the Eigenvector Centrality in those graphs is easily calculated.

**Lemma 2.** *Let  $G = (V, E, \omega) \in \mathcal{UG}$  be a unity graph. If  $G$  is not regular, then  $EV(G) = \emptyset$ . Otherwise, if  $G \in \mathcal{RUG}^\alpha$ , then:*

$$EV(G) = \left\{ \mathbf{x} \in \mathbb{R}_+^n : \forall (u, v) \in E x_v/x_u = \omega(u, v)/\alpha \right\}.$$

*Proof.* First, we show that if there exists a vertex in  $G$  without incoming edges, then  $EV(G) = \emptyset$ . To this end, observe that in a unity graph  $\lambda > 0$  – since a unity graph has at least one edge, say  $(u, v) \in E$ , from (1) we get  $\lambda = \omega(u, v)x_u/x_v$  for  $x_v, x_u, \omega(u, v) > 0$ . In result, if a vertex,  $v$ , has no incoming edges, (1) implies  $x_v = 0$ . However, since we assumed that centralities are positive vectors, we get  $EV(G) = \emptyset$ .

Now, assume that every vertex has exactly one incoming edge. Notice that the Eigenvector Centrality equation (1) simplifies to:

$$x_v = x_u \cdot \omega(u, v)/\lambda, \quad \text{for every } (u, v) \in E. \quad (3)$$

For a cycle,  $c = (v_1, v_2, \dots, v_k)$ , from (3) we get that

$$x_{v_1} = x_{v_k} \cdot \omega(v_k, v_1)/\lambda = \dots = x_{v_1} \cdot \omega(c)/\lambda^{|c|}.$$

Thus,  $\lambda = \sqrt[|c|]{\omega(c)}$ . In result, (3) has a solution if and only if there exists  $\alpha \in \mathbb{R}_+$  such that  $\omega(c) = \alpha^{|c|}$  for every cycle  $c \in C(G)$ , i.e., the graph is regular.

Assume now that  $G$  is regular, i.e.,  $G \in \mathcal{RUG}^\alpha$  for some  $\alpha \in \mathbb{R}_+$ . Then, for  $\lambda = \alpha$ , (3) transforms into

$$x_v = x_u \cdot \omega(u, v) / \alpha, \quad \text{for every } (u, v) \in E.$$

Now, for a cycle  $c = (v_1, \dots, v_k)$ , from  $\omega(c) = \alpha^{|c|}$  we know that there exists a solution and fixing  $x_{v_1}$  determines the values of other  $x_{v_2}, \dots, x_{v_k}$ . In particular, if  $(v, v) \in E$ , then  $x_v$  is any real number. This concludes the proof.  $\square$

The proof that Source Dependency, Endpoint Removal, Weak Set Locality and Convex Combination uniquely characterize the Eigenvector Centrality has the following structure. Assume  $F$  satisfies these axioms.

- In Lemma 3 and 4, we show conditions imposed on  $F$  in simple unity graphs, in which there exists at most one proper edge.
- In Lemma 5, we show that if graph is a regular unity graph, then  $F$  contains the Eigenvector Centrality.
- In Lemma 6, we prove that every graph such that for every vertex the sum of weights of all incoming edges (i.e., in-degree) is the same can be obtained as a linear combination of regular unity graphs.
- Finally, in Theorem 1, building upon Lemmas 3–6, we show that  $F$  is equal to the Eigenvector Centrality.

As mentioned, we begin by considering the simplest unity graphs – graphs in which every edge is a loop.

**Lemma 3.** *Let  $G = (V, E, \omega) \in \mathcal{UG}$  be a unity graph in which every edge is a loop, i.e.,  $E \subseteq \{(v, v) : v \in V\}$ . Assume  $F$  satisfies Source Dependency, Endpoint Removal, Weak Set Locality and Convex Combination. If  $G$  is regular, then  $F(G) = \mathbb{R}_+^n$ . Otherwise,  $F(G) = \emptyset$ .*

*Proof.* First, we will show that if  $G$  is regular, i.e., every vertex has a loop and all loops have identical weights, then  $F(G) = \mathbb{R}_+^n$ . Assume  $G \in \mathcal{RUG}^\alpha$  and  $V = \{u_1, \dots, u_n\}$ . Now, let  $v_1, \dots, v_n$  be vertices not from  $V$  and let us construct graphs  $G_1, \dots, G_n$  such that  $G_i = (\{u_i, v_i\}, \{(u_i, u_i), (u_i, v_i)\}, \omega_i)$ , where  $\omega_i(\cdot) = \alpha$ . From Source Dependency we have that  $F(G_i) = \{\mathbf{x} \in \mathbb{R}_+^2 : x_{u_i} = x_{v_i}\}$ . As these graphs are isomorphic to each other from Weak Set Locality we know that  $F(G_1 + \dots + G_n) = \prod_{i=1}^n F(G_i) = \{\mathbf{x} \in \mathbb{R}_+^{2n} : x_{v_i} = x_{u_i}\}$ . By using Endpoint Removal for vertices  $v_1, \dots, v_n$  we get  $F(G) = \mathbb{R}_+^n$ .

Now, assume that  $G$  is not regular, i.e., assume that there exists  $u, v \in V$  such that  $(u, u) \in E$  and  $\omega(u, u) > \omega(v, v)$  or  $(v, v) \notin E$ . We will prove by contradiction that  $F(G) = \emptyset$ . To this end, assume there exists a centrality  $\mathbf{x} \in F(G)$  and consider graph  $G|_{\{u, v\}}$ . From Endpoint Removal we know that there exists at least one centrality in  $F(G|_{\{u, v\}})$ , say  $\mathbf{x}' = (x_u, x_v)$ . We will now prove that there exists a graph,  $G_2 = (\{u, v\}, \{(u, u)\}, \omega_2)$ , such that  $\mathbf{x}' \in F(G_2)$ . If  $(v, v) \notin E$ , then  $G|_{\{u, v\}}$  is such a graph and we define  $\omega_2(u, u) = \omega(u, u)$ . Otherwise, consider graphs  $G_1, G_2$  such that  $G|_{\{u, v\}} = 1/2G_1 + 1/2G_2$  (see Figure 1):

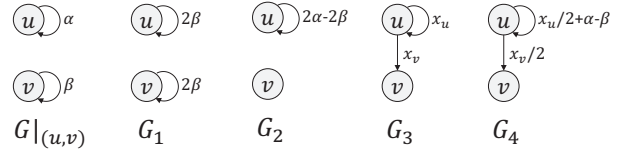


Figure 1: The graph construction used in the second part of the proof of Lemma 3.

$$G_1 = (\{u, v\}, \{(u, u), (v, v)\}, \omega_1), \quad \omega_1(\cdot) = 2\omega(v, v),$$

$$G_2 = (\{u, v\}, \{(u, u)\}, \omega_2), \quad \omega_2(\cdot) = 2(\omega(u, u) - \omega(v, v)).$$

From the first part of the proof we know that  $F(G_1) = \mathbb{R}_+^2$ , therefore  $\mathbf{x}' \in F(G_1)$ . Since  $\mathbf{x}' \in F(G|_{\{u, v\}})$ , from Convex Combination we have that  $\mathbf{x}' \in F(G_2)$ .

Finally, consider the following two graphs that satisfies  $1/2G_2 + 1/2G_3 = G_4$  (see Figure 1):

$$G_3 = (\{u, v\}, \{(u, u), (u, v)\}, \omega_3), \quad \omega_3(u, t) = x_t, \forall t \in \{v, u\},$$

$$G_4 = (\{u, v\}, \{(u, u), (u, v)\}, \omega_4), \quad \omega_4(u, u) = x_u/2 + \omega_2(u, u)/2, \quad \omega_4(u, v) = x_v/2.$$

According to Source Dependency  $\mathbf{x}' \in F(G_3)$ . Since we proved that  $\mathbf{x}' \in F(G_2)$  and  $\mathbf{x}' \in F(G_3)$ , from Convex Combination  $\mathbf{x}' \in F(G_4)$ . However, this contradicts Source Dependency, as  $x_u/x_v$  differs from  $\omega_4(u, u)/\omega_4(u, v)$ . This concludes the proof.  $\square$

Note that Lemma 2 and 3 implies that  $F(G) = EV(G)$ , for every unity graph in which every edge is a loop.

The next lemma considers regular unity graphs in which there is only one proper edge.

**Lemma 4.** *Let  $G = (V, E, \omega) \in \mathcal{RUG}^\alpha$  be a regular unity graph with only one proper edge  $(u, v)$ , i.e.,  $(u, v) \in E$  and  $E \setminus (u, v) = \{(t, t) : t \in V \setminus \{v\}\}$ . Assume  $F$  satisfies Source Dependency, Endpoint Removal, Weak Set Locality and Convex Combination. Then:*

$$F(G) \supseteq \{\mathbf{x} \in \mathbb{R}_+^n : x_v/x_u = \omega(u, v)/\omega(u, u)\}.$$

*Proof.* Assume  $V = \{u, v, t_1, \dots, t_{n-2}\}$  and consider  $G|_{\{u, v\}}$ . From Source Dependency we get that  $F(G|_{\{u, v\}}) = \{\mathbf{x} \in \mathbb{R}_+^2 : x_v = x_u \cdot \omega(u, v)/\omega(u, u)\}$ . Now, let  $s_1, \dots, s_{n-2}$  be vertices not from  $V$  and consider graphs  $G_1, \dots, G_{n-2}$  isomorphic to  $G|_{\{u, v\}}$ :  $G_i = (\{t_i, s_i\}, \{(t_i, s_i), (t_i, t_i)\}, \omega_i)$  with  $\omega_i(t_i, s_i) = \omega(u, v)$  and  $\omega_i(t_i, t_i) = \omega(u, u)$ . From Weak Set Locality we get that  $F(G|_{\{u, v\}} + G_1 + \dots + G_{n-2})$  consists of  $\mathbf{x} \in \mathbb{R}_+^{2n-2}$  such that  $x_v/x_u = x_{s_i}/x_{t_i} = \omega(u, v)/\omega(u, u)$  for every  $i \in \{1, \dots, n-2\}$ . By removing vertices  $s_1, \dots, s_{n-2}$  from

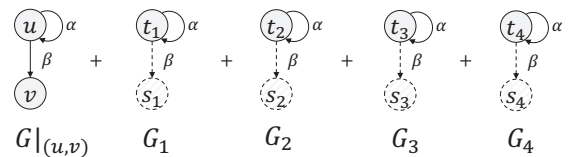


Figure 2: The construction used in the proof of Lemma 4.

$G|_{\{u,v\}} + G_1 + \dots + G_{n-2}$  we obtain graph  $G$  and from Endpoint Removal we get our thesis (see Figure 2).  $\square$

Finally, we consider all regular unity graphs.

**Lemma 5.** *If  $F$  satisfies Source Dependency, Endpoint Removal, Weak Set Locality and Convex Combination, then for every graph  $G \in \mathcal{RUG}^\alpha$ ,  $EV(G) \subseteq F(G)$ .*

*Proof.* Assume  $F$  satisfies all four axioms. Let  $G = (V, E, \omega) \in \mathcal{RUG}^\alpha$  be a regular unity graph and  $\mathbf{x} \in EV(G)$ . From Lemma 2 we know that  $\mathbf{x}$  satisfies  $x_v = x_u \cdot \omega(u, v)/\alpha$  for every  $(u, v) \in E$ . Our goal is to prove that  $\mathbf{x} \in F(G)$ .

For every edge  $(u, v) \in E$ , let us construct a graph

$$G_{(u,v)} = (V, \{(u, v)\} \cup \{(t, t) : t \in V \setminus \{v\}\}, \omega_{(u,v)}),$$

where  $\omega_{(u,v)}(u, v) = \omega(u, v)$  and  $\omega_{(u,v)}(t, t) = \alpha$  for every  $t \in V \setminus \{v\}$  (for an example see Figure 3). From Lemma 4:

$$F(G_{(u,v)}) \supseteq \{\mathbf{x} \in \mathbb{R}_+^n : x_v/x_u = \omega(u, v)/\alpha\}.$$

Let us denote an average of all graphs  $G_{(u,v)}$  by  $G'$ . Since  $F$  satisfies Convex Combination and for every  $(u, v) \in E$  we know that  $\mathbf{x} \in F(G'_{(u,v)})$ , we get that  $\mathbf{x} \in F(G')$ .

Finally, let  $G''$  be a graph that contains only loops of weight  $\alpha$ :  $G'' = (V, \{(v, v) : v \in V\}, \omega'')$  with  $\omega''(\cdot) = \alpha$ . Observe that  $G' = 1/|V| \cdot G + (|V| - 1)/|V| \cdot G''$  (see Figure 3). From Lemma 3 we get that  $F(G'') = \mathbb{R}_+^n$ , which implies  $\mathbf{x} \in F(G'')$ . In result, from Convex Combination we get that  $\mathbf{x} \in F(G)$ . This concludes the proof.  $\square$

In the next lemma, we show the key property of the proof of uniqueness. It states that every graph such that for every vertex the sum of weights of all incoming edges is the same is a linear combination of regular unity graphs (for an example see Figure 4).

**Lemma 6.** *For every graph  $G = (V, E, \omega)$  such that  $\sum_{u \in \mathcal{N}_G^-(v)} \omega(u, v) = \beta$ , for every  $v \in V$  and some  $\beta \in \mathbb{R}_+$ , there exist regular unity graphs  $G_1 = (V, E_1, \mathbf{1}), \dots, G_k = (V, E_k, \mathbf{1}) \in \mathcal{RUG}^1$  so that*

$$G = \alpha_1 G_1 + \dots + \alpha_k G_k,$$

for some real values  $\alpha_1, \dots, \alpha_k \in \mathbb{R}_+$ , with  $\sum_{i=1}^k \alpha_i = \beta$ .

*Proof.* We will prove this lemma by induction on the number of edges of the graph  $G$ . For the basis of induction, observe that since the sum of incoming edges of every vertex

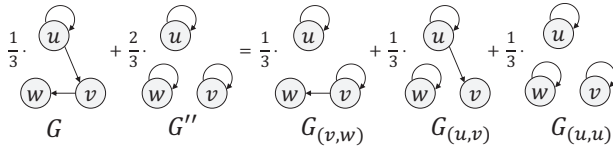


Figure 3: An illustration of the graph construction that is used in the proof of Lemma 5 for an exemplary graph  $G = (\{u, v, w\}, \{(u, u), (u, v), (v, w)\}, \omega)$ .

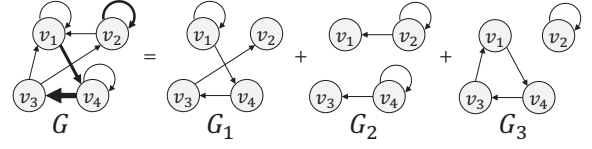


Figure 4: A decomposition of an exemplary graph  $G$  with the sum of weights of incoming edges equal to 3 for its every vertex in the way described by Lemma 6. Thicker edges have weight 2, and the thickest one – weight 3.

is larger than zero, we know that  $|E| \geq |V|$ . If  $|E| = |V|$ , then every vertex must have exactly one incoming edge of weight  $\beta$ . This means  $G$  is a regular unity graph and  $G = \beta \cdot (V, E, \mathbf{1})$ .

Now, assume that  $|E| = m > |V|$  and the thesis holds for every graph with less than  $m$  edges. Let  $\alpha$  be a minimal weight of an edge:  $\alpha = \min_{e \in E} \omega(e)$ . It is clear that  $\alpha < \beta$ , because otherwise all weights of all edges would have to be equal to  $\beta$  and we would get  $|E| = |V|$ .

Let  $E_s$  be any subset of  $E$  that contains at least one edge  $e$  with  $\omega(e) = \alpha$  and includes exactly one incoming edge for every vertex. Thus, graph  $G_s = (V, E_s, \mathbf{1})$  is a regular unity graph. Consider graph  $G' = (V, E', \omega')$  obtained from  $G$  by decreasing weight of each edge from  $E_s$  by  $\alpha$  (and removing edges that will have zero weight as a result). We have  $G = G' + \alpha G_s$ . In result, we get that  $\sum_{u \in \mathcal{N}_{G'}^-(v)} \omega'(u, v) = \beta - \alpha$  for every  $v \in V$  and  $|E'| < |E|$ . With the inductive assumption this concludes the proof.  $\square$

Finally, based on Lemmas 3–6, we prove the main theorem of this section.

**Theorem 7.** *If  $F$  satisfies Source Dependency, Endpoint Removal, Weak Set Locality and Convex Combination, then it is equal to Eigenvector Centrality Function.*

*Proof.* Assume that  $F$  satisfies these requirements. Firstly, let us consider graphs without any edges. For such a graph,  $G = (V, \emptyset, \omega)$ , we have  $EV(G) = \mathbb{R}_+^n$ , as for  $\lambda = 0$  equation (1) trivializes for every  $v \in V$ . Let us construct a graph  $G' = (V, \{(t, t) : t \in V\}, \mathbf{1})$  and see that  $1/2 \cdot G + 1/2 \cdot G' = 1/2 \cdot G'$ . From Lemma 3 we have that  $F(G) = F(1/2 \cdot G') = \mathbb{R}_+^n$ , thus from Convex Combination  $F(G) = EV(G)$ .

In the remainder of the proof we will consider graphs having at least one edge – first we prove that  $EV(G) \subseteq F(G)$  for every such a graph  $G$  and then we prove that  $F(G) \subseteq EV(G)$  as well.

Let  $G = (V, E, \omega)$  be a graph,  $\mathbf{x} \in EV(G)$  and  $\lambda$  be the corresponding eigenvalue, i.e., real value for which (1) is satisfied. We need to show that  $\mathbf{x} \in F(G)$ . Consider graph  $G' = (V, E, \omega')$  where  $\omega'(u, v) = \omega(u, v) \cdot x_u / \lambda x_v$ , for every  $(u, v) \in E$ . From (1) we get that the sum of weights of incoming edges of every node equals 1. Hence, on behalf of Lemma 6 we know that we can write  $G'$  as a sum:  $G' = \alpha_1 G'_1 + \dots + \alpha_k G'_k$ , where  $G'_i = (V, E_i, \mathbf{1}) \in \mathcal{RUG}^1$  for every  $i \in \{1, \dots, k\}$  and  $\sum_{i=1}^k \alpha_i = 1$ .

For every  $i \in \{1, \dots, k\}$ , let us construct a new graph,  $G_i = (V, E_i, \omega_i)$ , obtained from  $G_i$  by modifying weights of edges as follows:  $\omega_i(u, v) = \lambda \cdot x_v/x_u$  for every  $(u, v) \in E_i$ . Note that graphs  $G_1, \dots, G_k$  are also regular unity graphs – each graph has exactly one incoming edge for every vertex and the weight of every cycle,  $c$ , equals  $\lambda^{|c|}$ . Also, observe that  $\mathbf{x} \in EV(G_i)$ , since  $\sum_{u \in \mathcal{N}_{G_i}^-(v)} \omega_i(u, v) \cdot x_u = \lambda x_v$  for every  $v \in V$ , which means (1) is satisfied. From Lemma 5 we know that since  $\mathbf{x} \in EV(G_i)$ , then  $\mathbf{x} \in F(G_i)$ .

Finally, we show that  $G$  is a convex combination of graphs  $G_1, \dots, G_k$ . Since the weight of every edge,  $(u, v)$ , in  $\alpha_1 G_1 + \dots + \alpha_k G_k$  is equal to  $\omega(u, v) \cdot x_u/\lambda x_v$ , then we get that  $\alpha_1 G_1 + \dots + \alpha_k G_k = G$ . From Convex Combination we get  $\mathbf{x} \in F(G)$ . This concludes the proof that  $EV(G) \subseteq F(G)$ .

In the remainder of the proof, we show that  $F(G) \subseteq EV(G)$ . To this end, let us take any  $\mathbf{x} \in F(G)$ . We will prove by contradiction that it implies  $\mathbf{x} \in EV(G)$ . Assume  $G = (V, E, \omega)$  is the graph with the smallest number of proper edges such that  $\mathbf{x} \in F(G)$ , but  $\mathbf{x} \notin EV(G)$ . If there are no proper edges, i.e., all edges in  $G$  are loops, then from Lemma 2 and 3 we know that  $\mathbf{x} \in F(G)$  implies  $\mathbf{x} \in EV(G)$ . Assume otherwise. Let  $(u, v) \in E$  be a proper edge from  $G$ . Consider two graphs:

$$G_1 = (V, \{(t, t) : t \in V\}, \omega_1), \quad \omega_1(\cdot) = \omega(u, v) \cdot x_u/x_v,$$

$$G_2 = (V, \{(u, v)\} \cup \{(t, t) : t \in V \setminus \{v\}\}, \omega_2),$$

$$\omega_2(u, v) = \omega(u, v), \quad \forall_{t \in V \setminus \{v\}} \omega_2(t, t) = \omega(u, v) \cdot x_u/x_v.$$

From Lemma 3 we get that  $\mathbf{x} \in F(G_1)$ . Moreover, by checking condition  $x_u/x_v = \omega_2(u, u)/\omega_2(u, v)$ , from Lemma 4 we get that  $\mathbf{x} \in F(G_2)$ . Now, let us define graph  $G_3$  that satisfies the equation  $1/2 \cdot G + 1/2 \cdot G_1 = 1/2 \cdot G_2 + 1/2 \cdot G_3$ :

$$G_3 = (V, E \setminus \{(u, v)\}, \omega_3), \quad \omega_3(v, v) = \omega(v, v) + \omega(u, v) \frac{x_u}{x_v}$$

and  $\omega_3(e) = \omega(e)$ , for every  $e \in E \setminus \{(u, v), (v, v)\}$ . From Convex Combination we get that  $\mathbf{x} \in F(G_3)$  and, since  $G_3$  has less proper edges than  $G$ , we know that  $\mathbf{x} \in EV(G_3)$ . Thus, from (1) we know that there exists  $\lambda \in \mathbb{R}_+$  such that  $x_t = 1/\lambda \sum_{s \in \mathcal{N}_{G_3}^-(t)} \omega_3(s, t) x_s$ , for every  $t \in V$ .

Since only incoming edges of  $v$  differ between graphs  $G_3$  and  $G$ , then only the equation for  $t = v$  differs in both cases. Nevertheless, by looking at the definition of  $\omega_3$  we get that  $x_v = 1/\lambda \left( \omega(u, v) \cdot x_u + \sum_{s \in \mathcal{N}_{G_3}^-(v)} \omega(s, v) x_s \right) = 1/\lambda \sum_{s \in \mathcal{N}_G^-(v)} \omega(s, v) x_s$ . In result, we get  $\mathbf{x} \in EV(G)$ . This contradiction concludes the proof.  $\square$

It is worth mentioning that all four axioms are required to get uniqueness.

## Axiomatization of the Katz Centrality

In this section, we will provide our axiomatization of the Katz Centrality. What is important, this axiomatization is close to the axiomatization of the Eigenvector Centrality. Specifically, our axiomatization will consist of two previously introduced axioms, namely Endpoint Removal and

Convex Combination, and two new axioms, that we call *Compound Dependency* and *Set Locality*.

First new axiom, named *Compound Dependency*, is a modification of the Source Dependency, which constitutes a borderline case in the Eigenvector Centrality axiomatization. Using our journal citation interpretation assume there are only two journals, the second one not citing anyone, and the first one citing the other journal  $k$  times more often than itself. Source Dependency states that the second journal is  $k$  times more important. It is clear from the definition that because of the constant  $b$  added, the Katz Centrality does not satisfy this condition. However, what we can say, is that the proportion between relative difference between importance of both journals is some fixed part (equal to  $a$ ) of the difference between the numbers of citations.

**Compound Dependency:** *There exist  $a > 0$ , such that for every two vertices  $u, v$  and every graph  $G = (\{u, v\}, E, \omega)$  with  $E \subseteq \{(u, u), (u, v)\}$ :*

$$\mathbf{x} \in F(G) \Rightarrow \frac{x_v - x_u}{x_u} = a \cdot (\omega^*(u, v) - \omega^*(u, u)),$$

$$\text{and } F(G) = \emptyset \Leftrightarrow a \cdot \omega^*(u, u) \geq 1.$$

Our second new axiom is called *Set Locality*. Consider a graph that consists of several independent parts. Set Locality simply states that the centrality of this graph is a product of sets of centrality in these parts taken separately.

**Set Locality:** *For every  $G_1 = (V_1, E_1, \omega_1), \dots, G_k = (V_k, E_k, \omega_k)$  such that  $V_i \cap V_j = \emptyset$  for any  $i \neq j$ :*

$$F(G_1 + \dots + G_k) = F(G_1) \times \dots \times F(G_k).$$

The name of this axiom refers to *Locality*, proposed by Skibski et al. (2016). This axiom states that the centrality of a vertex depend solely on a connected part it belongs to. Thus, *Set Locality* is a version of this axiom for centrality functions which by the definition return not one centrality, but a set of centralities.

In what follows, we show that these two axioms along with Endpoint Removal and Convex Combination axiomatize the Katz Centrality. In other words, each centrality function that satisfies these axioms is the Katz Centrality for some constants  $a$  and  $b$ . We begin with a theorem that shows the Katz Centrality for every  $a$  and  $b$  satisfies all four axioms.

**Theorem 8.** *The Katz Centrality satisfies Compound Dependency, Endpoint Removal, Set Locality and Convex Combination.*

To show that the Katz Centrality is the unique centrality function that satisfies our axioms we will again use the unity graphs. However, in the Katz Centrality the relation between vertex centrality and centrality of its only predecessor in a unity graph is not as simple as in the Eigenvector Centrality:

**Lemma 9.** *For a centrality  $\mathbf{x}$  and a unity graph  $G = (V, E, \omega) \in \mathcal{UG}$  we have  $\mathbf{x} \in K_{a,b}(G)$  if and only if  $\omega(u, v) = r_{\mathbf{x}}(u, v)$  for every  $(u, v) \in E$  and  $|\mathcal{N}_G^-(v)| = 0 \Rightarrow x_v = b$  for every  $v \in V$ , where for centrality  $\mathbf{x}$  and constants  $a, b \in \mathbb{R}_+$  function  $r_{\mathbf{x}} : V \times V \rightarrow \mathbb{R}$  is defined as*

follows:

$$r_{\mathbf{x}}(u, v) = \frac{x_v - b}{a \cdot x_u}, \text{ for every } u, v \in V.$$

Moreover, if  $\mathbf{x} \in K_{a,b}(G)$ , then  $K_{a,b}(G) = \{\mathbf{x}\}$ .

*Proof.* Taking (2) for a vertex  $v$  without any predecessors, i.e.,  $|\mathcal{N}_G^-(v)| = 0$ , gives  $x_v = b$ . For every other vertex  $v$  and its only predecessor  $u$  it states  $x_v = a \cdot \omega(u, v)x_u + b$ , which is equivalent to  $\omega(u, v) = (x_v - b)/(a \cdot x_u)$ . The thesis follows from the system of these equations for every vertex  $v \in V$ .  $\square$

In the previous section, our axioms determined single centrality function that happened to be the Eigenvector Centrality. Here, our axioms lead to multiple centrality functions, but we show that each is the Katz Centrality  $K_{a,b}$  for some parameters  $a, b$ . Parameter  $a$  is determined by  $a$  from Compound Dependency. On the other hand, parameter  $b$  is equal to the centrality of a vertex in a graph with only one vertex.

**Lemma 10.** *If  $F$  satisfies Compound Dependency and Set Locality, then there exists  $b \in \mathbb{R}_+$  such that*

$$F(\{v\}, \emptyset, \omega) = \{b\}, \text{ for every } v.$$

*Proof.* Assume that  $F(\{v\}, \emptyset, \omega) = \emptyset$  for some vertex  $v$ . Take any vertex  $u \neq v$ . From Set Locality we get that  $F(\{u, v\}, \emptyset, \omega) = \emptyset$  which contradicts Compound Dependency. Thus, we proved that  $F(\{v\}, \emptyset, \omega) \neq \emptyset$ .

Now, assume that  $b_1 \in F(\{u\}, \emptyset, \omega)$ ,  $b_2 \in F(\{v\}, \emptyset, \omega)$  and  $b_1 \neq b_2$  for vertices  $u \neq v$ . Then, again from Set Locality we get that  $(b_1, b_2) \in F(\{u, v\}, \emptyset, \omega_3)$ , which contradicts Compound Dependency. This also means that  $F(\{u\}, \emptyset, \omega)$  cannot contain two centralities, as at least one of it would be different than a centrality from  $F(\{v\}, \emptyset, \omega)$  (which we know exists). This concludes the proof.  $\square$

In the remainder of this section, we will say that  $F$  satisfies Compound Dependency and Set Locality with constants  $a, b$  if (1)  $F$  satisfies Compound Dependency with constant  $a$ , (2)  $F$  satisfies Set Locality, and (3)  $F$  for a vertex in a graph with only one vertex equals  $b$  (from Lemma 10 we know such  $b$  exists). We will prove that the Katz Centrality  $K_{a,b}$  is the only centrality that satisfies Endpoint Removal, Convex Combination and Compound Dependency and Set Locality with constants  $a, b$ .

The scheme of our proof and is analogous to the corresponding proof for the Eigenvector Centrality. We begin with the lemma concerning graphs that consist only of loops.

**Lemma 11.** *Let  $G = (V, E, \omega) \in \mathcal{UG}$  be a unity graph in which every edge is a loop:  $E \subseteq \{(v, v) : v \in V\}$ . Assume  $F$  satisfies Compound Dependency and Set Locality with constants  $a, b$ . Then,*

$$F(G) = K_{a,b}(G) = \left\{ \mathbf{x} \in \mathbb{R}_+^n : \forall v \in V x_v = \frac{b}{1 - a \cdot \omega^*(v, v)} \right\}.$$

*Proof (sketch).*<sup>3</sup> For every vertex  $u \in V$  that has a loop, we consider a graph  $G_u = (\{u, v\}, \{(u, u)\}, \omega|_{\{u\}})$ , where  $v$

<sup>3</sup>Techniques used in Lemmas 11–13 and Theorem 14 are similar to those from the previous section. Due to space constraints, we present only sketches of the proofs.

is an arbitrary node not from  $V$ . From Compound Dependency, Weak Set Locality and Lemma 10, we obtain that  $F(G_u) = K_{a,b}(G_u)$ . Moreover, centrality functions  $F$  and  $K_{a,b}$  are also equal for the graph obtained from  $G_u$  by removing vertex  $v$ . Since  $G$  can be expressed as the sum of these graphs by using Set Locality we obtain the thesis.  $\square$

The next lemma considers unity graphs with exactly one proper edge.

**Lemma 12.** *Let  $G = (V, E, \omega) \in \mathcal{UG}$  be a unity graph with only one proper edge  $(u, v)$ , i.e.,  $(u, v) \in E$  and  $E \setminus \{(u, v)\} \subseteq \{(t, t) : t \in V\}$ . If  $F$  satisfies Compound Dependency, Endpoint Removal and Set Locality with constants  $a, b$ , then  $F(G) \supseteq K_{a,b}(G)$ .*

*Proof (sketch).* From Compound Dependency, Endpoint Removal and Lemma 11 we obtain that for the subgraph induced by vertices  $u$  and  $v$ , i.e.,  $G|_{\{u,v\}}$ , we have  $F(G|_{\{u,v\}}) = K_{a,b}(G|_{\{u,v\}})$ . In the graph induced by remaining vertices, i.e.,  $G|_{V \setminus \{u,v\}}$ , all of the edges are loops, thus from Lemma 11 we get  $F(G|_{V \setminus \{u,v\}}) = K_{a,b}(G|_{V \setminus \{u,v\}})$ . Since  $G$  is the sum of graphs  $G|_{\{u,v\}}$  and  $G|_{V \setminus \{u,v\}}$ , from Set Locality the thesis follows.  $\square$

Now, let us consider all unity graphs.

**Lemma 13.** *Let  $G = (V, E, \omega) \in \mathcal{UG}$  be a unity graph. If  $F$  satisfies Compound Dependency, Endpoint Removal, Set Locality and Convex Combination with constants  $a, b$ , then  $F(G) \supseteq K_{a,b}(G)$ .*

*Proof (sketch).* The proof follows the similar reasoning as the proof of Lemma 5. We assume that there exists a centrality  $\mathbf{x} \in K_{a,b}(G)$ . Using Lemma 12 for every edge  $e$  in  $G$  we construct a graph,  $G_e$ , that consists of this edge and loops with weights tailored so that  $\mathbf{x} \in F(G_e)$  (for an example see Figure 5). If we denote the average of these graphs by  $G'$ , then from Convex Combination we get  $\mathbf{x} \in F(G')$ . Using Lemma 11 we construct a graph  $G''$  that contains only loops with weights so that  $\mathbf{x} \in F(G'')$ . Since  $G$  is a convex combination of graphs  $G'$  and  $G''$ , the thesis follows from Convex Combination.  $\square$

Finally, using Lemmas 11-13 and Lemma 6 from previous section we show that any centrality function  $F$  satisfying all four axioms is the Katz Centrality.

**Theorem 14.** *If  $F$  satisfies Compound Dependency, Endpoint Removal, Set Locality and Convex Combination with*

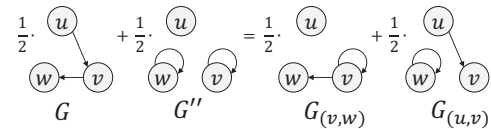


Figure 5: An illustration of the graph construction that is used in the proof of Lemma 13 for an exemplary graph  $G = (\{u, v, w\}, \{(u, v), (v, w)\}, \omega)$ .

constants  $a, b$ , then it is equal to Katz Centrality function with constants  $a, b$ .

*Proof (sketch).* First, we prove that  $K_{a,b}(G) \subseteq F(G)$ , for any graph  $G$ . To this end, we modify the edges and their weights in  $G$ , so that the sum of weights of all incoming edges to every vertex is equal to 1. For this modified graph we use Lemma 6 to get regular unity graphs  $G'_1, \dots, G'_k$  of which it is a convex combination. By modifying them in the opposite manner, we obtain unity graphs  $G_1, \dots, G_k$  and we prove, that taking the same convex combination of these modified unity graphs, will produce graph  $G$ . Using Convex Combination and Lemma 13 concludes this part of the proof.

Second, we consider the graph with the smallest number of proper edges for which there exists  $x$  such that  $x \in F(G)$  and  $x \notin K_{a,b}(G)$ . Then, by contradiction, we prove that it does not exist. If it is a graph with only loops – the contradiction is due to Lemma 11. If not, then we construct a graph,  $G'$ , with one proper edge less than  $G$ . From inductive assumption we know  $x \in F(G')$ . By showing that  $K_{a,b}(G) = K_{a,b}(G')$ , we obtain the thesis.  $\square$

It can be proven that all four axioms are required to obtain uniqueness.

## Related Work

Recently, there have been a number of papers concerning axiomatization of centrality measures (e.g., (Boldi and Vigna 2014; Skibski et al. 2016)). However, just a few of them concentrated on feedback centralities such as the Eigenvector, Katz and PageRank Centralities. We discuss them below.

Dequiedt and Zenou (2014) proposed axioms for the Eigenvector and Katz Centralities for undirected graphs. In their axiomatization, the authors extended the class of graphs by considering graphs with vertices of fixed centrality and used axioms that depends on the maximal eigenvalue  $\lambda$  of a graph. Our axiomatization is provided for directed graphs, does not go beyond the class of standard graphs and do not use eigenvalue in any axiom.

The Eigenvector Centrality has also been an object of one of the axiomatizations by Kitti (2016). Axioms proposed were algebraic properties of adjacency matrix. Kitti did not provide an axiomatization of the Katz Centrality.

Palacios-Huerta and Volij (2004) proposed an axiomatization of simplified version of the PageRank Centrality in a setting of journal citation network, which they called *Invariant method*. The Eigenvector Centrality in that setting was also mentioned (under the name *The Liebowitz-Palmer method*), however they did not provide its complete axiomatization. Altman and Tennenholtz (2005) provided the axiomatization of the same version of the PageRank Centrality in the general setting. The authors proposed several axioms based on simple graph operations. Unlike us, the authors axiomatized not the numerical values of the PageRank Centrality, but the ranking resulting from it.

## Conclusions

In this paper, we studied the axiomatic characterization of the Eigenvector and Katz Centralities. We proved that the

Eigenvector Centrality is the only one that satisfies four axioms – Source Dependency, Endpoint Removal, Weak Set Locality and Convex Combination. Moreover, replacing Weak Set Locality with Set Locality and Source Dependency with Compound Dependency we get the axiomatization of the Katz Centrality. In our future work, we plan to create a similar axiomatization for the PageRank Centrality.

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