

Properties of Position Matrices and Their Elections

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We study the properties of elections that have a given position matrix (in such elections each candidate is ranked on each position by a number of voters specified in the matrix). We show that counting elections that generate a given position matrix is #P-complete. Consequently, sampling such elections uniformly at random seems challenging and we propose a simpler algorithm, without hard guarantees. Next, we consider the problem of testing if a given matrix can be implemented by an election with a certain structure (such as single-peakedness or group-separability). Finally, we consider the problem of checking if a given position matrix can be implemented by an election with a Condorcet winner. We complement our theoretical findings with experiments.

1 Introduction

Studies of voting and elections are at the core of computational social choice [Brandt et al., 2016]. An (ordinal) election is represented by a set of candidates and a collection of voters who rank the candidates from the most to the least appealing one. Such preferences are sometimes shown in an aggregate form as a *position matrix*, which specifies for each candidate the number of voters that rank him or her on each possible position. Motivated by the connection of position matrices to the so-called maps of elections, and their similarity to weighted majority relations, we study the properties of elections with a given position matrix.

The idea of a map of elections, introduced by Szufa et al. [2020] and Boehmer et al. [2021b], is to collect a set of elections, compute the distances between them, and embed the elections as points in the plane, so that the Euclidean distance between points resembles the distance

between the respective elections. Such maps are useful because nearby elections seem to have similar properties (such as, e.g., running times of winner determination algorithms, scores of winning candidates, etc.; see, e.g., the works of Szufa et al. [2020], Boehmer et al. [2021a], and Boehmer and Schaar [2022]). However, there is a catch. The positionwise distance, which is commonly used in these maps, views elections with the same position matrix as identical. Hence there might exist very different elections that, nonetheless, have identical position matrices and in a map are placed on top of each other. We want to evaluate to what extent this issue constitutes a problem for maps of elections.

The second motivation for our studies is that position matrices are natural counterparts of weighted majority relations, which specify for each pair of candidates how many voters prefer one to the other. While weighted majority relations provide sufficient information to determine winners of many Condorcet-consistent voting rules,¹ position matrices provide the information needed by positional scoring rules (i.e., rules where each voter gives each candidate a number of points that depends on this candidate's position in his or her ranking). Together with Condorcet-consistent rules, positional scoring rules are among the most widely studied single-winner voting rules. While weighted majority relations are commonly studied and analyzed (even as early as in the classic theorem of McGarvey [1953]), position matrices have not been studied as carefully.

¹Rules that can be computed using only the weighted majority relation are called C2 by Fishburn [1977]; see also the overview of Zwicker [2015]. A Condorcet winner is preferred to every other candidate by a majority of voters. Condorcet-consistent rules always select Condorcet winners when they exist. Some non-Condorcet-consistent rules are also C2 (e.g., the Borda rule).

Our contributions regard three main issues. First, we ask how similar are elections that have the same position matrix. To this end, we would like to sample elections with a given position matrix uniformly at random. Unfortunately, doing so appears to be challenging. In particular, a natural sampling algorithm requires the ability to count elections that generate a given position matrix, and we show that doing so is $\#P$ -complete. While, formally, there may exist a different approach, perhaps providing only an approximately uniform distribution, finding it is likely to require significant effort (indeed, researchers have been trying to solve related sampling problems for quite a while, without final success as of now; see, e.g., the works of Jacobson and Matthews [1996] and Hong and Miklós [2021]). We design a simpler sampling algorithm, without hard guarantees on the distribution, and use it to evaluate how different two elections with a given position matrix can be. The algorithm, albeit not central to our study, might be of independent interest when considering sampling various preference distributions [Regenwetter et al., 2006, Tideman and Plassmann, 2012, Allen et al., 2017].

Second, we consider structural properties of elections that generate a given position matrix (or its normalized variant, called a frequency matrix). Specifically, given a matrix we ask if there is an election that generates it and whose votes come from a given domain (such as the single-peaked domain [Black, 1958], some group-separable domains [Inada, 1964, 1969], or a domain given explicitly vote-by-vote as part of the input). We show polynomial-time algorithms that, given a frequency matrix and a description of a domain (e.g., via a single-peaked axis or by listing the votes explicitly), decides if there is an election with votes from this domain that generates this matrix. We apply these algorithms to test which frequency matrices from the map of elections can be generated from elections with a particular structure.²

Finally, we consider the problem of deciding for a given position matrix if there is an election that implements the matrix and has a Condorcet winner (i.e., a candidate who is preferred to every other one by a strict majority of voters). We evaluate experimentally which matrices from our map have such elections, provide a necessary condition for such elections to exist, and check how often this condition is effective on the map of elections. Additionally, for each matrix from the map we compute for how many different candidates there is an election that generates this matrix and where this candidate is a Condorcet winner.

With our theoretical and empirical analysis, we ultimately want to answer the question how much information is contained in a position matrix and how much flex-

ibility is still left when implementing it.³

2 Preliminaries

For each $k \in \mathbb{N}_+$, by $[k]$ we denote the set $\{1, \dots, k\}$. Given a matrix X , by $X_{i,j}$ we mean its entry in row i and column j . For two equal-sized sets X and Y , by $\Pi(X, Y)$ we mean the set of one-to-one mappings from X to Y . S_n is a shorthand for $\Pi([n], [n])$, i.e., the set of permutations of $[n]$.

An *election* \mathcal{E} is a pair $(\mathcal{C}, \mathcal{V})$ consisting of a set $\mathcal{C} = \{c_1, c_2, \dots, c_m\}$ of *candidates* and a collection $\mathcal{V} = (v_1, v_2, \dots, v_n)$ of *votes*, i.e., complete, strict orders over the candidates. These orders rank the candidates from the most to the least appealing one according to a given voter (we use the terms “vote” and “voter” interchangeably). If some voter v prefers candidate c over candidate c' , then we write $c \succ_v c'$; we omit the subscript when it is clear from context. Given a vote $v_i: c_1 \succ c_2 \succ \dots \succ c_m$, we say that v_i ranks c_1 on the first position, c_2 on the second one, and so on. For two votes u and v over the same candidate set, their swap distance, $\text{swap}(u, v)$, is the smallest number of swaps of adjacent candidates necessary to transform u into v .

In an election $\mathcal{E} = (\mathcal{C}, \mathcal{V})$, a candidate $c \in \mathcal{C}$ is a *Condorcet winner* of the election if for every other candidate d more than half of the voters prefer c to d .

2.1 Position and Frequency Matrices

Let \mathcal{E} be some election and assume that the candidates are ordered in some way (e.g., lexicographically, with respect to their names). The *position matrix* of \mathcal{E} (with respect to this order) is a non-negative, integral $m \times m$ matrix X such that for each $i, j \in [m]$, $X_{i,j}$ is the number of voters that rank the j -th candidate on the i -th position. By $P(\mathcal{E})$ we denote the set of all position matrices of \mathcal{E} for all possible orderings of candidates. Note that the matrices in $P(\mathcal{E})$ only differ by the order of their columns.

For a position matrix $X \in P(\mathcal{E})$, where \mathcal{E} is an election with n voters, the corresponding *frequency matrix* is $Y := \frac{1}{n} \cdot X$. In other words, frequency matrices are normalized variants of the position ones, where each value $Y_{i,j}$ gives the fraction of voters that rank the j -th candidate on the i -th position. Every frequency matrix is *bistochastic*, i.e., the elements in each row and in each column sum up to one. Hence, we often refer to bistochastic matrices as frequency matrices, and to integral square matrices with nonnegative entries, where each row and each column sums up to the same value, as position matrices.

We say that an election \mathcal{E} *realizes* (or *generates*) a position matrix X (or, a frequency matrix Y) if $X \in P(\mathcal{E})$

²We form a map that is analogous to that used by Boehmer et al. [2021b], but which uses 8 candidates rather than 10 (using fewer candidates helps significantly with our computation times).

³The code for the experiments is available at: <https://github.com/Project-PRAGMA/Position-Matrices-AAAI-2023>.

(or, $n \cdot Y \in P(\mathcal{E})$, where n is the number of voters in \mathcal{E}). Boehmer et al. [2021b] showed that every position matrix X is realizable by some election (their result is a reinterpretation of an older result of Leep and Myerson [1999]). Yang and Guo [2016] also showed that position matrices are always realizable as part of a proof that they can be used to solve a Borda manipulation problem. Note that two distinct elections may generate the same position matrix.

Example 1. Consider an election \mathcal{E} with candidates a , b , c , and d and four votes shown below on the left. On the right we show a position matrix of this election (for the natural ordering of the candidates):

$$\begin{array}{l} v_1: a \succ b \succ c \succ d, \\ v_2: b \succ a \succ d \succ c, \\ v_3: a \succ b \succ d \succ c, \\ v_4: b \succ a \succ c \succ d. \end{array} \quad \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} a \quad b \quad c \quad d \\ \left[\begin{array}{cccc} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{array} \right] \end{array}$$

Note that this is also a position matrix for an election with two votes $a \succ b \succ c \succ d$ and two votes $b \succ a \succ d \succ c$.

2.2 Structured Domains

We are interested in elections where the votes have some structure. For example, the single-peaked domain captures votes on the political left-to-right spectrum (and, more generally, votes focused on a single issue, such as those regarding the temperature in a room or the level of taxation).

Definition 1. An election $\mathcal{E} = (\mathcal{C}, \mathcal{V})$ is single-peaked if there is an order \triangleright (the societal axis) over candidates \mathcal{C} such that for each vote $v \in \mathcal{V}$ and for each $\ell \leq |\mathcal{C}|$, the top ℓ candidates according to v form an interval with respect to \triangleright .

Intuitively, in a single-peaked election each voter first selects their favorite candidate and, then, extends his or her ranking step by step with either the candidate directly to the left or directly to the right (wrt. \triangleright) of those already ranked.

Group-separability captures settings where the candidates have some features and the voters have hierarchical preferences over these features. Let \mathcal{C} be a set of candidates and consider a rooted, ordered tree \mathcal{T} , where each leaf one-to-one corresponds to a candidate. A *frontier* of \mathcal{T} is a vote that we obtain by reading the names of the candidates associated with the leaves of \mathcal{T} from left to right. A vote is *compatible* with \mathcal{T} if it can be obtained as its frontier by reversing for some nodes in \mathcal{T} the order of their children. Intuitively, we view the internal nodes of \mathcal{T} as features and a candidate has the features that appear on the path from it to the root.

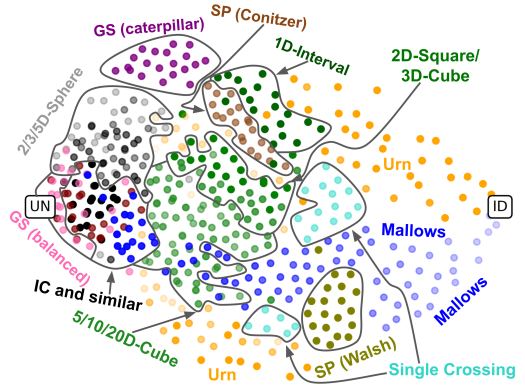


Figure 1: Map of elections visualizing the 8x80 dataset. Each dot represents an election and its color corresponds to the statistical model used to generate it. x D-Cube/Sphere models are Euclidean models where the points of the candidates and voters are chosen uniformly at random from an x -dimensional hypercube/hypersphere (1D-Interval is 1D-Cube; 2D-Square is 2D-Cube). For Mallows and Urn elections the transparency of the coloring indicates the value of the used parameter. For the other models, see Appendix A.

Definition 2. An election $\mathcal{E} = (\mathcal{C}, \mathcal{V})$ is group-separable if and only if there is a tree \mathcal{T} over candidate set \mathcal{C} such that each vote from \mathcal{V} is compatible with \mathcal{T} .

We focus on balanced trees (i.e., complete binary trees) and on caterpillar trees (i.e., binary trees where each non-leaf has at least one leaf as a child). If an election is group-separable for a balanced tree, then we say that this election is *balanced group-separable*. Analogously, we speak of *caterpillar group-separable elections*.

Example 2. The election from Example 1 is both single-peaked (for societal axis $c \triangleright a \triangleright b \triangleright d$) and balanced group-separable (for a tree whose frontier is $a \succ b \succ c \succ d$).

Single-peaked elections were introduced by Black [1958], and group-separable ones by Inada [1964, 1969]. We mention that Inada’s original definition is different from the one that we provided, but they are equivalent [Karpov, 2019] and the tree-based one is algorithmically much more convenient. We point readers interested in structured domains to the recent survey of Elkind et al. [2022].

2.3 Map of Elections

For our experiments, we use an 8×80 dataset that resembles those of Szufa et al. [2020], Boehmer et al. [2021b], and Boehmer et al. [2022b]. It contains 480 elections with

8 candidates and 80 votes generated using the same statistical models, with the same parameters, as the map of Boehmer et al. [2022b]. In particular, we used (i) impartial culture (IC), where each vote is equally likely, (ii) the Mallows and urn distributions, whose votes are more or less correlated, depending on a parameter, (iii) various Euclidean models, where candidates and voters are points in Euclidean spaces and the voters rank the candidates with respect to their geometric distance, and (iv) uniform distributions over balanced group-separable, caterpillar group-separable, and single-peaked elections (we refer to the uniform distribution of single-peaked elections as the Walsh model; we also use the model of Conitzer [2009] to generate single-peaked elections). See Appendix A for exact descriptions. We repeated all our experiments from Sections 4 and 5 on analogously composed datasets with a varying number of candidates and voters. Specifically, we considered elections with either 4 or 8 candidates and either 40, 80, or 160 voters. The results on those datasets were similar to those for the 8x80 one.

We present our dataset as a map of elections, i.e., as points on a plane, where each point corresponds to an election (see Fig. 1). The Euclidean distances between the points resemble positionwise distances between the respective elections. For a definition of the positionwise distance, we point the reader to the work of Szufa et al. [2020] or to Appendix A; an important aspect of this distance is that for two elections \mathcal{E} and \mathcal{F} (with the same numbers of candidates and voters) it depends only on $P(\mathcal{E})$ and $P(\mathcal{F})$. Hence, we will also sometimes speak of the distance between position matrices.

Our maps include two special position matrices, the uniformity one (UN), which corresponds to elections where each candidate is ranked on each position equally often, and the identity one (ID), which corresponds to elections where all votes are identical. ID models “perfect order,” whereas UN models “perfect chaos” (but note that there exist very structured elections whose position matrix is UN). UN and ID, as well as two other special points, were introduced by Boehmer et al. [2021b]. For each two elections, their positionwise distance is at most as large as the distance between UN and ID [Boehmer et al., 2022b].

3 Counting and Sampling Elections

Given a position matrix, it would be useful to be able to sample elections that realize it uniformly at random. Unfortunately, doing so seems challenging. Indeed, one of the natural sampling algorithms (presented in Appendix B.1) requires, among others, the ability to count elections that realize a given matrix, a task which we show to be $\#P$ -complete. While, formally, this does not pre-

clude the existence of a polynomial-time uniform sampler (and, certainly, it does not preclude the existence of an approximately uniform one), we believe that it suggests that finding such algorithms would require deep insights; for closely related problems such insights are still elusive [Jacobson and Matthews, 1996, Hong and Miklós, 2021].

Formally, in the $\#REALIZATIONS$ problem we are given an $m \times m$ position matrix X (and a candidate set $\{c_1, \dots, c_m\}$, where, for each i , candidate c_i corresponds to the i -th column of X) and we ask for the number of elections that realize X . Two elections are distinct if their voter collections are distinct when viewed as multisets.

Theorem 1. *$\#REALIZATIONS$ is $\#P$ -complete even if the realizing elections contain three votes.*

3.1 Preparing for The Proof of Theorem 1

We first provide the necessary background for our proof of Theorem 1. Given a graph G , directed or undirected, a t -edge coloring is a function that associates each of its edges with one of t colors. Such a coloring is proper if for each vertex the edges that touch it have different colors. A graph is r -regular if each vertex touches exactly r edges (for directed graphs, both incoming and outgoing edges count). The $\#P$ -hardness of $\#REALIZATIONS$ follows by a reduction from the problem of counting proper 3-edge colorings of a given 3-regular bipartite (simple) graph. We refer to this problem as 3-REG.-BIPARTITE-3-EDGE-COLORING. We start by establishing that this problem is $\#P$ -hard. To prove this, we will give a reduction from a specific Holant problem, which we will call HOLANT-SPECIAL. In this problem we are given a planar, 4-regular, directed graph G , where each vertex has two incoming edges and two outgoing ones. Further, we have an embedding of this graph on the plane, which has the following property: As we consider the edges touching a given vertex in the counter-clockwise order, every other edge is incoming and every other one is outgoing. Let \mathcal{C} be the set of all 3-edge-colorings of G . Given a vertex v , its four touching edges e_1, \dots, e_4 (listed in the counter-clockwise order, starting from some arbitrary one) and some coloring $\sigma \in \mathcal{C}$, we denote by $\sigma(v)$ the vector $(\sigma(e_1), \dots, \sigma(e_4))$. We define a function f so that:

1. $f(\sigma(v)) = 0$ if $\sigma(v)$ includes three different colors,
2. $f(\sigma(v)) = 2$ if all colors in $\sigma(v)$ are identical,
3. $f(\sigma(v)) = 1$ if $\sigma(v)$ includes two different colors and there are two consecutive edges in the counter-clockwise order that have the same color,
4. $f(\sigma(v)) = 0$ otherwise (i.e., if $\sigma(v)$ includes two different colors and each two consecutive edges in the counter-clockwise order have different colors).

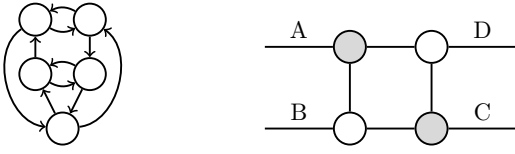


Figure 2: An example input graph (left) and the gadget used in the proof of Theorem 2. The letters label the gadget’s dangling edges. The colors illustrate its bipartiteness.

The goal is to compute $\sum_{\sigma \in \mathcal{C}} \prod_{v \in V(G)} f(\sigma(v))$. Cai et al. [2016] have shown that doing so is $\#P$ -complete (their results are far more general than this; the problem we consider is a variant of their $\langle 2, 1, 0, 1, 0 \rangle$ -HOLANT problem). The left-hand side of Fig. 2 shows an example input for HOLANT-SPECIAL.

Theorem 2. $\#3\text{-REG.-BIPARTITE-3-EDGE-COLORING}$ is $\#P$ -hard.

Proof. We give a reduction from HOLANT-SPECIAL to $\#3\text{-REG.-BIPARTITE-3-EDGE-COLORING}$. The construction is inspired by those used by Cai et al. [2016]. Let $G = (V, E)$ be the input graph and let the notation be as in the discussion preceding the theorem statement.

The high-level idea is to modify graph G by replacing each vertex $v \in V$ with a gadget, while keeping “copies” of edges from E . Then, the value of $f(\sigma(v))$ for some edge-coloring σ of the edges from E in G corresponds to the number of proper 3-edge-colorings in the gadget for v assuming the “copies” of E in the constructed graph are colored according to σ . Specifically, we replace each vertex v by the gadget depicted in the right-hand side of Fig. 2. Its four dangling edges implement the original four edges of v . However, we need some care in deciding which of the dangling edges we connect to which vertices from the gadgets corresponding to the neighbors of v in G (we will return to this issue after we explain how the gadget works).

For each of our gadgets, we name the dangling edges A , B , C , and D , as shown in Fig. 2. It is now easy to see that if all dangling edges are of the same color, say 1, then there are two colorings of the remaining edges of the gadget resulting in a proper coloring: Both “vertical” edges need to have the same color (either 2 or 3), and both “horizontal” edges need to have the same color (the single remaining one). Similarly, if edges A and B have the same color, and edges C and D have the same color, then there is a unique proper coloring of the other edges. By symmetry, the same holds if both edges A and D and edges B and C have the same color. Finally, if edges A and C have the same color, and edges B and D have the same color (or, the dangling edges have three different colors) then there are no proper colorings of the remaining edges

in the gadget. This way, for each vertex v and coloring σ , v ’s gadget implements the $f(\sigma(v))$ function.

Next we describe how we connect the dangling edges of the gadgets. If u and v are two vertices of G and there is a directed edge from u to v , then we merge one of the A and C dangling edges of u ’s gadget with one of the B and D dangling edges of v ’s gadget (which dangling edges we use is irrelevant for this proof). Since each vertex in G has two incoming and two outgoing edges, doing so is possible.

As the gadgets are bipartite themselves, and due to the way in which we connect their edges, the resulting graph G' is bipartite. It is also clear that it is 3-regular. Finally, due to the way in which 3-edge-colorings of G' can be extended to proper 3-edge-colorings of G (see the description of the gadgets), we see that the number of the latter is equal to the output of the HOLANT-SPECIAL for G . The reduction runs in polynomial-time and the proof is complete. \square

The above result also applies to 3-regular planar bipartite graphs. To see this, it suffices to appropriately arrange our gadgets in space (sometimes rotating them) and choose the dangling edges to connect more carefully.

3.2 The Proof of Theorem 1

The answer to an instance of $\#REALIZATIONS$ is the number of accepting paths of a non-deterministic Turing machine that constructs an election and then checks if it realizes the input matrix. As this machine works in (non-deterministic) polynomial time, $\#REALIZATIONS$ is in $\#P$.

To show $\#P$ -hardness, we give a reduction from $\#3\text{-REG.-BIPARTITE-3-EDGE-COLORING}$ to $\#REALIZATIONS$. Let $G = (U, V; E)$ be our input 3-regular bipartite graph, where U is the set of vertices on the left, V is the set of vertices on the right, and E is a set of edges. Since G is 3-regular, we have $|U| = |V|$. W.l.o.g., we let $U = \{u_1, \dots, u_m\}$ and $V = \{v_1, \dots, v_m\}$. We form an $m \times m$ matrix X , where each entry $X_{i,j}$ is either 1, if there is an edge between v_i and u_j , or 0, if there is no such edge. As G is 3-regular, X has exactly three ones in each row and in each column, so it is a position matrix and each election that realizes it contains three votes.

We now show that each proper 3-edge-coloring of G gives an election realizing matrix X . For a given coloring, the edges of the same color form a perfect matching in G . We interpret such a matching as a single vote. Specifically, we treat vertices from U as candidates and vertices from V as positions in the vote being constructed (e.g., if the matching contains an edge between v_i and u_j , then the vote ranks candidate u_j on position i). Hence, for each 3-coloring we get an election consisting of three votes,

one for each matching associated with one color. Since all edges must be part of some matching and each edge corresponds to a single 1-entry in X , the resulting election realizes X .

Each election realizing matrix X corresponds to six 3-edge-colorings of G . Indeed, taking one 3-edge-coloring, each of the six permutations of the colors gives raise to the same election. This holds, because for a single 3-edge-coloring, each color forms an edge-disjoint matching (as opposed to graphs with parallel edges, where this would not be true). So our reduction preserves the number of solutions with a multiplicative factor of 6. This completes the proof.

3.3 Experiments

We checked experimentally how diverse are elections that generate the same position matrix. To do so, we used the isomorphic swap distance, due to Faliszewski et al. [2019].

Definition 3. Let $\mathcal{E} = (\mathcal{C}, \mathcal{V})$ and $\mathcal{F} = (\mathcal{D}, \mathcal{U})$ be two elections, where $\mathcal{C} = \{c_1, \dots, c_m\}$, $\mathcal{D} = \{d_1, \dots, d_m\}$, $\mathcal{V} = (v_1, \dots, v_n)$, and $\mathcal{U} = (u_1, \dots, u_n)$. Their isomorphic swap distance is:

$$d_{\text{swap}}(\mathcal{E}, \mathcal{F}) = \min_{\sigma \in S_n} \min_{\pi \in \Pi(\mathcal{C}, \mathcal{D})} \sum_{i=1}^n \text{swap}(\pi(v_i), u_{\sigma(i)}),$$

where $\pi(v_i)$ is the vote v_i where each candidate $c \in \mathcal{C}$ is replaced with candidate $\pi(c)$.

Intuitively, the isomorphic swap distance between two elections is the summed swap distance of their votes, provided we first rename the candidates and reorder the votes to minimize this value. Maps of elections could be generated using the isomorphic swap distance instead of the positionwise one, and they would be more accurate than those based on the positionwise distance [Boehmer et al., 2022b], but the isomorphic swap distance is NP-hard to compute and challenging to compute in practice [Faliszewski et al., 2019]; indeed, we use a brute-force implementation.

Boehmer et al. [2022b] have shown that the largest isomorphic swap distance between two elections with m candidates and n voters is $\frac{1}{4}n(m^2 - m)$ (up to minor rounding errors; for this result, see their technical report). Whenever we give an isomorphic swap distance between two elections (with the same numbers of candidates and voters), we report it as a fraction of this value.

As we do not have a fast procedure for sampling (approximately) uniformly at random elections that realize a given matrix, we use the following naive approach (let X be an $m \times m$ position matrix):

1. We form an election $\mathcal{E} = (\mathcal{C}, \mathcal{V})$, where $\mathcal{C} = (c_1, \dots, c_m)$ and \mathcal{V} is initially empty. For each $i \in$

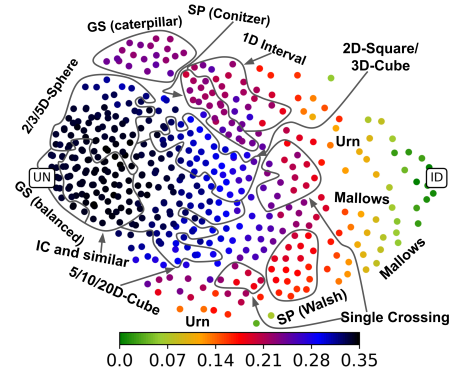


Figure 3: The maximum isomorphic swap distance found for elections realizing a given matrix in our 8x80 dataset.

$[m]$, candidate c_i corresponds to the i -th column of the matrix.

2. We repeat the following until X consists of zeros only: We form a bipartite graph with vertices c_1, \dots, c_m on the left and vertices $1, \dots, m$ on the right; there is an edge between c_j and i exactly if $X_{i,j} > 0$. We draw uniformly at random a perfect matching in this graph (it always exists; Leep and Myerson [1999])—we generate it relying on the standard self-reducibility of computing perfect matchings and using the classic reduction to computing the permanent [Valiant, 1979], which we compute using the formula of Ryser [1963].⁴ Given such a matching, we form a vote v where each candidate $c_j \in \mathcal{C}$ is ranked on the position to which he or she is matched. We extend \mathcal{E} with vote v and we subtract from X the position matrix of the election that contains v as the only vote.

In essence, the above procedure is a randomized variant of the algorithm presented by Boehmer et al. [2021b] to show that every position matrix is realized by some election.

We performed the following experiment: (i) For each election from the 8x80 dataset we computed its position matrix, (ii) using the naive sampler, we generated 100 pairs of elections that realize it, and, (iii) for each pair of elections, we computed their isomorphic swap distance. We report the results in Figure 3, where each dot has a color that corresponds to the farthest distance computed for the respective matrix.⁵ For elections close to UN, this

⁴We used a python module called *permanent* (<https://git.peteshadbolt.co.uk/pete/permanent>) by Pete Shadbolt. In principle, we could have used an approximately uniform sampler that runs in polynomial time [Jerrum et al., 2004, Bezáková et al., 2008], but they are too slow in practice.

⁵We tried 100 pairs for two reasons. First, each computation is quite expensive. Second, even with testing 10 pairs the results were very similar to those for 100 pairs (if we reported average distances, the results also would not change very much).

distance can be very large. Indeed, for about half of the elections (all located close to UN) this distance is larger than 20% of the maximum possible isomorphic swap distance. On the other hand, elections realizing position matrices in the vicinity of ID are much more similar to each other, which is quite natural.

While we used a naive sampling algorithm rather than a uniform one, the results are sufficient to claim that for many position matrices—in particular, those closer to UN than to ID—there are two elections that generate them, whose isomorphic swap distance is very large. If we had a uniform sampler, the distances could possibly increase, but the overall conclusion would not change. Indeed, we ran our experiment for an analogous dataset, but for elections with 4 candidates and 16 voters; in this case we computed maximum possible isomorphic swap distances by generating all elections realizing a given matrix. The results, presented in Appendix B.2, are analogous. (For this experiment we also counted how many elections realize a given matrix and the results were strongly correlated with the above described results for the maximum distance.)

4 Recognizing Structure

In this section we consider the problem of deciding if a given (arbitrary) position or frequency matrix can be realized by elections whose votes come from some domain (e.g., the single-peaked or group-separable one). Overall, we find that if a precise description of the domain is part of the input (e.g., if we are given the societal axis for the single-peaked domain), then for frequency matrices we can often solve this problem in polynomial time. For position matrices our results are less positive and less comprehensive. The reason why frequency matrices are easier to work with here is that they only specify fractions of votes where a given candidate appears on a given position, whereas position matrices specify absolute numbers of such votes and thus are less flexible.

Let us fix a candidate set \mathcal{C} . We consider sets \mathcal{D} of votes, called domains, specified in one of the following ways:

1. *explicit*: \mathcal{D} contains explicitly listed votes, or
2. *single-peaked*: \mathcal{D} contains all votes that are single-peaked with respect to an explicitly given axis \triangleright , or
3. *group-separable*: \mathcal{D} contains all group-separable votes that are compatible with a given rooted, ordered tree \mathcal{T} , where each leaf is associated with a unique candidate. We only consider balanced or caterpillar trees.

The next theorem is our main result of this section.

Theorem 3. *There is a polynomial-time algorithm that given a frequency matrix X and an explicit, single-peaked, or group-separable (balanced or caterpillar) domain \mathcal{D} , decides if there is an election that realizes X , and whose votes all belong to \mathcal{D} .*

For example, given a frequency matrix X and a societal axis \triangleright , we can check if there is an election that realizes X and is single-peaked with respect to \triangleright . A similar result for single-peakedness and a variant of weighted majority relations is provided by Spaanjar and Weng [2016].

The proof of Theorem 3 is quite involved and is relegated to Appendix C.1, but we mention two issues. First, some of our algorithms proceed by solving appropriate linear programs and, in principle, the elections that they discover might have exponentially many votes with respect to the length of the encoding of the input. This is not a problem as our algorithms do not build these elections explicitly. Second, while our algorithms need an explicit description of the domain, such as the societal axis or the underlying tree, for the balanced group-separable domain we can drop this assumption, and we can even deal with position matrices:

Theorem 4. *There is a polynomial-time algorithm that given a frequency (or position) matrix X decides if the matrix can be realized by a balanced group-separable election.*

Interestingly, if instead of taking the entire balanced group-separable domain (for a given tree) we only allow for an explicitly specified subset of its votes, the problem becomes NP-hard.

Theorem 5. *Given a set \mathcal{D} of votes, listed explicitly, and a position matrix X , it is NP-hard to decide if there is an election that realizes X and whose votes are all from \mathcal{D} . This holds even if the votes in \mathcal{D} are both single-peaked and balanced group-separable.*

Proof. We reduce from the NP-hard X3C problem, where we are given a set $\mathcal{U} = \{u_1, \dots, u_{3m}\}$ of $3m$ elements and a family $\mathcal{S} = \{S_1, \dots, S_n\}$ of size-3 subsets of \mathcal{U} . We ask if there are m sets from \mathcal{S} whose union is \mathcal{U} .

Let I be our input instance of X3C. We form a $6m \times 6m$ position matrix X with values $m-1$ on the diagonal, and where for each odd column there is value 1 directly below the $m-1$ entry, and for each even column there is value 1 directly above the $m-1$ entry (all other entries are equal to 0). The matrix looks as follows:

$$\begin{bmatrix} m-1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & m-1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & m-1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & m-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & m-1 \end{bmatrix}.$$

We let the candidate set be $\mathcal{C} = \{c_1, \dots, c_{6m}\}$, where for each $i \in [6m]$, candidate c_i corresponds to the i -th column. For each set $S_\ell = \{u_i, u_j, u_k\}$ we include in the domain \mathcal{D} a vote v_ℓ that is equal to $c_1 \succ c_2 \succ \dots \succ c_{6m}$ except that c_{2i} and c_{2i+1} are swapped, c_{2j} and c_{2j+1} are swapped, and c_{2k} and c_{2k+1} are swapped. We claim that there is an election that realizes X and whose votes all belong to \mathcal{D} if and only if I is a *yes*-instance of X3C.

Let us assume that there are m sets from \mathcal{S} whose union is \mathcal{U} and, w.l.o.g., that these sets are S_1, \dots, S_m . One can verify that election $(\mathcal{C}, (v_1, \dots, v_m))$ realizes X . Indeed, since S_1, \dots, S_m cover \mathcal{U} , for each candidate c_i there are $m - 1$ votes where c_i is ranked on the i -th position, and a single vote where c_i is either ranked one position higher or one position lower, depending on the parity of i .

For the other direction, let us assume that there is an election \mathcal{E} that realizes X and, w.l.o.g., that it contains votes v_1, \dots, v_m (all the votes must be distinct as otherwise some non-diagonal entry of this election's position matrix would have value greater than 1). Since \mathcal{E} realizes X , for each $i \in [3m]$ there is exactly one vote in \mathcal{E} that ranks c_{2i} below c_{2i+1} . This means that for each $u_i \in \mathcal{U}$, there is exactly one set among S_1, \dots, S_m that includes u_i . Hence, I is a *yes*-instance of X3C.

Finally, we observe that all votes in \mathcal{D} are single-peaked with respect to societal axis $c_{6m-1} \succ c_{6m-3} \succ \dots \succ c_3 \succ c_1 \succ c_2 \succ c_4 \succ \dots \succ c_{6m}$ and are balanced group-separable with respect to a balanced tree whose frontier is $c_1 \succ c_2 \succ \dots \succ c_{6m}$ (to be formally correct, we would need to have a number of candidates equal to a power of two, it is easy to achieve this by adding at most $6m$ dummy candidates and extending matrix X so that for these candidates the diagonal entry would be equal to m). \square

An Experiment. Using our algorithms from Theorems 3 and 4, we checked for each of the frequency matrices from our 8x80 dataset whether it is realizable by a single-peaked or a caterpillar/balanced group-separable election (for each election we tried all societal axes and all caterpillar trees). For all three domains we found that for each election in the dataset, its frequency matrix can be realized by an election from the domain only if the election itself belongs to this domain.⁶ This indicates that frequency matrices (and also position matrices) of elections from restricted domains have specific features that are not likely to be produced by elections sampled from other models.

⁶Elections from our dataset that are part of a restricted domain are almost exclusively sampled from models that are guaranteed to produce such elections.

5 Condorcet Winners

Our final set of results regards Condorcet winners in elections that realize a given position matrix.

First, we consider the problem of deciding if a given position matrix can be realized by an election where a certain candidate is a Condorcet winner. In general, the complexity of this problem remains open, but if we restrict our attention to elections that only contain votes from a given set we obtain a hardness result (even if the input matrix can always be realized using votes from the given set).

Theorem 6. *Given a set \mathcal{D} of votes, listed explicitly, a position matrix X (which can be realized by an election containing only votes from \mathcal{D})⁷, and a candidate c , it is NP-hard to decide if there is an election realizing X , in which c is a Condorcet winner and all votes come from \mathcal{D} .*

Making partial progress on the general problem, we provide a necessary condition for the existence of an election realizing a given position matrix in which a given candidate c is a Condorcet winner. Roughly speaking, for each $i \in [m]$, our condition looks for a set S of candidates (different from c) that frequently appear in the first i positions. If occurrences of candidates from S on the first i positions are “sufficiently frequent” compared to how often candidate c appears in the first $i - 1$ positions, and both S and i are “small enough,” then c cannot be a Condorcet winner in any election realizing the matrix.

Theorem 7. *For each position matrix X and each $c \in [m]$, if there is an election \mathcal{E} realizing X , where c is a Condorcet winner, then for every $i \in [m]$ and $S \subseteq [m]$, it holds that*

$$\sum_{j \in S} \sum_{k=1}^i X_{k,j} \leq |S| \cdot \lfloor \frac{n-1}{2} \rfloor + \sum_{k=1}^{i-1} (X_{k,c} \cdot \min(|S|, i-k)).$$

The condition can be checked in polynomial time.

Experiment 1. We tested our condition on the elections from the 8x80 dataset. We checked for each election and each candidate whether the condition is satisfied, but there is no election realizing the matrix in which the candidate is a Condorcet winner (using an ILP formulation of the problem). It turns out that this situation is very rare: among all 480 matrices in the 8x80 dataset (i.e., among the position matrices of the elections from the dataset), there were only 6 in which there was one candidate for which our condition gave the wrong answer (there were none with more than one such candidate). Thus, our condition appears to be quite an effective way to detect potential Condorcet winners.

⁷Verifying this condition is not part of the problem as, by Theorem 5, such a test is NP-hard. It is simply a feature of our reduction.

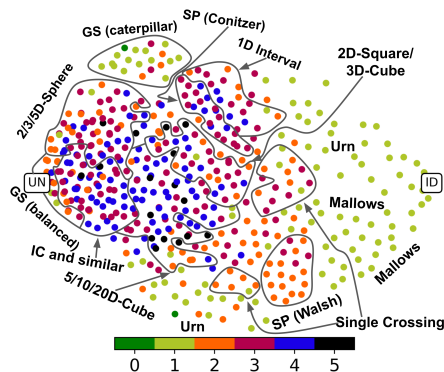


Figure 4: The number of candidates that can be Condorcet winners in elections realizing a given matrix in our 8x80 dataset.

Experiment 2. We conclude with an experiment where for each position matrix from our 8x80 dataset we count how many different candidates are a Condorcet winner in at least one election realizing the matrix. The results are in Fig. 4.

First, we observe that while 94 elections from our 8x80 dataset do not have a Condorcet winner, only two of them have a position matrix that cannot be realized by an election with a Condorcet winner. Second, examining Fig. 4, we see that for most matrices there are multiple different possible Condorcet winners, with the average number of Condorcet winners being 2.6 and 120 matrices having four or more possible Condorcet winners. The number of possible Condorcet winners is correlated with the position of the matrix on the map. Generally speaking, it seems that the closer a matrix is to UN, the more possible Condorcet winners we have. However, in the close proximity of UN there is a slight drop in the number of possible Condorcet winners. Overall, these results confirm that elections realizing a given position matrix can be very different from each other (in terms of pairwise comparisons of candidates).

6 Conclusions

We have analyzed various properties of elections that realize given position or frequency matrices. Among others, (i) we have shown algorithms for deciding if such elections can be implemented using votes from particular structured domains, and (ii) we have found that for a given matrix, such elections can be very diverse. The latter result is witnessed by the fact that two elections realizing a matrix may have large isomorphic swap distance and may have different Condorcet winners. Hence, while maps of elections (based on position matrices) certainly are very convenient tools for visualizing some experimental results

(including ours), for others their value might be limited. It would be interesting to find such experiments and establish their common features.

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Appendix

A Map of Elections

Given two vectors $x, y \in \mathbb{R}^n$ of nonnegative numbers that both sum up to the same value, their Earthmover’s distance, denoted $\text{emd}(x, y)$, is the smallest total cost of

transforming one to the other by using the following operations: For each $i, j \in [n]$ and $\delta > 0$, if the i -th entry of x is at least δ , then at cost $\delta \cdot |i - j|$ we can subtract δ from it and add it to its j -th entry.

Given a matrix X , by X_j we mean the row vector equal to X 's j -th column. Let \mathcal{E} and \mathcal{F} be two elections, each with m candidates and n voters. Let E and F be their (arbitrarily chosen) position matrices. The positionwise distance of \mathcal{E} and \mathcal{F} is:

$$d_{\text{pos}}(\mathcal{E}, \mathcal{F}) = \min_{\sigma \in S_m} \sum_{j=1}^m \text{emd}(E_j, F_{\sigma(j)})$$

(note that due to minimization over permutations σ , the distance does not depend on the choice of the position matrices).

A.1 Single-Crossing Elections

Among many types of elections introduced in the main part, our map additionally contains *single-crossing elections*. Similarly to single-peaked ones, they can also be understood as describing the left-to-right political spectrum, but this time from the perspective of the voters.

Definition 4. *An election $\mathcal{E} = (\mathcal{C}, \mathcal{V})$ with n voters ordered (v_1, v_2, \dots, v_n) is single-crossing with respect to this order if for each pair (c, c') of the candidates such that $c \succ_{v_1} c'$, there exists a positive integer t such that $\{i \in \{1, 2, \dots, n\} \mid c_1 \succ_{v_i} c_2\} = \{1, 2, \dots, t\}$. An election $\mathcal{E} = (\mathcal{C}, \mathcal{V})$ is single-crossing if there exists an ordering of the voters with respect to which the election is single-crossing.*

In single-crossing elections we can order the voters in a way that for each pair of candidates their relative order changes at most once when sweeping through the order votes.

A.2 Statistical Models

To generate elections, we use various statistical models, which we describe below.

Impartial Culture We draw each voter's preference order uniformly at random from the collection of all possible preference orders.

Pólya-Eggenberger Urn Model In the Urn model [Berg, 1985], we start with an urn containing every possible preference order. Constructing an election, we draw its votes iteratively (starting with an empty collection of votes), one vote per iteration. For some fixed parameter $\alpha \in [0, 1]$, in each iteration, we first draw a new voter's preference order uniformly at random from the urn. Then, we

return it to the urn together with $\alpha m!$ copies of the drawn preference order. For $\alpha = 0$, the Urn model is equivalent to the Impartial Culture.

Mallows Model The Mallows model [Mallows, 1957] is parameterized by a central preference order u^* and a dispersion parameter $\phi \in [0, 1]$. For fixed values of the parameters, we draw each preference order of a generated election independently. The probability of drawing a given preference order u is proportional to $\phi^{\text{swap}(u, u^*)}$ (recall that $\text{swap}(u, u^*)$ is the number of swaps required to transform u to u^*). For the special cases of $\phi = 0$ and $\phi = 1$, we obtain, respectively, an election with all voters having the central preference order and the IC model.

Normalized Mallows Model This model is an adaptation of the Mallows model [Boehmer et al., 2021b]. Instead of the dispersion parameter ϕ , it uses as a parameter the expected relative swap distance $\text{rel-}\phi \in [0, 1]$ of u^* from a drawn vote. Given some value of $\text{rel-}\phi$, we first compute the value of the dispersion parameter ϕ of the (standard) Mallows Model that yields the requested expected relative swap distance $\text{rel-}\phi$. Then, we use this parameterization of the (standard) Mallows Model, with the given central preference, to generate an election.

Euclidean Models In Euclidean models [Enelow and Hinich, 1984, 1990], we model the ideological space as the Euclidean space. Specifically, for a t -dimensional Euclidean model, we generate voters and candidates as points in the t -dimensional Euclidean space. Then, we construct the preference order of a voter by listing the candidates from the closest one to the farthest one. We consider two ways of generating the points:

- **Uniform Interval Model:** We place each point uniformly at random in a t -dimensional hypercube $[0, 1]^t$.
- **Sphere Model:** We place each point uniformly at random on a t -dimensional hypersphere with radius 1 centered at point $(0, 0, \dots, 0)$.

Single-Peaked Elections Models We consider two models, one by Conitzer [2009] and one by Walsh [2015]; hence, we call them the Walsh and the Conitzer models. Under both of them, we first randomly select the societal axis; for the sake of presentation, assume it to be $c_1 \triangleright c_2 \triangleright \dots \triangleright c_m$. Then we proceed as follows:

- **Conitzer Model:** To draw a preference order, we first select the top candidate c_i of the order. Then, we fill up the preference order in

$m - 1$ steps as follows. In each step, assuming that the preference order already consists of candidates from c_j to $c_{j'}$, $j < j'$, with the same probability we extend the preference order by either c_{j-1} or $c_{j'+1}$. It might happen that one of these candidates does not exist and then we take the existing one.

- **Walsh Model:** We draw preference orders uniformly at random from the space of all possible preference orders (conforming the selected axis), as described by Walsh [2015].

Single-Peaked on a Circle Elections Model

Following Szufa et al. [2020], to generate single-peaked on a circle vote we use the same procedure as for the Conitzer model, except that we take care of the fact that the societal axis is cyclical.

Group-Separable Elections Models We only generate group-separable elections based on caterpillar and balanced trees. Doing so, we take the approach of Boehmer et al. [2022b]. Consider an initial (respective) tree in which each leaf represents a unique candidate such that the leftmost leaf represents c_1 , the next leaf to the right represents c_2 , and so on. To generate a vote, we start from the initial tree and reverse the order of each node’s children with probability $\frac{1}{2}$. Then, the order of the candidates, from left to right, represents the preference order of the new vote.

Single-Crossing Elections Model We use this distribution only for the sake of completeness and compatibility with other datasets from the literature. Thus, we will skip the description of this model referring the reader to the work of Szufa et al. [2020], whose procedure we use.

A.3 Description of the 8x80 Dataset

The dataset contains 480 elections generated according to the models introduced in the previous section. The exact combination of election models and the quantities of elections generated using them is depicted in Table 1.

For the 80 elections generated using the urn model and the normalized Mallows model, we followed the protocol of Boehmer et al. [2021b, 2022b]. Hence, for each of the elections generated with the normalized Mallows Model, we drew the value of $\text{rel-}\phi$ uniformly at random from the $[0, 1]$ interval. The parameter for the urn model elections was drawn according to the Gamma distribution with the shape parameter $k = 0.8$ and the scale parameter $\theta = 1$. We refer to the work of Boehmer et al. [2021b] for a detailed discussion on the presented choice of parameters.

Table 1: The ingredients of the 8x80 dataset grouped by the election models.

model	#elections
Impartial Culture	20
single-peaked (Conitzer)	20
single-peaked (Walsh)	20
single-peaked on a circle	20
single-crossing	20
1D-Euclidean (uniform interval)	20
2D-Euclidean (uniform interval)	20
3D-Euclidean (uniform interval)	20
5D-Euclidean (uniform interval)	20
10D-Euclidean (uniform interval)	20
20D-Euclidean (uniform interval)	20
2D-Euclidean (sphere)	20
3D-Euclidean (sphere)	20
5D-Euclidean (sphere)	20
group-separable (balanced)	20
group-separable (caterpillar)	20
normalized Mallows model	80
urn model	80

B Additional Material for Section 3

B.1 Uniform Sampler

In this section we describe an algorithm that given a position matrix X samples an election that realizes it uniformly at random. The algorithm relies on the ability to count elections that realize a given position matrix and whose lexicographically first vote has a given prefix (if the prefix is empty, then this problem becomes $\#\text{REALIZATIONS}$). We refer to this counting problem as $\#\text{LEXREALIZATIONS}$.

Let X be our input $m \times m$ position matrix, whose each row and each column sums up to n . We let the candidate set be $\mathcal{C} = \{c_1, \dots, c_m\}$. We assume the natural ordering over the candidates, so c_1 corresponds to the first column of X , c_2 corresponds to the second one, and so on. We also use this order to compare votes (so, e.g., vote $c_3 \succ c_1 \succ c_2 \succ \dots$ precedes vote $c_3 \succ c_1 \succ c_4 \succ \dots$). Our sampler generates the output election vote by vote, in a lexicographic order. Initially, we assume that there is only one “virtual” vote $v_0: c_1 \succ c_2 \succ \dots \succ c_m$. Our algorithm will generate votes v_1, \dots, v_n such that $v_0 \leq v_1 \leq v_2 \leq \dots \leq v_n$ (for two votes x and y , by $x \leq y$ we mean that y is lexicographically greater or equal to x).

Below we describe the process of generating the votes. Let us say that we have already generated votes v_0, \dots, v_{i-1} and currently the goal is to generate v_i . Let Y_{i-1} be the position matrix (for the natural ordering of the candidates) of election $(C, (v_1, \dots, v_{i-1}))$ and let

$X_{i-1} = X - Y_{i-1}$. Let \mathcal{E}_i be a random variable equal to an election that realizes X_{i-1} and is selected uniformly at random among such elections whose lexicographically first vote is lexicographically greater or equal to v_{i-1} . We write u to denote the (random variable equal to the) lexicographically first vote in \mathcal{E}_{i-1} . For each $\ell \in [m-2] \cup \{0\}$ we define the random event T_ℓ such that:

1. length- ℓ prefix of u is equal to the length- ℓ prefix of v_{i-1} ,
2. length- $(\ell+1)$ prefixes of u and v_{i-1} are different, and
3. u is lexicographically greater than v_{i-1} .

Additionally, we let T_{m-1} be the event that $u = v_1$. Note that events T_0, \dots, T_{m-1} partition the space of elections from which \mathcal{E}_i is chosen. Further, computing the probability of each of these events is easy, provided that we have an algorithm for solving $\#\text{LEXREALIZATIONS}$ (although for each T_ℓ we might have to invoke this algorithm up to $O(m)$ times).

The first step of generating v_i is to choose a value $\ell \in [m-1] \cup \{0\}$ with probability equal to that of event T_ℓ . If $\ell = m-1$ then we let $v_i = v_{i-1}$ and we proceed to generating v_{i+1} . Otherwise, we fix the length- ℓ prefix of v_i to be equal to the length- ℓ prefix of v_{i-1} and we generate the remainder of the vote as follows (in a candidate-by-candidate manner). Let c be the candidate that v_{i-1} ranks on position $\ell+1$ and let D be the set of candidates d such that (a) v_{i-1} ranks d on a position greater than $\ell+1$ and (b) d is greater than c in our candidate ordering (at least one such candidate exists because otherwise the probability of selecting this value of ℓ would be 0). Then, we select a candidate $d \in D$ with probability equal to that of choosing uniformly at random an election \mathcal{E}'_i that realizes X_i and whose lexicographically first vote has $\ell+1$ -length prefix equal to length- ℓ prefix of v_{i-1} extended with d (again, we can compute these probabilities using $\#\text{LEXREALIZATIONS}$). We extend v_i with d . We continue choosing the candidates for the following positions of v_i in the same way, except that now in each iteration we let D be the set of candidates not-yet-ranked within v_i (for the $(\ell+1)$ -st position we had to be more careful to ensure that the length- $(\ell+1)$ prefix of v_i is lexicographically greater than the length- $(\ell+1)$ prefix of v_{i-1}). This completes the description of the sampler.

The algorithm generates elections that realize X uniformly at random because each random decision corresponds to partitioning the space of elections that generate X , and we make each decision with probability proportional to the size of the subspace to which we restrict our attention.

B.2 Experiments on 4x16 Dataset

We generated a 4x16 dataset used in this experiment analogously to the 8x80 one introduced in the main part of the paper (Section 2.3). That is, the 4x16 dataset includes 480 elections with 4 candidates and 16 voters; the election distributions and their counts are shown in Table 1.

We implemented a naive ILP program to compute all elections realizing a given position matrix. In the ILP program, we specified an integer variable for each possible 24 preference orders over four candidates. The value of each of these variables in a feasible solution corresponded to the number of voters with a given preference orders. Using constraints, we ensured that there are exactly 16 voters and that their preference orders conform to the given position matrix. Naturally, this approach does not scale well with the election size. Indeed, getting all elections realizing matrices with six voters turned out to be too computationally intensive in practice.

We performed two experiments on the 4x16 dataset. In the first experiment, for each election of the dataset we computed its position matrix and all elections realizing this matrix. We present the number of such realizing elections on the map in Fig. 5a. In the second experiment, for each position matrix in our dataset, we took all elections realizing the matrix and we computed isomorphic swap distances between all pairs of them. On the maps in Figs. 5b to 5d, we present the minimum, the maximum, and the average over these distances (for each matrix separately).

The results regarding the average and maximum distances (Figs. 5c and 5d) show that the matrices of elections closer to ID tend to yield elections with a very small average and maximum pairwise isomorphic distances. However, the distances increase as we get closer to UN. Indeed, elections near UN achieve the greatest value of the maximum distance of around 50% of the maximum achievable distance between two elections of four candidates and 16 voters. For each election in the dataset, a high value of the average and maximum distances coincide with a high number of elections realizing the election's position matrix (Fig. 5a) and vice versa. This is intuitive, as one would expect that with a greater diversity of elections realizing a single matrix, the average distance among them rises. In general, the results for the 4x16 dataset are qualitatively very similar to the results presented in Section 3.3 for the 8x80 dataset.

For the minimum distance we observe a (roughly) same-colored map, with almost all values indicating the minimum distance below 5% of the maximum achievable distance. This shows that irrespectively of the considered matrix, there always were two elections realizing a given matrix that were relatively close to each other with respect to the isomorphic swap distance.

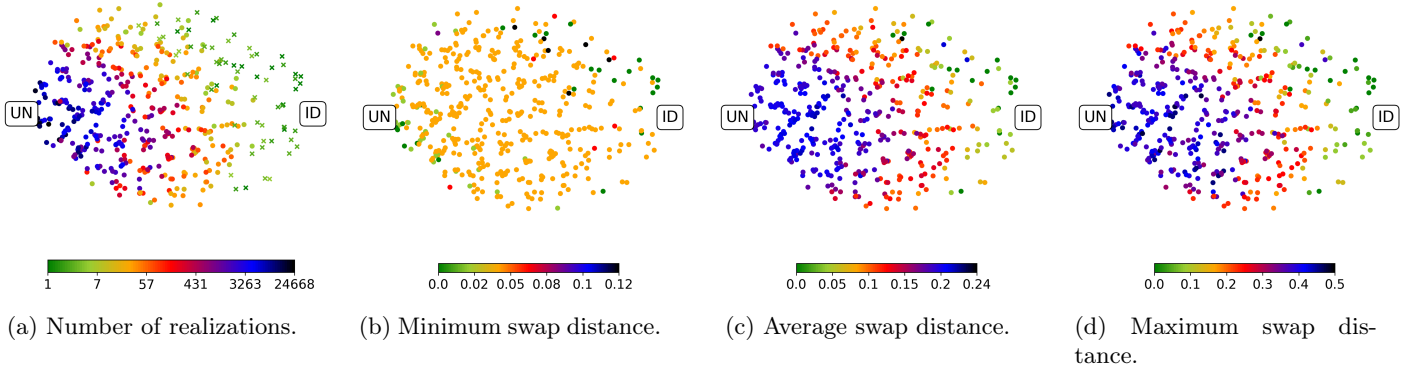


Figure 5: Maps with our experimental results for the 4x16 dataset. In Fig. 5a, the dot colors describe the number of realizations of the position matrix of the respective election. The crosses indicate that there were at most 5 such realizations. In Figs. 5b to 5d, the color shows, respectively, the minimum, average, and maximum isomorphic swap distance between all pairs of the elections realizing the position matrix of the respective election.

C Additional Material for Section 4

In this section, we provide the proofs of Theorems 3 and 4, which were omitted in the main body of the paper.

C.1 Proof of Theorem 3

Theorem 3. *There is a polynomial-time algorithm that given a frequency matrix X and an explicit, single-peaked, or group-separable (balanced or caterpillar) domain \mathcal{D} , decides if there is an election that realizes X , and whose votes all belong to \mathcal{D} .*

Proof. We split the proof, into four independent propositions, which we will subsequently prove:

- in Proposition 8 we focus on explicit domains \mathcal{D} ,
- in Proposition 9 we turn to elections in group-separable domain \mathcal{D} with balanced trees,
- in Proposition 10 we consider group-separable domain \mathcal{D} with caterpillar trees, and
- in Proposition 11 we look at single-peaked domain \mathcal{D} .

We will proceed with the propositions one-by-one. Let us start with domains explicitly given as a set of votes.

Proposition 8. *There is a polynomial-time algorithm that given a frequency matrix X and an explicit domain \mathcal{D} , decides whether there is an election that realizes X , and whose all votes belong to \mathcal{D} .*

Proof. In short, if $D = \{v_1, \dots, v_n\}$ then it suffices to find rational values y_1, \dots, y_n such that:

$$X = y_1 P(v_1) + y_2 P(v_2) + \dots + y_n P(v_n).$$

Doing so in polynomial time is a simple linear programming task. \square

Now, let us focus on group-separable domains. First let us consider these with a balanced tree. Here, our proof works in fact for both frequency and position matrices.

Proposition 9. *There is a polynomial-time algorithm that given a frequency or position matrix X and a balanced group-separable domain \mathcal{D} , decides whether there is an election that realizes X , and whose all votes belong to \mathcal{D} .*

Proof. First, let us consider position matrices and we will move to frequency matrices at the end of the proof.

For an arbitrary $2m \times 2m$ matrix $A = [a_{i,j}]_{i=1,j=1}^{2m,2m}$, by a *quarter* of A , let us denote each of the four $m \times m$ matrices obtained by “cutting” A in the middle horizontally and vertically, i.e., matrices $[a_{i,j}]_{i=1,j=1}^{m,m}$, $[a_{i,j}]_{i=m+1,j=1}^{2m,m}$, $[a_{i,j}]_{i=1,j=m+1}^{m,2m}$, and $[a_{i,j}]_{i=m+1,j=m+1}^{2m,2m}$. Now, we will say that A is *evenly-quartered* if the sum of entries in its *upper-left* quarter is equal to the sum of entries in its *bottom-right* quarter and the sum of entries in its *upper-right* quarter is equal to the sum of entries in its *bottom-left* quarter. Formally,

$$\sum_{i=1}^m \sum_{j=1}^m a_{i,j} = \sum_{i=m+1}^{2m} \sum_{j=m+1}^{2m} a_{i,j} \quad \text{and} \\ \sum_{i=m+1}^{2m} \sum_{j=1}^m a_{i,j} = \sum_{i=1}^m \sum_{j=m+1}^{2m} a_{i,j}.$$

Finally, we will say that $m \times m$ matrix A is *maximally-evenly-quartered* if $m = 2^k$ for some $k \in \mathbb{N}$ and A is either a one-element-matrix or it is evenly-quartered and each of its quarters is maximally-evenly-quartered.

Let X be an arbitrary $m \times m$ matrix. Without loss of generality, we will focus on balanced group-separable elections, $\mathcal{E} = (C, V)$, realizing X and compatible with the binary tree, \mathcal{T} , in which the order of the candidates at leaves aligns with the order of the corresponding columns in X (otherwise we can reorder the columns of X accordingly). We will prove that such election exists, if and only if, X is maximally-evenly-quartered (note that in this way we will prove also that each maximally-evenly-quartered matrix is a position matrix). Since checking if matrix is evenly-quartered can be done in polynomial time and there are $O(m^2)$ matrices that can be obtained from X by the sequence of taking the quarters $((4m^2 - 1)/3$ to be exact), this will imply the thesis.

First let us show that if X is a position matrix of some balanced group-separable election $\mathcal{E} = (C, V)$, then X is maximally-evenly-quartered. Since \mathcal{E} is balanced group-separable, $|C| = 2^k$ for some $k \in \mathbb{N}$. Let us proceed by an induction on k . If $k = 0$, the thesis holds trivially. Assume $k > 0$. Let us denote $C = \{c_1, \dots, c_{2^k}\}$. In order to prove that X is maximally-evenly-quartered, we have to prove that (1) X is evenly-quartered and (2) each of its quarters is maximally-evenly-quartered.

(1) Let us start by proving that X is evenly-quartered. To this end, let us denote $C_1 = \{c_1, \dots, c_{2^{k-1}}\}$ and $C_2 = \{c_{2^{k-1}+1}, \dots, c_{2^k}\}$. Since \mathcal{E} is balanced group-separable, we can split the voters in two disjoint sets, $V_1, V_2 \subseteq V$, such that $V_1 \cup V_2 = V$ and voters in V_1 prefer each candidate in C_1 over each candidate in C_2 and, conversely, voters in V_2 prefer each candidate in C_2 over each candidate in C_1 . Observe that in both upper-left and bottom-right quarters of X , the sum of entries is equal to $|V_1|$. Analogously, in both upper-right and bottom-left quarters of X , the sum of entries is equal to $|V_2|$. Hence, X is evenly-quartered.

(2) Now, let Y be an arbitrary quarter of X . We will construct elections $E' = (V', C')$ for which Y is a position matrix. First, let us take $C' = C_1$, if Y is upper- or bottom- left quarter, and $C' = C_2$, otherwise. Similarly, for V' let us take the set of all votes from V_1 restricted to C' , if Y is upper-left or upper-right quarter of X , and all votes from V_2 restricted to C' , otherwise. Then, $E' = (V', C')$ is also a balanced group-separable election, compatible with one of the two main branches of the original tree of E . Since $|C'| = 2^{k-1}$, by the inductive assumption, Y , is maximally-evenly-quartered. Therefore, X indeed is maximally-evenly-quartered.

In the remainder of the proof, let us show that if position matrix X is maximally-evenly-quartered, then there exists a balanced group-separable election realizing X .

We start by introducing some additional notation. For a vote v over candidates D and a vote u over candidates D' such that $D \cap D' = \emptyset$, by $v \circ u$ we denote the *concatenation*

of v and u . That is, the vote in which each candidate from D is preferred over each candidate from D' and each pair of candidates from one of the sets D or D' is ordered in the same way as in vote v or u , respectively.

We now present how to build a balanced group-separable election realizing X . Denote the upper-left, upper-right, bottom-left, bottom-right quarters of X by $Y^{\text{ul}}, Y^{\text{ur}}, Y^{\text{bl}},$ and Y^{br} , respectively. Since X is maximally-evenly-quartered, each of its quarter is also maximally-evenly-quartered. Hence, from the inductive assumption, for each quarter, there exists a balanced group-separable election with a binary tree in which candidates at consecutive leaves correspond to consecutive columns in this quarter. Let us denote such elections by $E^{\text{ul}} = (V^{\text{ul}}, C^{\text{ul}})$, $E^{\text{ur}} = (V^{\text{ur}}, C^{\text{ur}})$, $E^{\text{bl}} = (V^{\text{bl}}, C^{\text{bl}})$, and $E^{\text{br}} = (V^{\text{br}}, C^{\text{br}})$ for matrices $Y^{\text{ul}}, Y^{\text{ur}}, Y^{\text{bl}},$ and Y^{br} , respectively. Since quarters Y^{ul} and Y^{ur} share the same columns in the original matrix X , let us denote $C = C^{\text{ul}} = C^{\text{ur}}$ and, analogously, $C' = C^{\text{bl}} = C^{\text{br}}$.

Now, observe that since X is evenly-quartered, we get that $|V^{\text{ul}}| = |V^{\text{ur}}|$. Let us then denote $V^{\text{ul}} = \{v_1^{\text{ul}}, \dots, v_{n_1}^{\text{ul}}\}$ and $V^{\text{ur}} = \{v_1^{\text{ur}}, \dots, v_{n_1}^{\text{ur}}\}$. Then, we construct the following set of votes:

$$V = \{v_i^{\text{ul}} \circ v_i^{\text{ur}} : i \in [n_1]\}.$$

Analogously, $|V^{\text{bl}}| = |V^{\text{br}}|$. Hence, let us denote $V^{\text{bl}} = \{v_1^{\text{bl}}, \dots, v_{n_2}^{\text{bl}}\}$ and $V^{\text{br}} = \{v_1^{\text{br}}, \dots, v_{n_2}^{\text{br}}\}$. Again, we construct the following set of votes:

$$V' = \{v_i^{\text{bl}} \circ v_i^{\text{br}} : i \in [n_2]\}.$$

Eventually, we construct election $E = (V \cup V', C \cup C')$. Observe that because of our construction, the position matrix of E is equal to X . Moreover, every voter in V prefers each candidate in C over each candidate in C' . Similarly, every voter in V' prefers each candidate in C' over each candidate in C . Hence, since $E^{\text{ul}}, E^{\text{ur}}, E^{\text{bl}},$ and E^{br} are all balanced group-separable elections, E is also balanced group-separable and the thesis follows.

Finally, let us consider the frequency matrix case. Observe that if in a maximally-evenly-quartered matrix we multiply each element by a constant, the resulting matrix is still maximally-evenly-quartered. Hence, we can generalize our maximally-evenly-quartered property also to frequency matrices, by requiring the same equalities to hold. Since for every frequency matrix X there exists a constant n such that matrix X with each element multiplied by n is a position matrix, our proof holds also for frequency matrices. \square

Now, let us move to caterpillar group-separable domains.

Proposition 10. *There is a polynomial-time algorithm that given a frequency matrix X and a caterpillar group-separable domain \mathcal{D} , decides whether there is an election that realizes X , and whose all votes belong to \mathcal{D} .*

Proof. We will show that deciding whether for a given frequency matrix X there exists a caterpillar group-separable election profile \mathcal{E} with a given tree realizing X is equivalent to deciding whether a certain linear programming problem has a solution. Since the latter is known to be polynomial, we will obtain the thesis. Without loss of generality, we focus on caterpillar elections compatible with a caterpillar tree \mathcal{T} in which candidates at consecutive leaves correspond to consecutive columns of matrix X (otherwise the columns of X can be rearranged).

For every $k, \ell \in \mathbb{Z}$, by $[k, \ell]$ let us denote set $\{k, k+1, \dots, \ell\}$. Fix an arbitrary frequency matrix X . To define our linear program, for each column $j \in [m]$ and row $i \in [m]$, we introduce two rational variables $\ell_{i,j}$ and $r_{i,j}$. Intuitively, $\ell_{i,j}$ (or $r_{i,j}$) will be a fraction of votes in which the j th candidate is ranked at position i and it is the most preferred (or the least preferred) candidate among candidates from the subtree of \mathcal{T} rooted in the parent of this candidate, i.e., all candidates from j th to the m th one. Now, consider the following linear programming constraints (since we are only interested in the existence of a solution, we give no objective function):

$$\begin{cases} \ell_{i,j} + r_{i,j} = x_{i,j} \\ \text{for every } j, i \in [m]; & (1) \\ \ell_{i-1,j} + r_{i+m-j,j} = \ell_{i,j+1} + r_{i+m-j-1,j+1}; \\ \text{for every } j \in [m-1] \text{ and } i \in [-m, m]; & (2) \\ \ell_{i,j}, r_{i,j} \geq 0, \\ \text{for every } j \in [m] \text{ and } i \in [m]; & (3) \end{cases}$$

where $\ell_{i,j} = r_{i,j} = 0$, for each $j \in [m]$ and $i \in \mathbb{Z} \setminus [m]$. In what follows, we will show the following claim:

Claim 1. *For a given frequency matrix X , there exists a caterpillar group-separable election \mathcal{E} realizing X , if and only if, there exist rational variables $\ell_{i,j}, r_{i,j}$, for every $i, j \in [m]$, satisfying conditions (1)–(3)*

We will consider candidates $\{c_1, \dots, c_m\}$ and assume that they appear in the natural order in the caterpillar tree \mathcal{T} (see Fig. 6 for an illustration).

For every $j \in [0, m-1]$ and $i \in [1, j+1]$, by $S_{i,j}(\mathcal{E})$ let us denote the subset of votes in \mathcal{E} such that candidates c_{j+1}, \dots, c_m occupy positions $[i, i+m-j-1]$ (not necessarily in this order). Formally,

$$S_{i,j}(\mathcal{E}) = \{v \in V : \forall_{k \in [j+1, m]} \text{pos}_v(c_k) \in [i, i+m-j-1]\}.$$

We will skip \mathcal{E} for brevity. We also add also set $S_{.,m} = V$ and denote the collection of all such sets by \mathcal{S} .

Now, for a given caterpillar tree \mathcal{T} , we will construct a directed graph, $G(\mathcal{T})$, with the nodes from \mathcal{S} (here we do not treat them as sets, but as classes of votes; hence, although technically, for some \mathcal{E} some of the sets in \mathcal{S} can be equal, for the purpose of our graph construction we will distinguish each of them as a separate node). We illustrate our construction in Figure 6.

For each $j \in [m-1]$ and $i \in [m]$, let us denote the set of voters $L(\mathcal{E})_{i,j}$ such that every voter in $L(\mathcal{E})_{i,j}$ ranks candidate c_j at position i and prefers c_j over c_{j+1} . Formally,

$$L(\mathcal{E})_{i,j} = \{v \in V : \text{pos}_v(c_j) = i, c_j \succ_v c_{j+1}\}.$$

If \mathcal{E} is a caterpillar group-separable election, $j \in [m-2]$, and $i \in [j+1]$, then $v \in L(\mathcal{E})_{i,j}$ prefers c_j over c_k , for all $k \in [j+1, m]$. Thus, candidates c_{j+1}, \dots, c_m occupy positions $[i+1, i+m-j-1]$. Furthermore, candidates c_j, c_{j+1}, \dots, c_m occupy positions $[i, i+m-j-1]$. Conversely, both facts imply that c_j is at position i and preferred over c_{j+1} , thus

$$L(\mathcal{E})_{i,j} = S_{i,j-1} \cap S_{i+1,j}, \quad (4)$$

for each $j \in [m-2]$ and $i \in [j+1]$. Hence, let us add to graph $G(\mathcal{T})$ an arc from $S_{i,j-1}$ to $S_{i+1,j}$ that corresponds to set $L(\mathcal{E})_{i,j}$.

Analogously, for each $j \in [m-1]$ and $i \in [m]$, let us denote the set of voters $R(\mathcal{E})_{i,j}$ such that every voter in $R(\mathcal{E})_{i,j}$ ranks candidate c_j at position i , but prefers c_{j+1} over c_j , i.e.,

$$R(\mathcal{E})_{i+m-j,j} = \{v \in V : \text{pos}_v(c_j) = i, c_{j+1} \succ_v c_j\}.$$

If \mathcal{E} is a caterpillar group-separable election, $j \in [m-2]$, and $i \in [j+1]$, then $v \in R(\mathcal{E})_{i+m-j,j}$ prefers c_k over c_j for all $k \in [j+1, m]$. Hence, candidates c_{j+1}, \dots, c_m occupy position $[i, i+m-j-1]$ and candidates c_j, c_{j+1}, \dots, c_m positions $[i, i+m-j]$. Since, conversely, both facts imply that c_j is at position $i+m-j$ and c_{j+1} preferred over it, we get that

$$R(\mathcal{E})_{i+m-j,j} = S_{i,j-1} \cap S_{i,j}, \quad (5)$$

for each $j \in [m-2]$, and $i \in [j+1]$. Thus, let us add to graph $G(\mathcal{T})$ an arc from $S_{i,j-1}$ to $S_{i,j}$ that corresponds to set $R(\mathcal{E})_{i+m-j,j}$.

In sets $S_{1,m-1}, \dots, S_{m,m-1}$, the position of candidate c_m is uniquely determined. Hence, let us add to G and arc from each of these sets to $S_{.,m}$.

If \mathcal{E} is a caterpillar group-separable election, then the fact that in vote v candidates c_j, \dots, c_m occupy positions $[i, i+m-j-1]$, implies that candidate c_j has to be either at position i (and preferred over c_{j+1}) or at position $i+m-j-1$ (and c_{j+1} is preferred over it). Hence, $S_{i,j} \subseteq L(\mathcal{E})_{i,j+1} \cup R(\mathcal{E})_{i+m-j-1,j+1}$. On the other hand,

from equations (4) and (5) we get that $S_{i,j} \supseteq L(\mathcal{E})_{i,j+1} \cup R(\mathcal{E})_{i+m-j-1,j+1}$. Thus,

$$S_{i,j} = L(\mathcal{E})_{i,j+1} \cup R(\mathcal{E})_{i+m-j-1,j+1}, \quad (6)$$

for every $j \in [0, m-1]$ and $i \in [1, j+1]$. Moreover, in such vote v , candidate c_{j-1} has to be either at position $i-1$ (and preferred over c_j) or at position $i+m-j$ (and c_j is preferred over it). Thus, analogously we get that

$$S_{i,j} = L(\mathcal{E})_{i-1,j} \cup R(\mathcal{E})_{i+m-j,j}, \quad (7)$$

for every $j \in [1, m-1]$ and $i \in [1, j+1]$.

Building upon our construction of graph $G(\mathcal{T})$, let us prove Claim 1. Let us start by showing that if X is a frequency matrix realizable by some caterpillar group-separable election $\mathcal{E} = (C, V)$, then there exist rational constants $\ell_{i,j}, r_{i,j}$, for every $i, j \in [m]$, satisfying conditions (1)–(3). To this end, let us set $\ell_{i,j} = |L(\mathcal{E})_{i,j}|/|V|$ and $r_{i,j} = |R(\mathcal{E})_{i,j}|/|V|$, for every $j \in [m-1]$ and $i \in [m]$, and $\ell_{i,m} = x_{i,m}$ and $r_{i,m} = 0$, for every $i \in [m]$. Observe that all defined variables are indeed rational.

Now, observe that for every $j \in [m-1]$, in every $v \in V$, candidate c_j is either preferred to c_{j+1} or c_{j+1} is preferred over c_j . This means that $L(\mathcal{E})_{i,j}$ and $R(\mathcal{E})_{i,j}$ are disjoint and $L(\mathcal{E})_{i,j} \cup R(\mathcal{E})_{i,j}$ is a set of exactly all votes in which c_j is ranked at position i . Hence, we get that $\ell_{i,j} + r_{i,j} = x_{i,j}$, for every $j \in [m-1]$ and $i \in [m]$. For $j = m$, we obtain the same equation directly from our choice of $\ell_{i,m}$ and $r_{i,m}$. Hence, condition (1) is satisfied.

Next, let us fix $j \in [m-2]$. From equations (6) and (7), we get that

$$L(\mathcal{E})_{i-1,j} \cup R(\mathcal{E})_{i+m-j,j} = L(\mathcal{E})_{i,j+1} \cup R(\mathcal{E})_{i+m-j-1,j+1}.$$

Since $i < i+m-j-1$, sets $L(\mathcal{E})_{i,j+1}$ and $R(\mathcal{E})_{i+m-j-1,j+1}$ are disjoint. The same is true for sets $L(\mathcal{E})_{i-1,j}$ and $R(\mathcal{E})_{i+m-j,j}$. Thus, we obtain

$$|L(\mathcal{E})_{i-1,j}| + |R(\mathcal{E})_{i+m-j,j}| = |L(\mathcal{E})_{i,j+1}| + |R(\mathcal{E})_{i+m-j-1,j+1}|.$$

Dividing both sides by $|V|$ yields

$$\ell_{i-1,j} + r_{i+m-j,j} = \ell_{i,j+1} + r_{i+m-j-1,j+1}.$$

For $j = m-1$, observe that we have

$$|L(\mathcal{E})_{i-1,m-1}| + |R(\mathcal{E})_{i+1,m-1}| = |S_{i,m-1}|,$$

for every $i \in [m]$. Now, observe that set $S_{i,m-1}$ is a set of exactly these votes in which candidate c_m is at position i . Thus, we get

$$\ell_{i-1,m-1} + r_{i+1,m-1} = x_{i,m}.$$

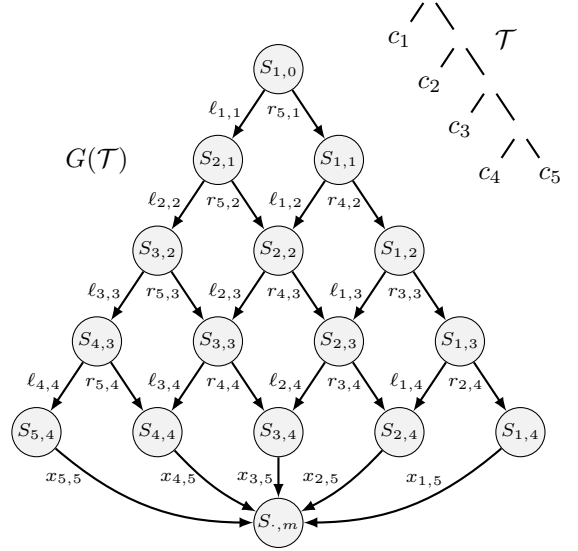


Figure 6: A tree of a caterpillar group-separable election with 5 candidates (\mathcal{T}) and a graph constructed for such election as in proof of Proposition 10 (G).

And since we took $\ell_{i,m} = x_{i,m}$ and $r_{i,m} = 0$, we get

$$\ell_{i-1,m-1} + r_{i+1,m-1} = \ell_{i,m} + r_{i,m}.$$

Hence, condition (2) is satisfied.

Finally, since the cardinality of sets is always nonnegative, condition (3) is also satisfied.

In the remainder of the proof, let us show that if for a given frequency matrix X there exist rational constants $\ell_{i,j}, r_{i,j}$, for every $i, j \in [m]$, satisfying conditions (1)–(3), then there exists a caterpillar group-separable election \mathcal{E} that realizes X .

To this end, observe that for every $j \in [0, m-1]$ and $i \in [j+1]$, each edge outgoing from $S_{i,j}$ in graph $G(\mathcal{T})$ corresponds to placement of candidate c_j at particular position. Hence, each path from $S_{1,0}$ to $S_{i,m}$ in graph $G(\mathcal{T})$ corresponds to placements of all candidates c_1, \dots, c_m , i.e., a vote. Moreover, each such vote can be present in a caterpillar group-separable election compatible with tree \mathcal{T} . Thus, if all votes in an election, \mathcal{E} , would be obtained in this way, then \mathcal{E} would be a caterpillar group-separable election compatible with \mathcal{T} . We will show that it is possible, by considering flows from $S_{1,0}$ to $S_{i,m}$ on graph $G(\mathcal{T})$.

Since for every $i, j \in [m]$ constants $\ell_{i,j}$ and $r_{i,j}$ are rational, there exists $n \in \mathbb{N}$ such that $\ell_{i,j} \cdot n$ and $r_{i,j} \cdot n$ are integers for every $i, j \in [m]$. Let us define capacity function, $c : E(G(\mathcal{T})) \rightarrow \mathbb{R}_{\geq 0}$, that for each arc returns the maximal number of flows that can go through this

arc. Specifically, we set

$$\begin{cases} c(S_{i,j}, S_{i+1,j+1}) = \ell_{i,j+1} \cdot n, \\ \quad \text{for all } j \in [0, m-2], i \in [1, j+1], \\ c(S_{i,j}, S_{i,j+1}) = r_{i+m-j-1,j+1} \cdot n, \\ \quad \text{for all } j \in [0, m-2], i \in [1, j+1], \\ c(S_{i,m-1}) = x_{i,m} \cdot n, \\ \quad \text{for all } i \in [m] \end{cases}$$

(see Fig. 6 for an illustration).

Observe that the sum of capacities of arcs incoming to $S_{\cdot,m}$ is equal to n . Condition (2) ensures that the summed capacity of arcs incoming to $S_{i,j}$ is equal to the summed capacity of arcs outgoing from $S_{i,j}$, for every $j \in [m-1]$ and $i \in [j+1]$. Therefore, there exist a (possibly multi-) set of n paths on G starting in $S_{1,0}$ and ending in $S_{\cdot,m}$, such that each edge $e \in E(G(\mathcal{T}))$ is traversed by exactly $c(e)$ paths. Each such path can be identified with a particular preference order of candidates in C . Let us denote all n of these preference orders by V . Then election $\mathcal{E} = (C, V)$ is a caterpillar group-separable election compatible with \mathcal{T} . Furthermore, from condition (1) we know that \mathcal{E} realizes matrix X , which concludes the proof. \square

Due to a certain relation between caterpillar group-separable and single-peaked elections, the above results immediately implies the next one (in short, a position matrix of a single-peaked election is a transposition of a position matrix of a caterpillar group-separable one).

Proposition 11. *There is a polynomial-time algorithm that given a frequency matrix X and a single-peaked domain \mathcal{D} , decides whether there is an election that realizes X , and whose all votes belong to \mathcal{D} .*

Proof. Boehmer et al. [2022a] have shown a one-to-one correspondence between single-peaked elections with a given societal axis and caterpillar group-separable elections compatible with a given tree. Moreover, a transposition frequency matrix of a single-peaked election is the frequency matrix of the corresponding caterpillar group-separable election. Therefore, to check whether a frequency matrix X can be realized by a single-peaked election, we can transpose it and use an algorithm described in Proposition 10 to check if it is realizable by the corresponding caterpillar group-separable election. \square

Propositions 8 to 11 combined imply the claim of the theorem, which concludes the proof. \square

C.2 Proof of Theorem 4

Theorem 4. *There is a polynomial-time algorithm that given a frequency (or position) matrix X decides if the matrix can be realized by a balanced group-separable election.*

Proof. The algorithm for checking if a given position matrix can be realized by a balanced group-separable election \mathcal{E} is given as Algorithm 1. The algorithm is based on the condition from the proof of Proposition 9 that a frequency matrix X can be realized by a balanced group-separable election compatible with a tree in which candidates on consecutive leaves correspond to consecutive columns of X , if and only if, X is maximally-evenly-quartered (see the proof of Proposition 9 for the definition of a maximally-evenly-quartered matrix). The algorithm checks if it is possible to change the order of columns of X in such a way that it is maximally-evenly-quartered. Throughout this proof, we will identify candidates with the columns of matrix X , i.e., we set $C = [m]$.

Let us first describe the algorithm and then prove its correctness. The algorithm works recursively. We start by checking the borderline case in which matrix X is a one-element matrix (lines 1–3). If it is so, then we return True, as X can be realized by a trivial election with one candidate; clearly, such election is balanced group-separable. If X is $m \times m$ matrix for $m = 2^k$, where $k > 0$, we list all possible pairs of candidates in the PossibleSiblings set (lines 4–5). In this set, we store the pairs of candidates that can end up as sibling leaves in the tree of balanced group-separable election that realizes X . If two candidates, j and j' , are sibling leaves in the tree of balanced group-separable election, then for every $i \in [m/2]$ it must hold that

$$x_{2i,j} = x_{2i-1,j'} \quad \text{and} \quad x_{2i,j'} = x_{2i-1,j}. \quad (8)$$

This is because, matrix

$$Q_{j,j'} = \begin{bmatrix} x_{2i-1,j} & x_{2i-1,j'} \\ x_{2i,j} & x_{2i,j'} \end{bmatrix}$$

can be obtained by taking the quarters from X when we reorder its columns in the order of leaves in the candidate tree. Hence, if such reordered X is maximally-evenly-quartered, then matrix $Q_{j,j'}$ has to be evenly-quartered, which is equivalent to equation (8). Thus, for every pair of candidates and every $i \in [m/2]$ we check if equation (8) holds. If for some pair $\{j, j'\} \in C$ and some $i \in [m/2]$ it does not, we remove this pair from PossibleSiblings (lines 6–12).

Next, we check if there is a perfect matching in the graph of candidates with the set of edges that is the final PossibleSiblings set. If the perfect matching exists, we store it in list M (lines 13–14). A perfect matching can be found greedily, i.e., we can match each candidate to the first unmatched candidate it is connected to. To see why, observe that if candidate j is connected to two distinct candidates c and c' , i.e., $\{j, c\}, \{j, c'\} \in \text{PossibleSiblings}$, then equation (8) implies that $x_{i,c} = x_{i,c'}$, for every $i \in$

Algorithm 1 RealizabilityByBalanced

Input: $m \times m$ position matrix X , where $m = 2^k$ for some $k \in \mathbb{N}$

Output: Does there exist a balanced group-separable election, \mathcal{E} , that realizes X ?

```
1: if  $m = 1$  then
2:   return True
3: end if
4:  $C \leftarrow [m]$ 
5: PossibleSiblings  $\leftarrow \{S \subseteq C : |S| = 2\}$ 
6: for  $i$  in  $[m/2]$  do
7:   for  $\{j, j'\}$  in PossibleSiblings do
8:     if not ( $x_{2i,j} = x_{2i-1,j'}$  and  $x_{2i-1,j} = x_{2i,j'}$ )
9:       then
10:        PossibleSiblings  $\leftarrow$  PossibleSiblings  $\setminus \{\{j, j'\}\}$ 
11:      end if
12:    end for
13: end for
14: if HasPerfectMatching( $C$ , PossibleSiblings) then
15:    $M \leftarrow$  PerfectMatching( $C$ , PossibleSiblings)
16:    $Y \leftarrow m/2 \times m/2$  matrix
17:   for  $i$  in  $[m/2]$  do
18:     for  $j$  in  $[m/2]$  do
19:        $\{c, c'\} \leftarrow M[j]$ 
20:        $Y[i, j] \leftarrow x_{2i,c} + x_{2i,c'}$ 
21:     end for
22:   end for
23:   return RealizabilityByBalanced( $Y$ )
24: else
25:   return False
26: end if
```

$[m]$. Hence, columns c and c' are identical and matching j with c or with c' is equivalent.

If there is no perfect matching, this means that it is impossible to pair candidates to obtain the lowest level of binary tree of candidates. Hence, we return False (lines 23–25).

If there is a perfect matching of candidates that can be siblings, then we construct $m/2 \times m/2$ matrix Y in which each column j corresponds to, each time different, matching $\{c, c'\} = M[j]$, in an arbitrary order (lines 15–21). For each $i, j \in [m/2]$ and $\{c, c'\} = M[j]$, we set entry $Y[i, j]$ of matrix Y to the sum of entries in matrix $Q_{c,c'}$ divided by 2, which, by equation (8), is equal to

$$x_{2i,c} + x_{2i,c'}.$$

This way, matrix Y is also a position matrix.

Finally, we repeat the algorithm, this time for matrix Y (line 22).

Now, let us prove the correctness of our algorithm. To this end, we will follow the induction by k , i.e., the binary

logarithm of the number of candidates. If $k = 0$, then our algorithm always returns True, and matrix X is always realizable by a trivial election with one candidate, hence the thesis holds. Let us then assume that our thesis holds for all $2^k \times 2^k$ position matrices and let us consider an arbitrary $2^{k+1} \times 2^{k+1}$ position matrix X .

First, let us show that if there exists a balanced group-separable election $\mathcal{E} = (C, V)$ that realizes X , then our algorithm returns True. Let \mathcal{T} be a binary tree with which \mathcal{E} is compatible. Observe that the result of Algorithm 1 does not depend on the order of the columns of the input position matrix. Thus, without loss of generality, we can assume that the order of the candidates at the leaves of \mathcal{T} aligns with the order of corresponding columns in matrix X (otherwise the columns of X can be reordered). Let us denote the candidates in \mathcal{E} as $C = [m]$, where the order $1, \dots, m$ is the order in which they appear at the leaves of \mathcal{T} . Then, our algorithm will assign to M a perfect matching of candidates that is $(\{1, 2\}, \{3, 4\}, \dots, \{m-1, m\})$ (or equivalent if some columns in X are identical). Consequently, matrix Y will be a $2^k \times 2^k$ matrix in which every entry is equal to the sum of four neighboring elements of matrix X divided by 2, i.e.,

$$y_{i,j} = x_{2i,2j} + x_{2i,2j-1},$$

for each $i, j \in [m/2]$. Now, the outcome of our algorithm for matrix X will be the same as the outcome for matrix Y . Hence, we have to show that the outcome for Y is True. By the inductive assumption, it suffices to construct balanced group-separable election that realizes Y . To this end, let us denote $C' = \{\{1, 2\}, \{3, 4\}, \dots, \{m-1, m\}\}$, i.e., the candidates C' are matched pairs of candidates from C . Since \mathcal{E} is a balanced group-separable election, for every voter $v \in V$ and two distinct pairs of candidates $\{j, j+1\}, \{j', j'+1\} \in C'$, it holds that either v prefers both j and $j+1$ over both j' and $j'+1$ or the converse is true, i.e., v prefers both j' and $j'+1$ over both j and $j+1$. Thus, we can define an *aggregated vote* v , denoted by $f(v)$, as a preference order on C' in which

$$\begin{cases} \{j, j+1\} \succ_{f(v)} \{j', j'+1\}, & \text{if } j \succ_v j', \\ \{j', j'+1\} \succ_{f(v)} \{j, j+1\}, & \text{if } j' \succ_v j, \end{cases}$$

for every $\{j, j+1\}, \{j', j'+1\} \in C'$. Furthermore, let

$$V' = \{f(v) : v \in V\}.$$

Then, election $\mathcal{E}' = (C', V')$ is also a balanced group-separable election compatible with tree \mathcal{T} with its leaves removed. Observe that for every $i, j \in [m/2]$ candidate $\{2j-1, 2j\}$ is ranked at position i by voter $f(v) \in V'$, if and only if, either candidate $2j-1$ or candidate $2j$ is ranked at position $2i$ by voter v . Hence, election \mathcal{E}' realizes matrix Y . Therefore, by the inductive assumption,

our algorithm returns True for Y , which means that it also returns True for X .

Finally, let us prove that if our algorithm returns True, then there exists a balanced group-separable election that realizes X . Since we return True and $k > 0$, there exists a perfect matching M of candidates such that for every $\{c, c'\} \in M$ and every $i \in [m/2]$ equation (8) holds. Without loss of generality, let us assume that $M = \{\{1, 2\}, \{3, 4\}, \dots, \{m-1, m\}\}$ (otherwise, we can reorder the columns of matrix X). Thus, we can construct matrix Y as in the algorithm. Then, from the inductive assumption we know that there exists a balanced group-separable election, $\mathcal{E}' = (C', V')$, that realizes Y in which $C' = M$.

Now, based on election \mathcal{E}' , let us construct election $\mathcal{E}'' = (C, V'')$ in which we exchange each candidate $\{c, c'\}$ for a pair for candidates c, c' . Specifically, for a preference order v on candidates C' , let us define a *disaggregated vote* v , denoted by $g(v)$, as a preference order on C in which $c \succ_{g(v)} d$, for every $c, d \in C$ such that $\{c, d\} \notin C'$ and $\{c, c'\} \succ_v \{d, d'\}$ for some $c', d' \in C'$, and $2j-1 \succ_{g(v)} 2j$ for every $j \in [m/2]$. Then, we set $V'' = \{g(v) : v \in V'\}$. Observe that election \mathcal{E}'' is still balanced group-separable election and it realizes position matrix

$$Y' = \begin{bmatrix} y_{1,1} & 0 & y_{1,2} & 0 & y_{1, \frac{m}{2}} & 0 \\ 0 & y_{1,1} & 0 & y_{1,2} & \cdots & 0 & y_{1, \frac{m}{2}} \\ y_{2,1} & 0 & y_{2,2} & 0 & y_{2, \frac{m}{2}} & 0 \\ 0 & y_{2,1} & 0 & y_{2,2} & 0 & y_{2, \frac{m}{2}} \\ \vdots & & & \ddots & & \vdots \\ y_{\frac{m}{2},1} & 0 & y_{\frac{m}{2},2} & 0 & y_{\frac{m}{2}, \frac{m}{2}} & 0 \\ 0 & y_{\frac{m}{2},1} & 0 & y_{\frac{m}{2},2} & \cdots & 0 & y_{\frac{m}{2}, \frac{m}{2}} \end{bmatrix}.$$

In what follows, we will construct a sequence of elections, $\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_{m/2}$, such that $\mathcal{E}_0 = \mathcal{E}'$ and $\mathcal{E}_{m/2}$ will realize matrix X . For each $j \in [m/2]$, we will construct election \mathcal{E}_j from \mathcal{E}_{j-1} , by swapping the positions of candidates $2j-1$ and $2j$ in some of the votes (observe that they are always at consecutive positions). Let us describe how we obtain election \mathcal{E}_j from \mathcal{E}_{j-1} in more detail. Let us denote $\mathcal{E}_{j-1} = (C, V_{j-1})$. Next, let us split set V_{j-1} into $m/2$ sets $V_{j-1}^2, V_{j-1}^4, \dots, V_{j-1}^m$ depending on the position of candidate $2j$ in a vote. Specifically, for each $i \in [m/2]$, let

$$V_{j-1}^{2i} = \{v \in V_{j-1} : \text{pos}_v(2j) = 2i\}.$$

Then, from each set V_{j-1}^{2i} we construct set U_{j-1}^{2i} by arbitrarily choosing $x_{2i-1, 2j}$ votes in which we swap the positions of candidates $2j$ and $2j-1$ (so, now $2j-1$ is ranked at position $2i$ and $2j$ at position $2i-1$). Then, we set $V_j = \bigcup_{i=1}^{[m/2]} U_{j-1}^{2i}$ and $\mathcal{E}_j = (C, V_j)$. Since we always swap only the candidates within one pair, \mathcal{E}_j is balanced group-separable election for each $j \in [m/2]$. Moreover,

for every $j \in [m/2]$, in each election $\mathcal{E}_{j'}$ for $j' \geq j$, candidate j is at position i in exactly $x_{i,j}$ votes. Hence, the position matrix of election $\mathcal{E}_{m/2}$ is indeed matrix X . This concludes the proof. \square

D Additional Material for Section 5

In this section we provide the proofs of Theorems 6 and 7 and additional details of experiments concerning the Condorcet winners in elections realizing a given position matrix.

D.1 Proof of Theorem 6

Theorem 6. *Given a set \mathcal{D} of votes, listed explicitly, a position matrix X (which can be realized by an election containing only votes from \mathcal{D})⁸, and a candidate c , it is NP-hard to decide if there is an election realizing X , in which c is a Condorcet winner and all votes come from \mathcal{D} .*

Proof. We reduce from EXACT COVER BY 3-SETS where given a $3t$ -element universe $\mathcal{U} = \{u_1, u_2, \dots, u_{3t}\}$ and a family of 3-element subsets $\mathcal{S} = \{S_1, S_2, \dots, S_k\}$, the question is whether there exists a t -size set $K \subseteq \mathcal{S}$ forming an exact cover of \mathcal{U} , that is, $\bigcup_{S \in K} S = \mathcal{U}$. For convenience, we assume that elements of subsets S are ordered ascending. Furthermore, we call the smallest element of a subset S the *representative* of S . Consequently, for some exact cover K , we refer to the set of representatives of subsets belonging to K as to $\text{repr}(K)$. We note that $\text{repr}(K)$, consisting of exactly t distinct elements, yields too little information to uniquely define K .

We transform an instance \mathcal{I} of EXACT COVER BY 3-SETS into instance $\mathcal{I}' = (\mathcal{C}, X, \mathcal{D}, p)$ of a problem in which we ask whether there exists a realization of X using only votes from \mathcal{D} such that p is the Condorcet winner. Our reduction guarantees that there is always at least one feasible realization of X using votes from \mathcal{D} .

D.1.1 Construction

Regarding the candidates of \mathcal{I}' , for each element $u_i \in \mathcal{U}$, we add three candidates $\{c_i, e_i, f_i\}$ that we call, respectively, *element*, *executive*, and *filler* candidates of element u_i . Furthermore, we add the *preferred* candidate p and the *despised* candidate d . Overall, we have $9t+2$ candidates.

For the sake of presentation, we define our matrix X (of size $(9t+2) \times (9t+2)$) as the sum of two matrices A_1 and A_2 . For reasons of clarity, when presenting a matrix, we leave zero-entries blank.

⁸Verifying this condition is not part of the problem as, by Theorem 5, such a test is NP-hard. It is simply a feature of our reduction.

Claim 2. *In every election $\mathcal{E} = (C, V)$ realizing matrix X , in which all votes belong to domain \mathcal{D} , the voters can be split into two disjoint sets $V_1, V_2 \subseteq V$, such that $V_1 \cup V_2 = V$ and:*

1. V_1 consists of $4t + 11$ votes v_p and $7t - 1$ votes v_d , and
2. V_2 consists of $9t$ votes solely from \mathcal{D}_2 .

Building upon Claim 2, we consider elections $\mathcal{E}_1 = (V_1, C)$ and $\mathcal{E}_2 = (V_2, C)$. Since in both votes v_p and v_d positions $3, 4, \dots, 9t+2$ are occupied by candidates $e_1, \dots, e_{3t}, f_1, c_1, \dots, f_{3t}, c_{3t}$, respectively, \mathcal{E}_1 realizes matrix A_1 . Hence, in the remainder of the proof, we mainly focus on the construction of election \mathcal{E}_2 restricted to \mathcal{D}_2 that realizes matrix A_2 .

(I) We construct such election $\mathcal{E}_2 = (V_2, C)$ realizing A_2 as follows. We start with empty V_2 and we add to V_2 one copy of each of the votes $v_1^-(u_i), v_2^-(u_i), v_3^-(u_i)$, for every $u_i \in \mathcal{U}$.

Now, let us show that \mathcal{E}_2 indeed realizes matrix A_2 . For every $u_i \in \mathcal{U}$, the preferred candidate p is at positions $6t + 2$ in $v_1^-(u_i)$, $6t + 1$ in $v_2^-(u_i)$, and $6t + 3$ in $v_3^-(u_i)$. Hence, in each of these positions it appears $3t$ times in total. The same is true for the despised candidate d (it appears at position $6t + 1$ in $v_1^-(u_i)$, $6t + 3$ in $v_2^-(u_i)$, and $6t + 2$ in $v_3^-(u_i)$, for every $u_i \in \mathcal{U}$). Thus, candidates p and d appear at positions as indicated by matrix A_2 .

For every $i \in [3t]$, executive candidate e_i appears at position $6t + 3$ in $v_1^-(u_i)$, $6t + 2$ in $v_2^-(u_i)$, and $6t + 1$ in $v_3^-(u_i)$. In every other vote, i.e., vote $v_\ell^-(u_j)$ for some $u_j \in (\mathcal{U} \setminus \{u_i\})$ and $\ell \in [3]$, it appears in either position $6t + 2 + i$, if $j < i$, or position $6t + 3 + i$, if $j > i$. Hence, in total, it appears once at positions $6t + 1, 6t + 2$, and $6t + 3$; $3i - 3$ times at position $6t + 2 + i$; and $9t - 3i$ times at position $6t + 3 + i$. This conforms to matrix A_2 .

Finally, for every $i \in [3t]$, $u_j \in \mathcal{U}$ and $\ell \in [3]$, element and filler candidates, f_i and c_i , appear at positions $2i - 1$ and $2i$, respectively, in every vote $v_\ell^-(u_j)$, unless $j = 1$ and $\ell = 3$, where f_i appears at positions $2i$ and c_i in $2i - 1$. Thus, positions of these candidates also agree with matrix A_2 . This concludes the proof of part (I).

(II) Now, let us show that if \mathcal{I} admits an exact cover K , then there exists election \mathcal{E} restricted to domain \mathcal{D} realizing matrix X such that p is the Condorcet winner. Without loss of generality, we assume $K = \{S_1, S_2, \dots, S_t\}$. Building upon Claim 2, we focus on constructing $\mathcal{E}_2 = (C, V_2)$ restricted to domain \mathcal{D}_2 realizing matrix A_2 . To this end, we include in V_2 the following votes:

1. For each $S \in K$, we add voter $v^+(S)$, which gives, in total, t votes;
2. For each $u_i \in (\mathcal{U} \setminus \text{repr}(K))$, we add voter $v_3^+(u_i)$, which gives, in total, $2t$ votes;

3. For each $u_i \in \mathcal{U}$, we add voters $v_1^+(u_i)$ and $v_2^+(u_i)$, which gives, in total, $6t$ votes.

Let us show that such defined election \mathcal{E}_2 indeed realizes matrix A_2 .

For every $u_i \in \mathcal{U} \setminus \text{repr}(K)$, the preferred candidate p appears at positions $6t + 1, 6t + 2, 6t + 3$ in votes $v_1^+(u_i), v_3^+(u_i)$, and $v_2^+(u_i)$, respectively. Since each element $u_i \in \text{repr}(K)$ belongs to exactly one $S_j \in K$, we also have that p appears at positions $6t + 1, 6t + 2, 6t + 3$ respectively in votes $v_1^+(u_i), v^+(S_j)$, and $v_2^+(u_i)$. Hence, in total, p appears $3t$ times in each position $6t + 1, 6t + 2$, and $6t + 3$. By analogous reasoning, this is also true for the despised candidate d , which agrees with position matrix A_2 .

For each executive candidate e_i , observe that e_i appears at positions $6t + 1, 6t + 2$, and $6t + 3$ in votes $v_1^+(u_i), v_3^+(u_i), v_2^+(u_i)$, respectively (or in votes $v_1^+(u_i), v^+(S_j), v_2^+(u_i)$ if u_i is the representative of some set $S_j \in K$). In the remaining votes it occupies either position $6t + 2 + i$ or position $6t + 3 + i$, depending on the index of element from \mathcal{U} , as in part (I). Thus, again, its positions conform to matrix A_2 .

Finally, consider filler candidate f_i and element candidate c_i . Observe that f_i is at position $2i - 1$ and c_i at position $2i$ in every vote except $v^+(S_j)$ such that $S_j \in K$ and $u_i \in S_j$. Since there is exactly one such S_j for every $i \in [3t]$, we obtain that f_i appears $9t - 1$ times at position $2i - 1$ and once at position $2i$, whereas c_i appears $9t - 1$ times at position $2i$ and once at position $2i - 1$. This agrees with matrix A_2 , so A_2 is realized by election \mathcal{E}_2 .

It remains to show that the preferred candidate p is indeed the Condorcet winner in \mathcal{E} . To this end, observe that in votes from V_1 candidate p wins $11t$ times with each candidate in $C \setminus \{p, d\}$. However, there is only $9t$ votes in V_2 , so p cannot lose against any candidate from $C \setminus \{p, d\}$. As per candidate d , p wins only $4t + 1$ times with d in votes from V_1 . Hence, since $|V_1 \cup V_2| = 20t$, we have to show that among votes in V_2 , candidate p is preferred over d at least $6t$ times (this means that p wins with d exactly $10t$ times in total). Observe that this is exactly the case, as p is preferred over d in:

- vote $v_1^+(u_i)$, for every $u_i \in \mathcal{U}$ ($3t$ votes),
- vote $v_3^+(u_i)$, for every $u_i \in \mathcal{U} \setminus \text{repr}(K)$ ($2t$ votes), and
- vote $v^+(S_j)$, for every $S_j \in K$ (t votes).

This concludes the proof of part (II).

(III) Now, let us assume that there exists an election $\mathcal{E} = (C, V)$ restricted to domain \mathcal{D} realizing X such that the preferred candidate p is the Condorcet winner. We show that this implies that the original instance \mathcal{I} admits an exact cover.

From part (II) above we know that candidate p wins with every candidate in $C \setminus \{p, d\}$ at least $11t$ times. However, only in $4t + 1$ votes from V_1 it is preferred over d . The fact that p is the Condorcet winner implies that it is preferred over d in at least $10t + 1$ votes in V . Hence, p wins with d in at least $6t$ votes in V_2 .

From the fact that \mathcal{E}_2 realizes matrix A_2 , we see that in votes from V_2 both p and d occupy only positions $6t + 1$, $6t + 2$, and $6t + 3$, each $3t$ times. Thus, in all $3t$ votes in which d is at position $6t + 1$, candidate d has to be preferred over p . However, since in all votes from V_2 candidate p is preferred over d at least $6t$ times, this implies that p wins with d in all remaining votes (in all of them d is not at position $6t + 1$). This means that in every vote in which candidate p is at position $6t + 1$ or $6t + 2$ candidate d occupies the next position: $6t + 2$ or $6t + 3$, respectively. Therefore, V_2 can include only votes denoted with a plus sign: $v_1^+(u_i)$, $v_2^+(u_i)$, $v_3^+(u_i)$, for each $u_i \in \mathcal{U}$; and $v^+(S_i)$, for each $S_i \in \mathcal{S}$.

Now, consider an element candidate c_i , for some fixed $i \in [3t]$. Since \mathcal{E}_2 realizes matrix A_2 , c_i appears at position $2i - 1$ exactly once. However, among votes denoted with a plus sign, c_i appears at position $2i - 1$ only in $v^+(S_j)$, for these $S_j \in \mathcal{S}$ for which $u_i \in S_j$. Let us denote by K the collection of such sets $S \in \mathcal{S}$ for which $v^+(S)$ is a vote in election \mathcal{E}_2 , i.e.,

$$K = \{S \in \mathcal{S} : v^+(S) \in V_2\}.$$

Then, for every $u_i \in \mathcal{U}$, since in \mathcal{E}_2 candidate c_i appears at position $2i - 1$ exactly once, we have that u_i belongs to exactly one set $S \in K$. Therefore, K is an exact cover. This, concludes the proof. \square

D.2 Proof of Theorem 7

Theorem 7. *For each position matrix X and each $c \in [m]$, if there is an election \mathcal{E} realizing X , where c is a Condorcet winner, then for every $i \in [m]$ and $S \subseteq [m]$, it holds that*

$$\sum_{j \in S} \sum_{k=1}^i X_{k,j} \leq |S| \cdot \left\lfloor \frac{n-1}{2} \right\rfloor + \sum_{k=1}^{i-1} (X_{k,c} \cdot \min(|S|, i-k)).$$

The condition can be checked in polynomial time.

Proof. First, let us prove the condition and then we will show that it can be checked in polynomial time.

Let $\mathcal{E} = (C, V)$ be arbitrary elections with a Condorcet winner, $c \in C$, and let X be its position matrix (for notational convenience, by $X_{i,j}$, we will denote number of voters that rank candidate $j \in C$ at position i). Let us also take an arbitrary set of candidates $S \subseteq C \setminus \{c\}$ and number $i \in [m]$.

Now, for each $k \in [i - 1]$, by $V_k \subseteq V$ let us denote the set of votes in which candidate c is at position k . Then,

by N_k let us denote the number of times candidates from S appear at positions $k + 1, k + 2, \dots, i$ in votes in V_k . Formally,

$$N_k = \sum_{v \in V_k} |\{j \in S : k < \text{pos}_v(j) \leq i\}|.$$

Since in each vote, there are total of $i - k$ such positions, we have $|\{j \in S : k < \text{pos}_v(j) \leq i\}| \leq i - k$. Moreover, there can be at most $|S|$ candidates from S at these positions, we get $|\{j \in S : k < \text{pos}_v(j) \leq i\}| \leq |S|$. Hence,

$$N_k \leq \sum_{v \in V_k} \min(|S|, i - k) = X_{k,c} \cdot \min(|S|, i - k).$$

Summing for all $k \in [i - 1]$ we obtain

$$\sum_{k=1}^{i-1} N_k \leq \sum_{k=1}^{i-1} X_{k,c} \cdot \min(|S|, i - k). \quad (9)$$

Now, by $N_{k,j}$ let us denote the number of votes from V_k in which candidate j appears at positions $k+1, k+2, \dots, i$. Formally,

$$N_{k,j} = |\{v \in V_k : k < \text{pos}_v(j) \leq i\}|.$$

Since c is a strong Condorcet winner, for every candidate $j \in S$ there are at least $\lfloor (n+1)/2 \rfloor$ votes in which c is ranked before j . In particular, if j is ranked at position i or smaller in $X_{j, \leq i}$ votes, then there exist at least $\max(0, \lfloor (n+1)/2 \rfloor - (n - X_{j, \leq i}))$ votes in which j is at position i or smaller and c is ranked before j . We count each such vote in one of sets $N_{1,j}, \dots, N_{i-1,j}$, hence we get that

$$\begin{aligned} \sum_{k=1}^i N_{k,j} &\geq \max\left(0, X_{j, \leq i} - \left\lfloor \frac{n-1}{2} \right\rfloor\right) \\ &\geq X_{j, \leq i} - \left\lfloor \frac{n-1}{2} \right\rfloor \\ &= \sum_{k=1}^i X_{k,j} - \left\lfloor \frac{n-1}{2} \right\rfloor. \end{aligned}$$

Observe that $\sum_{j \in S} N_{k,j} = N_k$. Therefore, we get that

$$\begin{aligned} \sum_{k=1}^i N_k &\geq \sum_{j \in S} \left(\sum_{k=1}^i X_{k,j} - \left\lfloor \frac{n-1}{2} \right\rfloor \right) \\ &= \sum_{j \in S} \sum_{k=1}^i X_{k,j} - |S| \cdot \left\lfloor \frac{n-1}{2} \right\rfloor. \end{aligned}$$

Combining this with inequality (9) we obtain the thesis.

Now, let us focus on proving that the condition can be checked in polynomial time. To this end, let us fix $i \in [m]$

and sort the candidates in $C \setminus \{c\}$ in the order of how often they appear in first i positions. Formally, let o be a strict linear order on $C \setminus \{c\}$ such that $j \succ_o j'$ if and only if either

$$\sum_{k=1}^i X_{k,j} > \sum_{k=1}^i X_{k,j'}$$

or

$$\sum_{k=1}^i X_{k,j} = \sum_{k=1}^i X_{k,j'} \quad \text{and} \quad j > j'.$$

Now, for each $j \in C \setminus \{c\}$, let us denote by $S_{i,j}$ a set of candidates other than c that is j and candidates before j in the order o . Formally,

$$S_{i,j} = \{k \in C \setminus \{c\} : k \succeq_o j\}.$$

Observe that for every $i \in [m]$ and $S \subseteq C \setminus \{c\}$ it holds that the left hand side of condition for set S is smaller or equal to the left hand side of condition for set of $|S|$ candidates other than c most frequently appearing in first i positions, i.e.,

$$\sum_{j \in S} \sum_{k=1}^i X_{k,j} \leq \sum_{j \in S_{i, \text{pos}_o(|S|)}} \sum_{k=1}^i X_{k,j}.$$

Hence, it suffices to check the condition only for sets $S_{i,j}$ for all $i \in [m]$ and all $j \in C \setminus \{c\}$, which can be done in polynomial time. \square

D.3 Experiments

D.3.1 Computing the Number of Possible Condorcet Winners

As described in the main body, given a position matrix, we computed the number of different candidates which are the Condorcet winner in some realization of the matrix. To solve this problem, we derive an Integer Linear Program (ILP) for the following closely related problem: Given an $m \times m$ position matrix X over a candidate set C (where rows and columns sum up to n) and a candidate $c^* \in C$, is there an election realizing X in which c^* is a Condorcet winner.

To model this problem as an ILP, we introduce for each $c \in C$, $i \in [m]$, and $k \in [n]$ a binary variable $x_{c,k,i}$. Setting $x_{c,k,i}$ to true corresponds to putting candidate c on position i in vote k . Let y be an n -dimensional vector which for each $i \in [m]$ contains X_{i,c^*} -times the number i . Because the ordering of votes is clearly irrelevant, we can start by fixing the position in which c^* is ranked in each vote:

$$x_{c^*,k,y_k} = 1.$$

To enforce that every position in each vote is taken by exactly one candidate and that each candidate appears in

each vote exactly once we add the following constraints:

$$\begin{aligned} \sum_{c \in C} x_{c,k,i} &= 1, & \forall i \in [m], k \in [n] \\ \sum_{i \in [m]} x_{c,k,i} &= 1 & k \in [n], c \in C \end{aligned}$$

Moreover, we add constraints enforcing that each candidate appears in every position as often as specified in its position vector:

$$\sum_{k \in [n]} x_{c,k,i} = X_{i,c}, \quad i \in [m], c \in C$$

Lastly, we ensure that candidate c^* wins the pairwise comparison against each other candidate. We can easily enforce this because we know in each vote the position on which c^* appears:

$$\sum_{k \in [n]: i \in [y_k - 1]} x_{c,k,i} \leq \left\lfloor \frac{n+1}{2} \right\rfloor, \quad c \in C$$