Erratum to “Computational Aspects of Extending the Shapley Value to Coalitional Games with Externalities”

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In this erratum, we correct the theorem statement that appeared in the original paper. All changes concern Sections 5.1 and 5.3. We state below the corrected version of these sections and emphasize the changes with emboldened font weight.

Theorem 5.2
Let $(A, \mathcal{WR})$ represent w, ***and assume set $\{B^l_k : |P^l_k| > 1\}$ is not empty, and if consist of rules from one block l, other rules in this block are pairwise compatible****. Algorithm 1 transforms $\mathcal{WR}$ into a corresponding set of simple rules $\mathcal{R}$ (from which $ESV_i^{PdN}(w)$ can be computed for all $a_i \in A$ in time linear in $|\mathcal{R}|$ as shown in [14]). Furthermore, for each $WR \in \mathcal{WR}$ the running time of Algorithm 1 is $O(|WR|^2|)$. Therefore, it is linear in the size of the representation $|\mathcal{WR}|$. Finally, it holds that $|\mathcal{R}| \leq |A| \times |\mathcal{WR}|$.*

Proof: We denote by $\mathcal{B}^*$ an interim expression which we use in the process of building a simple rule. For every $WR \in \mathcal{WR}$:

(i) If for more than one l there exist $\mathcal{B}^l_k : |P^l_k| > 1$ then WR cannot be met by any coalition structure of the form $\{C, \textit{singletons}\}$; thus, WR is disregarded;

(ii) If for exactly one l there exists $\mathcal{B}^l_k : |P^l_k| > 1$ then we need to ensure that, for this l, all expressions $\mathcal{B}^l_k : |P^l_k| > 1$ are compatible as they have to be met by the same coalition C. Thus:

  (a) If $\ominus\{\mathcal{B}^l_k : |P^l_k| > 1\}$ then WR is disregarded;

  (b) Otherwise, $\exists C \subseteq A : (\{C, \textit{singletons}\} \models WR) \wedge (C \models \mathcal{B}^*)$, where $\mathcal{B}^* = \bigwedge_{k:|P^l_k|>1} \mathcal{B}^l_k)$. Now, what is left is to ensure that the other conditions in WR that affect C are preserved. As for $\mathcal{B}^{l'}_{k'} : l' \neq l$, where $l' \in \{1, \ldots, s\} \setminus \{l\}$ and $k' \in \{1, \ldots, r_{l'}\}$, the only conditions that these expressions place on C is that $C \cap P^{l'}_{k'} = \emptyset$. Thus, $\mathcal{B}^* \leftarrow \mathcal{B}^* \wedge \neg p^{l'}_{k'}$. As for $\mathcal{B}^l_{k'} : k' \in \{1, \ldots, r_l\} \wedge |P^l_{k'}| = 1$, whenever $\ominus\{\mathcal{B}^*, \mathcal{B}^l_{k'}\}$ it places a condition on C that $C \not\models \mathcal{B}^l_{k'}$ since $C \models \mathcal{B}^*$. **Since we assumed other rules in this block can be arbitrary combined with each other and therefore coalition C,** at this point, we have incorporated in $\mathcal{B}^*$ all the conditions necessary for C to meet. Thus, $(\mathcal{B}^*, \sum_{k:|P^l_k|>1} v^l_k)$ becomes our first simple rule. However, if there exist $\mathcal{B}^l_{k'}$ such that $\oplus\{\mathcal{B}^*, \mathcal{B}^l_{k'}\}$ then they contribute to the value of C if $C \models \mathcal{B}^* \wedge \mathcal{B}^l_{k'}$ (see Steps 12-13).

(iii) **(deleted, since we assume set $\{B^l_k : |P^l_k| > 1\}$ is not empty).**

It is clear that the maximum number of simple rules that can be created from a single $WR \in \mathcal{WR}$ is $|A|$; thus, $|\mathcal{R}| \leq |A| \times |\mathcal{WR}|$. The running time comes from the fact that every WR contains at most $|A|$ expressions. □

Algorithm 1: : $f(\mathcal{WR})$

{**Note:** Given WR as in (6), unless stated differently, we assume that $l \in \{1, \ldots, s\}$ and, given l, $k \in \{1, \ldots, r_l\}$}

1: $\mathcal{R} \leftarrow \emptyset$;
2: **for** $WR \in \mathcal{WR}$ **do**
3: **if** $\exists l : (|P^l_k| > 1) \wedge (\forall l' \neq l, |P^{l'}_k| = 1) \wedge (\oplus\{\mathcal{B}^l_k : |P^l_k| > 1\})$ **then**
4: $\mathcal{B}^* \leftarrow \bigwedge_{k:|P^l_k|>1} B^l_k$;
5: **for** $(l' \in \{1, \ldots, s\} \setminus \{l\}) \wedge (k' = \{1, \ldots, r_{l'}\})$ **do**
6: $\mathcal{B}^* \leftarrow \mathcal{B}^* \wedge \neg p^{l'}_{k'}$;
7: **for** $k' \in \{1, \ldots, r_l\} : |P^l_{k'}| = 1$ **do**
8: **if** $\ominus\{\mathcal{B}^*, \mathcal{B}^l_{k'}\}$ **then**
9: $\mathcal{B}^* \leftarrow \mathcal{B}^* \wedge \neg p^l_{k'}$;
10: $\mathcal{R} \leftarrow \mathcal{R} \cup \{(\mathcal{B}^*; \sum_{k:|P^l_k|>1} v^l_k)\}$;
11: **for** $k' \in \{1, \ldots, r_l\} : |P^l_{k'}| = 1$ **do**
12: **if** $\oplus\{\mathcal{B}^*, \mathcal{B}^l_{k'}\}$ **then**
13: $\mathcal{R} \leftarrow \mathcal{R} \cup \{(\mathcal{B}^* \wedge \mathcal{B}^l_{k'}; v^l_{k'})\}$;

Theorem 5.4
Let $(A, \mathcal{WR})$ represent w, and assume that $(s = 1)$ ***and*** *$N^1_j \cap N^1_k = \emptyset$* ***and*** *$N^1_j \cap P^1_k = \emptyset$* ***for every*** *$j \neq k$, $j, k = 1, \ldots, r_1$ for every $WR \in \mathcal{WR}$. Algorithm 3 computes the ESV_i^M for all $a_i \in A$. Furthermore, for each $WR \in \mathcal{WR}$, the running time of Algorithm 3 is $O(|A| \times |WR|)$. Therefore, it is linear in $|\mathcal{WR}|$.*

Proof: Following Lemma 5.1, we focus on the case when $|\mathcal{WR}| = 1$ and rewrite the weighted rule as $k = 1, \ldots, r_1$ weighted rules of the form:

$$(\mathcal{B}^1_1; 0) \ldots (\mathcal{B}^1_{k-1}; 0)(\mathcal{B}^1_k; v)(\mathcal{B}^1_{k+1}; 0) \ldots (\mathcal{B}^1_{r_1}; 0) \qquad (1)$$

Now, the game can be represented by the following rules which consist of only positive literals:

$$(p^1_1 \wedge n'^1_1; 0) \ldots (p^1_k \wedge n'^1_k; (-1)^{|N'^1_1| + \ldots + |N'^1_{r_1}|} \cdot v) \ldots (p^1_{r_1} \wedge n'^1_{r_1}; 0) \quad (2)$$

for $N'^1_j \subseteq N^1_j$ and $j = 1, \ldots, r_1$. This follows from the following formula $\sum_{0 \leq l \leq r_1} (-1)^l \binom{r_1}{l} = 0$. **Note that from the assumption that $N^1_j \cap N^1_k = \emptyset$ and $N^1_j \cap P^1_k = \emptyset$ for every $j \neq k$, $j, k = 1, \ldots, r_1$ we know that each rule is feasible. Thus,** according to Proposition 5.4, for each k, $ESV_i^M(w)$ is:

$$\sum_{(N'^1_1, \ldots, N'^1_{r_1}) : N'^1_j \subseteq N^1_j} (-1)^{|N'^1_1| + \ldots + |N'^1_{r_1}|} \cdot \frac{v}{|P^1_k| + |N'^1_k|}$$

$$= \sum_{N'^1_k \subseteq N^1_k} (-1)^{|N'^1_1|} \cdot \frac{v}{|P^1_k| + |N'^1_k|} \times \sum_{(N'^1_1, \ldots, N'^1_{r_1}) : N'^1_{j \neq k} \subseteq N^1_j} (-1)^{|N'^1_{1 \neq k}| + \ldots + |N'^1_{1 \neq k}|}$$

The sum $\sum_{(N'^1_1, \ldots, N'^1_{r_1}) : N'^1_{j \neq k} \subseteq N^1_j}$ has non zero value iff $\forall j \neq 2$, $N^1_j = \emptyset$. Now, let us assume that $N^1_j = \emptyset$ for $j \neq k$. Then, the sum $\sum_{(N_1'^1,\ldots,N_{r_1}'^1):N_{j\neq k}'^1\subseteq N_j^1}(-1)^{|N_{1\neq k}'^1|+\ldots+|N_{1\neq k}'^1|}=1$; thus

$$ESV_i^M(w)=\sum_{N_k'^1\subseteq N_k^1}(-1)^{|N_k'^1|}\cdot\frac{v}{|P_k^1|+|N_k'^1|}=f(|P_k^1|,|N_k^1|).$$

From (2) we have that $ESV_i^M(w)=0$ for $a_i\notin P_k^1$. The agents in N_k^1 are equally valued by ESV^M due to symmetry axiom. Since A is always a carrier, we have that $\sum_{a_i}ESV_i^M(w)=w(A,\{A\})$; thus $ESV_i^M(w)=-\frac{|P_k^1|}{|N_k^1|}f(|P_k^1|,|N_k^1|)$ for $a_i\in N_k^1$. The running time comes from the fact that, in every rule, $|WR|$ expressions have to be checked against every $a_i\in A$. $\square$