Complainer's Dilemma

By: Greg Leo*

In assessing complaint policy, it is important to understand the strategic environment facing complainers. Complaining is costly, and there is incentive to free-ride. I model complainer's incentives as a generalization of the Volunteer's Dilemma. Any who complain pay a cost c, if at least m complain, all receive benefit b. The equilibria of this game have interesting implications for complaint policy. Even without being able to observe the benefit of addressing particular problems, the administrator can prevent complaints that are not worth addressing by carefully setting the cost and required number of complaints. I present the properties of this family of policies, including novel results for large constituencies. Notably, policies which minimize the cost of complaining while requiring a suitably large number of complaints are most efficient. This result relies on a useful transcendental function known as Lambert-W, and, in proving the result, I generalize a Poisson tail inequality originally due to Teicher (1955). Because these two aspects of the proof may be of broader interest, I discuss the use of Lambert-W in economics and applications of this Poisson tail inequality in separate appendix sections. *(JEL* C72, C73, D82*)*

1 Introduction

"...but when complaints are freely heard, deeply consider'd and speedily reform'd, then is the utmost bound of civil liberty attain'd..."

-From *Areopagitica* by John Milton Milton (1886).

Advancements in technology have created new ways to complain. Under president Obama, the White House introduced an online petition system called "We The People" to make it easier for citizens to voice concerns.[1] Aitamurto (2012) calls this "democratic crowdsourcing". At the civic level, reporting systems (or broadly "Constituency Relationship Management" systems) have evolved from call-in hotlines to smartphone apps (Gordon and Baldwin-Philippi, 2013). For instance, in Boston, the Commonwealth Connect application and Cambridge's iReport allow citizens to report issues ranging from roadway surface problems to graffiti and even rodent sightings directly from their smartphones.[2,3] Houston, Chicago, Washington DC and others have also implemented mobile reporting systems.[4] Private companies offer similar services. AT&T and Sprint provide smartphone apps that allow customers to report areas with weak wireless signal.[5,6]

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[1] Source: "Petition the White House with We the People" https://www.whitehouse.gov/blog/2011/09/22/petition-white-house-we-people

[2] Source: City of Boston, Commonwealth Connect. http://www.cityofboston.gov/DoIT/apps/commonwealthconnect.asp

[3] Source: City of Cambridge iReport - City of Cambridge. https://www.cambridgema.gov/iReport

[4] Source: See Click Fix, Mayor Quotes. http://gov.seeclickfix.com/customers/mayor-quotes

[5] Source: AT&T. http://www.research.att.com/articles/featured_stories/2010_09/201009_MTS.html?fbid=MCc-Zj9a1ms

[6] Source: Sprint. http://support.sprint.com/support/article/Diagnose-network-problems-service-and-coverage-issues-with-Sprint-Zone/c98c1442-0a59-4fce-97e5-d512ab2f2078

Inevitably, lowering the cost of complaining attracts more complaints about a wider variety of issues. However, this does not necessarily mean it is easier to get problems addressed. There are explicit costs to addressing issues, and when administrative resources are limited, the opportunity cost of addressing one issue is not addressing another. Some problems are not worth addressing because than the cost (including opportunity) is higher than the social benefit. Increasing the number of complaints by lowering cost has the downside of forcing the administrators to sift through additional noise. For instance, the White House received a petition to "have President Obama do the Hokey Pokey[7]" after implementing the "We The People" system.

This additional noise implies that these systems are likely to be accompanied by policies of ignoring problems which do not reach a certain threshold of complaints. Indeed, the Hokey-Pokey petition fell short of the 25,000 signature threshold for the We The People system, and it is probably only a partial coincidence that the White House raised the threshold to 100,000 a few months later.[8]

The White House is explicit about their threshold. Other's acknowledge a threshold but are not so explicit. For instance, the website for "Street Bump", an app for reporting potholes in Boston, says: "We won't inspect a roadway problem unless we get multiple reports[9]." However, even in the absence of such acknowledgment, it is likely that complainers understand that such policies are in place.

The fact that lower cost of complaining is associated with increased complaint thresholds muddles the efficiency picture. This is further complicated by the fact that complainers face a strategic environment. Under these systems, there is a strong incentive to free-ride. Suppose there are several who would benefit of having a particular problem addressed, and this benefit is greater than the cost of complaining. If the administrator's policy is to fix any problem that receives at least one complaint, then any would be willing to complain if he knew for sure no one else would. However, since only one complaint is needed, everyone prefers someone else do it.

This scenario is elegantly modeled by Volunteer's Dilemma game (Diekmann, 1985). In this game, the only symmetric equilibrium is mixed. All individuals volunteer (or complain in this case) with some probability that is decreasing in the cost/benefit ratio. This suggests why a reduction of complaining costs is so appealing to the complainers. A reduction in cost reduces the free-riding problem, and so there is a better chance that *someone* will complain. However, the cost reduction also creates incentives to complain about new and potentially trivial problems, (as long as the personal benefit exceeds the, now smaller, cost). The reduction in free-riding, clearly beneficial for complainers, is a problem for the administrator, *a problem of noise*.

The intuitive solution to this noise is to raise the complaint threshold. However, with a higher threshold, the volunteer's dilemma no longer captures the strategic incentives of the complainers. In section two of this paper, I present a game, the m-volunteers dilemma, that captures the decision about whether to volunteer (complain) when the threshold is larger than one. This game has a very interesting property which implies that this intuitive solution to noise is a valid one.

When only one complaint is required, complaints can be received (in equilibrium) about any problem for which the individual benefit exceeds the cost of complaining. However, when more than one complaint is required, there is a benefit level*, strictly above the cost of complaining*, below which complaints are *never* received in a symmetric equilibrium. That is, there are some problems that the complainers might like to have addressed (and in fact would complain about if only one complaint were required) which they will never complain about in equilibrium when more than one complaint is required.

This property of the strategic environment can be leveraged by the administrator to overcome the asymmetric infor-

[7] See: https://petitions.whitehouse.gov/petition/have-president-obama-do-hokey-pokey
[8] Source: "Petition the White House with We the People" https://www.whitehouse.gov/blog/2011/09/22/petition-white-house-we-people
[9] Source: Streetbump. http://www.streetbump.org/about

mation problem. By carefully selecting the cost of complaining and the number of complaints, the administrator can indirectly force the complainers to internalize the social cost of fixing problems. In fact, complaints about trivial problems can be completely eliminated. However, it is possible to do this in many ways- using higher complaint costs with lower thresholds or lower complaint cost with higher thresholds.

Low costs are appealing, and clearly the trend, but are they really better when we account for higher thresholds and the subtle free-riding incentives in equilibrium? At least among the particular class of policies studied here, the answer is *yes*. Among policies that rule out the same set of trivial complaints, problems with net social benefit are more likely to be fixed under policies which use a *low cost and high threshold*. From the standpoint of fixing non-trivial problems, lowering complaint costs and raising the complaint threshold is more efficient.[10]

The paper is structured as follows. Section 2 presents the m-volunteer's dilemma game, a generalization of Diekmann's Volunteer's Dilemma, and presents it's relevant equilibrium properties. Section 3 focuses on the administrator's problem of choosing a policy for a finite constituency. Section 4 presents approximations of the results in sections 2 and 3 for large constituencies. The novel results in this section rely on a useful transcendental function known as Lambert-W and demonstrate it's application in game theory. The paper concludes in section 5.

2 The Dilemma

An administrator is responsible for n constituents. Problems arise which are observed by the constituents but must be addressed by the administrator. Each problem is characterized by the value b which the constituents receive if the problem is addressed. The administrator can address problems at cost of γ (measured in per-capita terms to make it comparable to b). Unlike b, γ is fixed for all problems and known to the administrator. Under these conditions, the social optimum is to fix all problems for which $b > \gamma$.

The administrator cannot directly observe problems but instead learns about them through complaints. Furthermore, even upon receiving complaints, the administrator cannot directly assess the value b. This may be unrealistic in some environments. For instance, the White House knew that President Obama's failure to do the Hokey Pokey was not a high-benefit problem. However, this assumption maximizes the asymmetry of information and isolates how the strategic environment induced by the policy choices can be leveraged by the policy-maker.

A complaint policy consists of two elements chosen by the administrator: the cost of complaining c[11] and a threshold level m- the number of complaints the administrator requires before addressing a problem. A particular policy induces a game on the constituents. All who complain pay a cost of c. If there are at least m volunteers, the administrator addresses the problem and each receives a benefit b. These games are marked by freeriding incentives. When $m = 1$, the game is the classic Volunteer's Dilemma (Diekmann, 1985). When $b > c$ any player would be willing to complain if she knew no one else would, but if she believes someone else will complain, she would prefer not to. When $m > 1$, the constituents play a related game which I refer to as the m-volunteer's dilemma.[12]

The administrator can take advantage of the incentives of the game to partially overcome the asymmetric information problem. As I will demonstrate, even when the administrator only knows γ and n, it is possible to completely eliminate addressing "trivial" problems ($b < \gamma$) by carefully choosing the complaint policy. Before turning to this choice, I first establish the important properties of the game played under a fixed policy which are used in the results.

[10] In large groups, the cost of complaints can largely be ignored in determining efficiency since any the complaint costs, endured by only a small fraction of the group, are swamped by efficiency changes in addressing problems which affect the entire group. However, in numerical tests with small groups, it is often true that efficiency is higher for low cost policies even when the costs of complaining are included.

[11] The administrator has an ample catalog of paper forms and apps involving various levels of annoyance to choose from.

[12] Archetti and Scheuring (2011) demonstrate cooperators and defectors can coexist in a stable equilibrium among a population playing a generalized dilemma which includes the m-volunteer's dilemma.

2.1 Equilibria

A non-degenerate symmetric equilibrium[13] of the game has a simple form. By complaining, a player always bears cost c, but complaining only changes the outcome when a player is pivotal, and a player is pivotal when $m-1$ out of the $n-1$ others have complained. Thus, the expected value of complaining is b weighted by the pivotal probability. When the cost of complaining, c, is equal to the expected benefit, the player indifferent and willing to mix. This occurs when the pivotal probability is equal to $\frac{c}{b}$. Let X represent the number of players out of any $n-1$ who complain, the equilibrium condition is[14]:

$$Pr\left(X = m-1\right) = \frac{c}{b} \tag{1}$$

In a symmetric equilibrium, X is a binomial random variable. Letting p be the probability that an *individual* complains, the equilibrium condition can be written:

$$\binom{n-1}{m-1} p^{m-1}\left(1-p\right)^{n-m} = \frac{c}{b} \tag{2}$$

Taking $m=1$ provides the equilibrium condition of Diekmann's Volunteers Dilemma: $1-p=q=\left(\frac{c}{b}\right)^{\frac{1}{n-1}}$[15]. For $m \geq 2$, the left side of equation 2 is a polynomial which is equal to 0 at both $p=1$ and $p=0$ and reaches a maximum of $z_{m,n} = \binom{n-1}{m-1}\left(\frac{m-1}{n-1}\right)^{m-1}\left(\left(\frac{n-m}{n-1}\right)\right)^{n-m}$ at $p=\frac{m-1}{n-1}$. Since it has a single stationary point (a maximum) in the domain $p \in (0,1)$, it is quasiconcave. The maximum $z_{m,n}$ proves useful for determining when mixed-strategy equilibria exist. It can be interpreted as the largest probability that a player could possibly believe he is pivotal under symmetric strategies.

2.1.1 Existence and Stability of Equilibrium

The next three results characterize the symmetric equilibria for this game. For fixed n and m, the number and location of equilibra depend on the cost/benefit ratio $\frac{c}{b}$. If the benefit is low relative to the cost of complaining there are no symmetric equilibria involving non-zero probability of complaining (*proposition 1*).[16] This can occur even if the benefit is higher than the cost of complaining. On the other hand, if the benefit of having the problem fixed is high enough, there exist *two* symmetric equilibria involving non-zero probability of complaints (*proposition 2*). When this is the case, only one of the equilibria is stable and it is the equilibrium which involves a higher probability of complaining (*proposition 3*).

Proposition 1. *There does not exist a symmetric equilibrium with non-zero probability of volunteering when $\frac{c}{b} > z_{m,n}$.*

Proof. An equilibrium occurs where $\binom{n-1}{m-1}p^{m-1}q^{n-m} = \frac{c}{b}$. The function $\binom{n-1}{m-1}p^{m-1}q^{n-m}$ has a maximum of $z_{m,n}$. If $\frac{c}{b} > z_{m,n}$ then there is no $p \in [0,1]$ which makes this true. This is demonstrated graphically in figure 1. □

[13] There is always a symmetric equilibrium for $m \geq 2$ where everyone volunteers with zero probability.

[14] A player is indifferent when: $b\left[P\left(X \geq m-1\right)\right] - c = b\left[P\left(X \geq m\right)\right]$ this is equivalent to $b\left[P\left(X \geq m-1\right) - P\left(X \geq m\right)\right] = c$ which simplifies to $b\left[P\left(X = m-1\right)\right] = c$.

[15] Note that the probability there are no volunteers is simply $q^n = \left(\frac{1}{b}\right)^{\frac{n}{n-1}}$ which is an increasing function of n and approaches $\frac{1}{b}$. The probability that none volunteer is increasing in group size. This negative group-size result first pointed out by Diekmann (1985) also holds in games where only one of volunteer is randomly chosen to bear the cost or when all volunteers share the cost Bergstrom and Leo (2015). A similar result holds more generally for the m-volunteers dilemma Bergstrom and Leo (2015).

[16] When $m \geq 2$ there is always a pure-strategy symmetric Nash equilibrium in which no one complains.

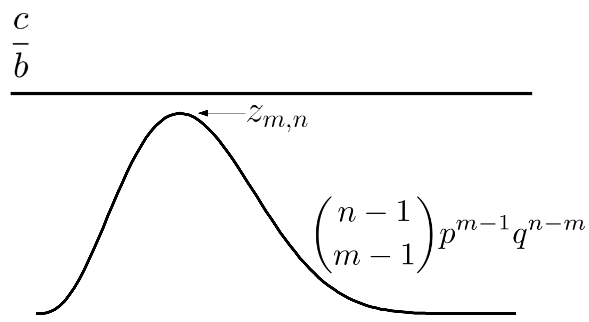

Figure 1: No Mixed Equilibrium Exist $\frac{c}{b} > z_{m,n}$

This puts a bound on the types of problems which receive complaints. Interestingly, since $z_{m,n}$ is decreasing in n (proof in appendix), smaller constituencies complain about a larger set of problems (under a fixed policy). For example, when $m = 2$ and $c = 1$, constituents complain (at least probabilistically) about any problems for which $b > \left(\frac{n-1}{n-2}\right)^{n-2}$. When $n = 3$ this is $b > 2$. As $n \to \infty$, this converges to $b > e$. This result is related in nature to the group size results of Bergstrom and Leo (2015); Diekmann (1985). Larger groups are more strongly affected by the free-riding inherent in this environment. The next result characterizes the equilibria that exist when $\frac{c}{b} < z_{m,n}$.

Proposition 2. *For $m \geq 2$ and any $\frac{c}{b} < z_{m,n}$ there are two symmetric non-degenerate mixed-strategy equilibria p_1^* and p_2^*. $p_1^* < \frac{m-1}{n-1}$ and $p_2^* > \frac{m-1}{n-1}$*

Proof. When $\frac{c}{b} < z_{m,n}$, the maximum of $\binom{n-1}{m-1}p^{m-1}q^{n-m}$ is larger than $\frac{c}{b}$. Since $\binom{n-1}{m-1}p^{m-1}q^{n-m}$ is quasiconcave in p and equal to zero at $p = 0$ and $p = 1$, it must attain the value $\frac{c}{b}$ exactly twice on either side of the maximum value of $\binom{n-1}{m-1}p^{m-1}q^{n-m}$ which occurs at $p = \frac{m-1}{n-1}$. This is demonstrated graphically in figure 2. □

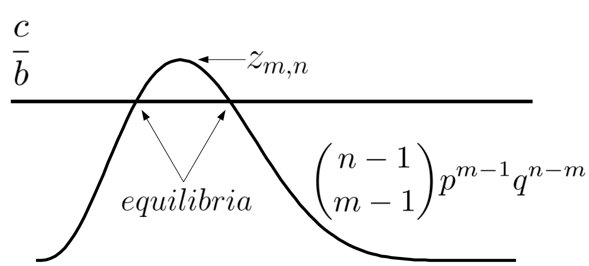

Figure 2: Two Mixed Equilibrium Exist $\frac{c}{b} > z_{m,n}$

For example, suppose $n = 3$, $m = 2$ and $b = 4$ and $c = 1$, the equilibrium condition is $2p(1-p) = \frac{1}{4}$. This has solutions $\frac{2-\sqrt{2}}{4} \approx .146$ and $\frac{2+\sqrt{2}}{4} \approx .854$. In the low-probability case it is very likely that there will be no complainers among any two randomly chosen (about .73 probability) and very unlikely that both will volunteer (about .02). Precisely the reverse is true in the high probability case. However, in both equilibria, there is exactly a $\frac{1}{4}$ probability of having one complainer among two randomly chosen players. The remaining player's complaint is pivotal.

The next result provides a remedy for the multi-equilibrium problem. The equilibrium which involves the higher probability of volunteering is stable in the sense of Palm (1984).

Proposition 3. *When there are two equilibria with $p > 0$, only one is stable and it is the equilibrium with higher p. For this equilibrium, there is a neighborhood $N(p) \ni p$ such that the utility from complaining with probability p when others complain with probability $p' \in N(p)$ is larger than complaining with probability p'. (Proof in Appendix)*

Looking at figure 1, in the pivotal probability function is increasing below $z_{m,n}$. Small increases in the probability that each complains increases the pivotal probability- increasing the incentive for complaining and creating more pressure to increase p. This positive-feedback is responsible for the instability of the low-probability equilibrium. On the other hand, since the pivotal probability is decreasing in p in the high-probability equilibrium, increases in the the complaint probability lower the incentive to complain creating a stabilizing feedback. The next section provides comparative statics for how the probability of complaining changes with each parameter.

2.1.2 Comparative Statics

These comparative statics in this section form the basis for analyzing complaint policies. There are four parameters in an m-volunteers dilemma: the threshold m, the number of players n, the cost c, and the benefit b. Here I consider the effects of changes to these parameters on the equilibrium complaint probability. In the *stable equilibrium* (high p), the compliant probability reacts in an intuitive way to changes in each parameter. Increasing the required number of complaints (m) or the benefit (b) increases the equilibrium complaint probability . Increasing the number of players (n) or the cost (c) decreases the probability. However, the unstable equilibrium (low p) has some unintuitive comparative statics when it comes to changes in costs and benefits. In the unstable equilibrium, the complaint probability *increases* when the cost increases or the benefit decreases. The unintuitive comparative statics of the unstable equilibrium provides more reason to focus on the stable equilibrium in the remaining analysis in this paper.

Proposition 4. *The probability of volunteering increases with m in the stable/unstable equilibrium. (Proof in Appendix)*

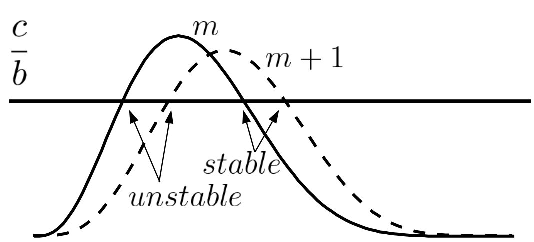

Figure 3: Comparative Statics in m

Proposition 4 is demonstrated graphically in the figure above. As m increases, the function $\binom{n-1}{m-1}p^{m-1}q^{n-m}$ shifts to the right. $\frac{c}{b}$ intersects the new function at higher p in both the unstable and stable equilibrium. A similar result, also given in the appendix demonstrates that the probability of volunteering is decreasing in n. As mentioned above, the comparative statics with respect to cost and benefit are a little less straight-forward and depend on which equilibrium is under consideration.

Proposition 5. *The probability of complaining decreases/increases with cost of complaining in the stable/unstable equilibrium. The probability of complaining decreases/increases with benefit in the stable/unstable equilibrium.*

Proof. $\binom{n-1}{m-1}p^{m-1}q^{n-m}$ is increasing in p over $[0, z_{m,n}]$ and decreasing over $[z_{m,n}, 1]$. An equilibrium solves $\binom{n-1}{m-1}p^{m-1}q^{n-m} = \frac{c}{b}$ by the implicit function theorem, the change in the equilibrium p^* is increasing/decreasing with c in the range in which $\binom{n-1}{m-1}p^{m-1}q^{n-m}$ increasing/decreasing. The unstable equilibrium occurs below $z_{m,n}$ and the stable equilibrium occurs above $z_{m,n}$. Thus, the unstable equilibrium is increasing with c and the stable equilibrium is decreasing with c. By the same logic, the opposite is true for benefit b. $\square$

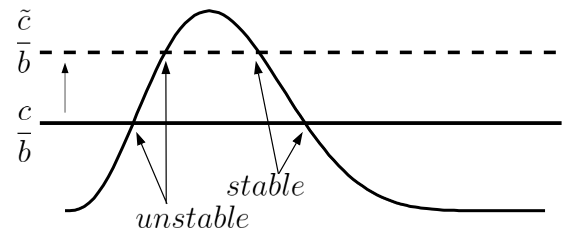

Figure 4: Comparative Statics in m

This proposition is demonstrated in the figure above. The quasi-concave shape of the function $\binom{n-1}{m-1}p^{m-1}q^{n-m}$ and the fact that the two equilibria occur on either side of the maximum imply that as $\frac{c}{b}$ increases, the function $\binom{n-1}{m-1}p^{m-1}q^{n-m}$ intersects it at a higher p below the $\left(\frac{m-1}{n-1}\right)$ (*unstable equilibrium*) and a lower p above the maximizer (*stable equilibrium*).

Of course, of the four parameters, only the cost of complaining and the threshold are under control of the administrator. The next section demonstrates how the administrator can adjust these values in combination to take advantage of the equilibrium structure to overcome the asymmetric equilibrium problem and address problems more efficiently.

3 Complaint Policy

Any choice of c and m, induces a particular m-volunteer's dilemma on potential complainers when a problem with value b arises. Again, while c, m are set by the administrator and n is known, b cannot be verified by the administrator. However, if there is a complaint, it must have been worthwhile to complain. For instance, suppose there are three constituents ($n = 3$), the administrator requires two complaints ($m = 2$) and the cost of complaining is $c = 1$. Any problems that receive complaints must have a value $b > 2$ (see section 2.1).[17] For lower benefit levels, there are no symmetric equilibria that involve complaining. The administrator can leverage this fact in choosing a policy. Suppose γ, the cost of addressing problems, is 2 (per-capita). The policy of requiring two complaints with a cost of complaining $c = 1$ *guarantees* that only socially beneficial problems are addressed.[18] Suppose there are two types of problems. Some problems are trivial $b = 1$ and some are significant $b = 10$. Under this policy, trivial problems never receive complaints and are never addressed. The probability an important problem is addressed (there are two or more complaints) is approximately 99%.

Of course, there are many pairs of c and m that will rule out complaints about problems with $b < \gamma$. In the next section, I derive the class of policies that rule out trivial complaints.

[17] The administrator can also deduce the expected value of the benefit from from the number of complaints. However, if the constituents know he will use this information, it distorts their incentives, making the initial assessment unreliable.

[18] Of course, this policy is not perfectly efficient since a problem worth fixing might not receive enough complaints.

3.1 *Eliminating Trivial Complaints*

The administrator can to eliminate complaints about all trivial problems $(b < \gamma)$ if c and m are chosen such that $\frac{c}{b} = z_{m,n}$ when $b = \gamma$. Doing so ensures there are no symmetric equilibria involving a non-zero complaint probability when $b < \gamma$. This is demonstrated in figure 5.

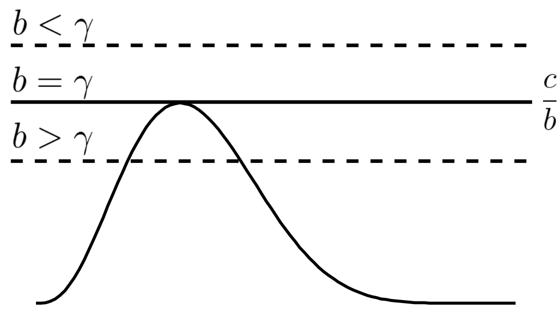

Figure 5: Eliminating Trivial Complaints.

Such a policy can be achieved by setting the following cost for any chosen threshold:

$$c_m = \gamma z_{m,n} = \gamma \binom{n-1}{m-1} \left(\frac{m-1}{n-1}\right)^{m-1} \left(\frac{n-m}{n-1}\right)^{n-m} \tag{3}$$

For instance, when $n = 10$,

	$m = 2$	3	4	5	6	7	8	9
$\frac{c_m}{\gamma}$	0.390	0.306	0.273	0.260	0.260	0.273	0.306	0.390

For any fixed size of the constituency, the cost that the administrator must set is decreasing in the threshold as long as the threshold is less than half of the size of the constituency $\left(m < \frac{n}{2}\right)$. As long as the threshold is small relative to the size of the constituency, the cost of complaining and the complaint threshold are inversely related for this class of policies. This is demonstrated in figure 6. If the administrator decreases the cost of complaining, the threshold must be increased to continue eliminating the same trivial complaints. This demonstrates more formally the intuitive arguments discussed in the introduction.

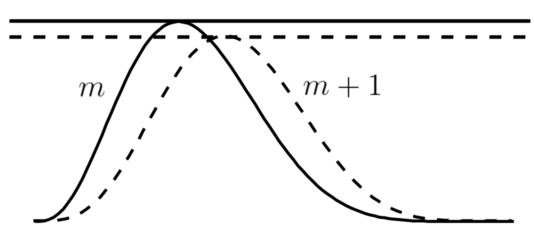

Figure 6: Higher m requires lower c when $m < \frac{n}{2}$.

3.2 Policy Choice

Even when focusing on the class of policies that eliminate all trivial complaints, there are many options for the administrator.[19] For small constituencies, the administrator's objective should account both for the the expected net social benefits $(b-\gamma)$ of problems addressed and the expected aggregate cost of complaining imposed by the policy. With more detailed knowledge of the distribution of problem values (b), it is possible to optimize over c and m. This may be interesting for particular problems, but would likely be primarily a numerical exercise given the polynomial form of the equilibrium condition in equation 2.

While small constituencies are relevant, for instance, in modeling the administration of a small condo building or a single academic department, administrators often oversee very large constituencies such as is the case for governments, large corporations, universities, etc. Beyond the relevance for these important environments, there are several benefits to analyzing complaint policy for large constituencies ($n \to \infty$). As I demonstrate below, for large constituencies, the equilibria of the m-volunteer's dilemma game can be solved in "near" closed form.[20] Further, as n grows large, the probability of complaining shrinks to zero (this is discussed further section 4) and the effect of complaint costs are dominated by changes in the probability of having problems addressed. For large constituencies, it is thus convenient (and approximately correct) to ignore the costs of complaining in assessing the efficiency of policies. This simplifies the administrators objective and permits an analytic rather than numeric approach to optimizing among policies.

4 Large Constituencies and The Lambert-W Correspondence

In this section, I derive the properties of the limit of m-volunteer dilemma equilibria as the group size grows large. In the limit, the equilibrium expected number of volunteers can be solved in closed-form, where solving for these values in finite groups involves finding polynomial roots. The motivation for working with the limit of these games is that this limit can serve as an approximation for what happens in large finite groups, such as those that might be faced by an administrator at a large institution. Fortunately, much of the structure of the finite constituency analysis applies here as well.

4.1 Equilibria

In finite constituencies, each player asses the likelihood that his complaint is pivotal. There, the number of other people who complain (X) is a binomial random variable. As n grows large, X can be approximated by a Poisson random variable. The following condition provides an approximation of the equilibrium condition for large group sizes.

$$\frac{((n-1)\,p)^{m-1}\,e^{-(n-1)p}}{(m-1)!} = \frac{c}{b} \tag{4}$$

Proposition 6. *The solutions to this approximation (4) are given by the following. (Proof in Appendix)*

$$p = -\frac{m-1}{n-1} W\left(-\frac{1}{m-1}\left[(m-1)!\frac{c}{b}\right]^{\frac{1}{m-1}}\right) \tag{5}$$

[19]Complaint threshold m is clearly a choice. While it may seem like the available choices for complaint cost c are constrained by available technology (paper forms, phone-in hotline, online form, smart phone apps) further control is possible through all kinds of modern annoyances: account logins, requesting frivolous details, increasing hold-time, etc.

[20]The solution involves the Lambert-W correspondence, an exponential function closely related to *log*.

The W in the expression above is worth remark. W, the Lambert-W correspondence provides the inverse solution of $y = xe^x$. A graph of the Lambert-W correspondence is provided in figure 7. This close relative of the *log* function has found applications in many areas of science from solving fuel-consumption problems, to modeling the spread of epidemics (Corless et al., 1996). Lambert-W is also used in solving for similar limits in the *Coordinated Volunteer's Dilemma* (Bergstrom and Leo, 2015).

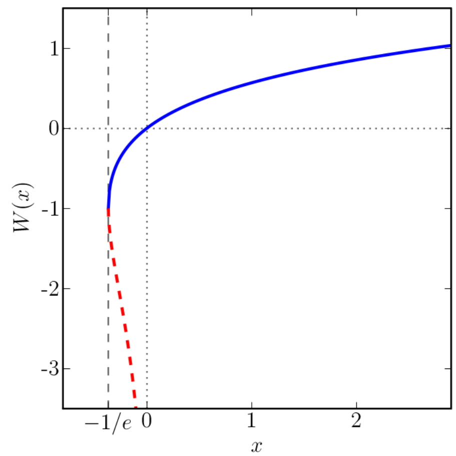

Figure 7: Lambert-W Correspondence

For large n, (5) is uninformative since the right side approaches zero. That is, the probability any one person volunteers approaches zero in a large group. This has to be the case in equilibrium, otherwise the probability of receiving at least m complaints would be 1 and none would have incentive to volunteer- a disequilirbium outcome. It is more informative to consider the approximation of the expected number of volunteers:

$$p(n-1) = -(m-1)W\left(-\frac{1}{m-1}\left[(m-1)!\frac{c}{b}\right]^{\frac{1}{m-1}}\right) \tag{6}$$

Now the right side is invariant in n and the left is approximately the expected number of volunteers for large n. Denote this expectation by λ_m. In the limit:

$$\lambda_m = -(m-1)W\left(-\frac{1}{m-1}\left[(m-1)!\frac{c}{b}\right]^{\frac{1}{m-1}}\right) \tag{7}$$

Recall that for small-groups, the game has two equilibria with volunteering (when any exist). This is true in the limit as well. The two equilibria correspond with the two branches of W. Along the upper branch, the expected number of volunteers approaches zero as b grows. This corresponds to limit of the low-probability family of equilibria. Along the lower branch, the expected number of volunteers approaches infinity as b grows. Thus, the benefit is very high, there is plenty of redundancy to ensure that the problem is addressed. This corresponds to the limit of the high-probability family of equilibria from the finite constituency case.

Suppose $m = 2$. The expected number of complaints in equilibrium is:

$$\lambda_2 = -W\left(-\frac{c}{b}\right) \tag{8}$$

Note that W is not defined below $-\frac{1}{e}$. This corresponds to the fact that as $n \to \infty$ for $m = 2$, the region in which no equilibrium with volunteering exists converges to $\left(\frac{1}{e}, 1\right)$, as shown previously in section 2.1.1 by taking the limit of $z_{m,n}$. For comparison, it is possible to determine the expected number of volunteers in equilibrium for Diekmann's game ($m = 1$). From taking the limit as $n \to \infty$ of equation 5 with $m = 1$:

$$\lambda_1 = -log\left(\frac{c}{b}\right) \tag{9}$$

This provides some context to the relationship between *log* and *Lambert-W* in a game-theoretic context.

4.2 Eliminating Trivial Complaints

Equilibrium condition (7) is only defined on the domain where the term inside the Lambert-W correspondence is larger than$-\frac{1}{e}$. When the term is smaller than $-\frac{1}{e}$, there are no equilibria with a non-zero expected number of complaints. Since this term is increasing in b, if c is set such that this term is exactly $-\frac{1}{e}$ for a problem with benefit γ, then the administrator can eliminate complaints about all trivial problems ($b < \gamma$). This is the cost that solves:

$$-\frac{1}{m-1}\left[(m-1)!\frac{c}{\gamma}\right]^{\frac{1}{m-1}} = -\frac{1}{e} \tag{10}$$

This solution can be given explicitly:

$$c_m = \gamma\frac{\left((m-1)\,\frac{1}{e}\right)^{m-1}}{(m-1)!} \tag{11}$$

This corresponds to the limit of the analogous finite constituency cost giving in equation 3 of the previous section. It is noted that c_m is decreasing in m and approaches 0 as $m \to \infty$. To maintain a policy where problems with benefit just less than γ are ruled out as m increases, the cost of complaining must be lowered. Again, if the administrator asks a lot of each complainer, few complaints should be required. If little is asked of each complainer, many complaints should be required.

4.3 Policy Choice

Since the probability of complaining shrinks to zero, or alternatively since the expected number of complaints is finite, the aggregate welfare effects of complaint costs of are swamped by changes in the probabilities of having problems addressed. Thus, the administrators objective is simply a function of the probability that not enough complaints are received for a particular problem (Q). In a perfectly efficient outcome, $Q = 1$ for problems where $b < \gamma$ and $Q = 0$ for problems where $b > \gamma$.

By construction, in the class of policies defined by (11), $Q = 0$ for any problem in which $b < \gamma$. Thus, for optimizing within this class of policies, the administrators objective is to minimize Q for problems in which $b > y$.[21] Plugging (11) into equation (7) provides the expected number of complainers under the policies of this class for any benefit b,

[21] Of course, this is just one class of policies available. With more detailed knowledge of the distribution of problem values b, the administrator could choose to lower costs below those in condition 3. This would decreases Q_m for all non-trivial problems but require the administrator to occasionally address trivial problems. It is an interesting question whether it might sometimes be efficient to make such a trade-off. On the other hand, setting complaint cost according to condition 3 and eliminating complaints about all trivial problems is likely to be most efficient for large groups when the distribution of problems is such that trivial problems are much more frequent than non-trivial problems.

which can then be simplified into the following expression:

$$\lambda_{m,b} = -(m-1)\,W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right) \tag{12}$$

Note that for any $b < \gamma$, $\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}} > 1$ and thus, $-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}} < -\frac{1}{e}$. However, Lambert-W is undefined below $-\frac{1}{e}$. This again demonstrates that there are no equilibria which involve trivial complaints. On the other hand, for $b > \gamma$, the probability not enough volunteer is given by the the probability that a Poisson random variable with mean $\lambda_{m,b}$ takes a value of $m-1$ or less. This can be expressed as follows:

$$Q_{m-1}\left(\lambda_{m,b}\right) = \sum_{k=0}^{m-1} \frac{e^{-\lambda_{m,b}}\left(\lambda_{m,b}\right)^k}{k!} \tag{13}$$

Table 1 reports computed values of $Q_{m-1}\left(\lambda_{m,b}\right)$ for various benefit levels with $\gamma = 2$. Even at moderate levels of b, the failure to reach the complaint threshold is rare for all policies. However, notice that moving from $m = 2$ to $m = 10000$ offers substantial improvement in all probabilities. This failure rate is more than cut in half for most of the reported benefit levels. From (11), moving from threshold of 2 to 10000 is associated with reducing complaint cost approximately one-hundred-fold. For comparison, this is about the difference in effort for writing and mailing a letter to the president (about an hour) and signing an online White House petition on *"We the People"* (thirty seconds).

	2	3	5	10	20	50	100	1000	10000
$b = 2$	0.736	0.677	0.629	0.587	0.561	0.538	0.527	0.508	0.503
3	0.357	0.310	0.274	0.244	0.225	0.210	0.202	0.190	0.186
5	0.196	0.166	0.143	0.125	0.113	0.103	0.099	0.091	0.089
10	0.092	0.076	0.065	0.055	0.049	0.044	0.042	0.038	0.037
100	0.008	0.007	0.006	0.005	0.004	0.003	0.003	0.003	0.003
1000	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.000

Table 1: Computed Probabilities Not Enough Volunteer for $\gamma = 2$.

Notice that in this chart, the improvement by moving from small threshold policies to large threshold (and low cost) policies is monotonic in m for all levels of b. In fact, this is a general result. Reducing cost and raising the threshold is *always* more efficient within this class of policies. *It is better to ask for less effort from more people.*

Theorem 1. *Among the class of policies that just rule out complaints about problems with $b < \gamma$, the probability not enough complain in the stable equilibrium is decreasing in m for any fixed b.*

Proof. The proposition is equivalent to $Q_{m-1}\left(\lambda_{m,b}\right) > Q_m\left(\lambda_{m+1,b}\right)$ for $m \geq 2$ and $b \geq \gamma$. Plugging in for $\lambda_{m,b}$ and $\lambda_{m+1,b}$:

$$Q_{m-1}\left(-(m-1)\,W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)\right) > Q_m\left(-(m)\,W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)\right) \tag{14}$$

Let $z \equiv -W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)$ and $y \equiv \frac{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)}{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)}$. (14) is equivalent to:

$$Q_{m-1}\left((m-1)\,z\right) > Q_m\left(myz\right) \tag{15}$$

Note $y \geq 1$ since $-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}$ is decreasing in m and $W(x)$ is decreasing in x along the minor branch. Since $Q_m(\lambda)$ is decreasing in λ, $Q_m(myz) \geq Q_m(mz)$. Thus, a sufficient condition for (15) is:

$$Q_{m-1}((m-1)z) > Q_m(mz) \tag{16}$$

The condition above is now a rather straight-forward inequality relating Poisson CDFs. It is proven for $z = 1$ by Teicher (1955); Adell and Jodra (2005). I prove the generalization of this result to $z \geq 1$ in appendix section 8. Interestingly, the proof of this result also relies on the properties of Lambert-W. □

5 Discussion

The m-volunteer's dilemma, like the volunteer's dilemma Diekmann (1985) and the coordinated Volunteer's Dilemma Bergstrom and Leo (2015) are marked by free-riding incentives. These free-riding incentives are an important aspect to take into account in analyzing any scenario involving one of these games. The complaint scenario studied here provides a rich and interesting environment in which to study the m-volunteer's dilemma game. The results can also inform analysis in other environments where the effort of several is needed to provide a public good.

For instance in fund raising, matching funds are sometimes contingent on reaching a certain number of donations. Gee and Schreck (2016) call this matching procedure a "threshold match" and study behavior of contributors under this procedure in field and laboratory experiments. They find that subject's behavior is consistent with the kind of pivotal calculations which are key to the equilibrium analysis in the results I have presented here. They also find that matching funds available through such a threshold match are very effective in increasing donations. The results I have presented here suggest that when using a threshold match procedure to raise adequate funds, it may be better to set a higher threshold with a lower minimum donation. In fact, this has been quite successful in practice as well. For instance, in 2013, *Radiotopia*, an online public radio collective, was successful in meeting a threshold of 10,000 donations with a $1 minimum for $20,000 in matching funds.[22] A similar campaign the next year met a threshold of 20,000 backers for $25,000 in matching funds.[23] Most interestingly, the total donations amounted to more than $375,000 from 11,693 backers in the first campaign and $620,000 from 21,808 backers in the second campaign.[24] The recent trend of crowd funding itself and the success of business like kickstarter and indigogo is built on the ask-less-from-more model of collective action (Agrawal et al., 2014).

There are several relevant extensions to consider. Here, I have assumed that complaining happens simultaneously. This is natural when the complainers cannot see or are not informed about the number of complaints that have already been filed. However, often this number is visible as is the case with the "We The People" petition platform and popular crowd-funding sites. Dynamic games related to the volunteer's dilemma have been studied in Bergstrom (2017); Shapira and Eshel (2000); Bilodeau and Slivinski (1996); Weesie (1994); Bliss and Nalebuff (1984). Extending the m-volunteer's dilemma game in this way may provide additional insights. Another interesting area to explore asymmetric players, either in terms of costs or benefits. This has been studied in the standard volunteer's dilemma by Bergstrom and Leo (2015); Diekmann (1993) and in an experimental setting by Pate and Healy (2016).

[22]See: https://www.kickstarter.com/projects/1748303376/99-invisible-season-4-weekly.

[23]See: https://www.kickstarter.com/projects/1748303376/radiotopia-a-storytelling-revolution/posts/1035882

[24]Without a counterfactual it is impossible to determine the exact impact of the threshold match in these campaigns. However, the fact that the number of backers was close to the threshold in both cases suggests that the matching funds were motivating.

6 Bibliography

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7 Appendix

7.1 *Proof that $z_{m,n}$ is decreasing in n.*

$$\binom{n}{m-1}\left(\frac{m-1}{n}\right)^{m-1}\left(\frac{n-m}{n}\right)^{n-m+1} < \binom{n-1}{m-1}\left(\frac{m-1}{n-1}\right)^{m-1}\left(\left(\frac{n-m}{n-1}\right)\right)^{n-m} \tag{17}$$

$$\binom{n}{m-1}\left(\frac{m-1}{n}\right)^{m-1}\left(\frac{n-m}{n}\right)^{n-m}\left(\frac{n-m}{n}\right) < \binom{n-1}{m-1}\left(\frac{m-1}{n-1}\right)^{m-1}\left(\left(\frac{n-m}{n-1}\right)\right)^{n-m} \tag{18}$$

$$\frac{\binom{n}{m-1}}{\binom{n-1}{m-1}}\left(\frac{n-m}{n}\right) < \frac{\left(\frac{m-1}{n-1}\right)^{m-1}\left(\left(\frac{n-m}{n-1}\right)\right)^{n-m}}{\left(\frac{m-1}{n}\right)^{m-1}\left(\frac{n-m}{n}\right)^{n-m}} \tag{19}$$

$$\frac{n}{n-m+1}\left(\frac{n-m}{n}\right) < \left(\frac{n}{n-1}\right)^{m-1}\left(\frac{n}{n-1}\right)^{n-m} \tag{20}$$

$$\frac{n-m}{n-m+1} < \left(\frac{n}{n-1}\right)^{n-1} \tag{21}$$

This is true since $\frac{n}{n-1} > 1$ and $\frac{n-m}{n-m+1} < 1$.

7.2 *Proof of Proposition 3*

Let $\pi(p_1, p_2, ..., p_n)$ be the payoff function (a function of the probability each individual volunteers) to a player. By Palm (1984) (prop 4), stability is equivalent to $\pi(p, x, p, ..., p) > \pi(x, x, p, ..., p)$ for every $x \in [0,1] \setminus p$ in this case where $\pi(p, p, p, ..., p) = \pi(x, p, p, ..., p)$ (which is true when p is a mixed-strategy equilibrium).

The probability that $m-1$ volunteer out of $n-1$ when one player volunteers with probability x and the others with probability p is:

$$xb(m-2, n-2, p) + (1-x)b(m-1, n-2, p)$$

$b(m-1, n-2, p) > (<) b(m-2, n-2, p)$ when $p > (<) \frac{m-1}{n-1}$.

For the low-probability equilibrium p_l, $b(m-1, n-2, p) < b(m-2, n-2, p)$ when one other volunteers with probability $x < p_l$ the probability that $m-1$ volunteer is smaller than in equilibrium where everyone else volunteers with probability p_l. Best response is to not volunteer. Since $x < p_l$ puts more weight on not-volunteering: $\pi(x, x, p_l, ..., p_l) > \pi(p_l, x, p_l, ..., p_l)$. Thus, the low probability equilibrium is not stable.

For the high-probability equilibrium p_h, $b(m-1, n-2, p) > b(m-2, n-2, p)$ when one other volunteers with probability $x < p_h$ the probability that $m-1$ volunteer is higher than in equilibrium where everyone else volunteers with probability p_h. Best response is to volunteer for sure. Since p_h puts more weight on volunteering, $\pi(p_h, x, p_h, ..., p_h) >$

$\pi(x,x,p_h,...,p_h)$. Similarly when $x > p_h$, the best response is to not volunteer. p_h puts less weight on volunteering and so $\pi(p_h,x,p_h,...,p_h) > \pi(x,x,p_h,...,p_h)$ there as well. □

7.3 *Proof of Proposition 4* and related result for comparative statics in n.

This proof is based on several lemmas given below. When m increases, from m to $m+1$, the two equilibrium conditions are $\binom{n-1}{m-1}p^{m-1}(1-p)^{n-m} = \frac{c}{b}$ and $\binom{n}{m-1}p^{m-1}(1-p)^{n-m+1} = \frac{c}{b}$. By Lemma 1 and 3, among the four equilibria, the small-probability equilibrium for m complaints must be the lowest, and the high-probability equilibrium for $m+1$ must be the highest. This implies that both the high and low-probability equilibria increase with m.

When n increases, from n to $n+1$, the two equilibrium conditions are $\binom{n-1}{m-1}p^{m-1}(1-p)^{n-m} = \frac{c}{b}$ and $\binom{n-1}{m}p^{m}(1-p)^{n-m-1} = \frac{c}{b}$. By Lemma 2 and 3, among the four equilibria, the small-probability equilibrium for $n+1$ complainers must be the lowest, and the high-probability equilibrium for n must be the highest. This implies that both the high and low-probability equilibria decrease with n.

Lemma 1. *$\binom{n-1}{m-1}p^{m-1}(1-p)^{n-m} > (<) \binom{n-1}{m}p^{m}(1-p)^{n-m-1}$ when $p < (>) \frac{n}{m}$. Equality is achieved at $p = \frac{m}{n} \in \left[\frac{m-1}{n-1}, \frac{m}{n-1}\right]$.*

Proof.

$$\binom{n-1}{m-1}p^{m-1}(1-p)^{n-m} > \binom{n-1}{m}p^{m}(1-p)^{n-m-1} \tag{22}$$

$$\frac{m}{n-m} > \frac{p}{1-p} \tag{23}$$

$$p < \frac{m}{n} \tag{24}$$

□

Lemma 2. *$\binom{n}{m-1}p^{m-1}(1-p)^{n-m+1} > (<) \binom{n-1}{m-1}p^{m-1}(1-p)^{n-m}$ when $p < (>) \frac{m-1}{n}$. Equality is achieved at $p = \frac{m-1}{n}$.*

Proof.

$$\binom{n-1}{m-1}p^{m-1}(1-p)^{n-m} < \binom{n}{m-1}p^{m-1}(1-p)^{n-m+1} \tag{25}$$

$$\frac{n-m+1}{n} < 1-p \tag{26}$$

$$p < \frac{m-1}{n} \tag{27}$$

□

Lemma 3. Let $f(x)$ and $g(x)$ be quasi-concave functions such that there are distinct $x_1, x_2 y_1, y_2$ with $f(x_1) = f(x_2) = g(y_1) = g(y_2) = c$. Without loss of generality suppose $x_1 < x_2$, $y_1 < y_2$ and $x_1 < y_1$. If there is a unique z such that such that $f(z) = g(z)$ with $arg.max_x g(x) = x^* \geq z \geq y^* = arg.max_x f(x)$ then $x_2 < y_2$.

Proof. Suppose otherwise. $x_1 < y_1 < y_2 < x_2$. This implies that $g < f$ on the domain $[y^*, x^*]$ which contradicts $x^* \geq z \geq y^*$. □

7.4 *Proof of Proposition 6.*

(28)
$$\frac{((n-1)p)^{m-1} e^{-(n-1)p}}{(m-1)!} = \frac{c}{b}$$

This is equivalent to:

(29)
$$-\frac{((n-1)p)}{m-1} e^{-\frac{(n-1)p}{m-1}} = -\frac{1}{m-1}\left[(m-1)!\frac{c}{b}\right]^{\frac{1}{m-1}}$$

The equation is of the form $xe^x = z$ and can be inverted using the Lambert-W correspondence:

(30)
$$-\frac{(n-1)p}{m-1} = W\left(-\frac{1}{m-1}\left[(m-1)!\frac{c}{b}\right]^{\frac{1}{m-1}}\right)$$

Isolating p provides the following solution:

(31)
$$p = -\frac{m-1}{n-1} W\left(-\frac{1}{m-1}\left[(m-1)!\frac{c}{b}\right]^{\frac{1}{m-1}}\right)$$

8 Appendix For Poisson Tail Inequality Result

Teicher (1955) proved that the probability a Poisson distribution with mean k takes on a value of k or less is monotonically decreasing in k. I extend this inequality by proving that the probability a Poisson distribution with mean zk takes on a value of k or less is monotonically decreasing for $z \geq 1$.

Let X_λ be a Poisson distributed random variable with mean λ. This section considers the monotonicity of $P(X_{zk} \leq k)$ in k. Teicher (1955) proves the following (see also: Adell and Jodra, 2005):

(32)
$$P(X_k \leq k) > P\left(X_{(k+1)} \leq k+1\right)$$

Proposition 7. $P(X_{zk} \leq k) > P\left(X_{z(k+1)} \leq k\right)$ *for* $z \geq 1$.

Proof. Let $G_k(z) \equiv P(X_{zk} \leq k) - P\left(X_{z(k+1)} \leq k\right)$. The proposition is equivalent to $G_k(z) > 0$ for $z \geq 1$. Note that $G_k(1)$ by Adell, Jodra (2005). Furthermore, $lim_{z\to\infty} G_k(z) = 0$. Note that $G_k(z)$ can be rewritten by the following steps:

$$G_k(z) = P(X_{zk} \leq k) - P\left(X_{z(k+1)} \leq k+1\right) = \tag{33}$$
$$P(X_{zk} \leq k+1) - P\left(X_{z(k+1)} \leq k+1\right) - P(X_{zk} = k+1)$$

By the Gamma-Poisson relationship, the lower tail of X_λ can be written as the upper-tail of a random variable with distribution $\Gamma(k+1,1)$. Thus, $P(X_\lambda \leq k) = P(Y > \lambda) = \frac{\int_\lambda^\infty u^k e^{-u} du}{k!}$. Using this relationship, 33 can be written:

$$G_k(z) = \frac{\int_{zk}^\infty u^{(k+1)} e^{-u} du}{(k+1)!} - \frac{\int_{z(k+1)}^\infty u^{(k+1)} e^{-u} du}{(k+1)!} - \frac{(zk)^{k+1} e^{-(zk)}}{(k+1)!} \tag{34}$$

The first two terms have the same integrand but different bounds. The expression can be re-written as:

$$G_k(z) = \frac{\int_{zk}^{z(k+1)} u^{(k+1)} e^{-u} du}{(k+1)!} - \frac{(zk)^{k+1} e^{-(zk)}}{(k+1)!} \tag{35}$$

The derivative of $(k+1)!G_k(z)$ with respect to z is:

$$(k+1)!G_k'(z) = (k+1)(z(k+1))^{(k+1)} e^{-z(k+1)} - (k)(zk)^{k+1} e^{-zk} \tag{36}$$
$$- k(k+1)(zk)^k e^{-zk} + k(zk)^{k+1} e^{-zk}$$
$$= (k+1)(z(k+1))^{(k+1)} e^{-z(k+1)} - k(k+1)(zk)^k e^{-zk} \tag{37}$$

Thus $G_k'(z) > 0$ iff:

$$\left(\frac{k+1}{k}\right)^{(k+1)} > \frac{1}{z} e^z \tag{38}$$

For $z = 1$ this is $\left(\frac{k+1}{k}\right)^{(k+1)} > e$ which is true for all $k > 0$. Thus, $G_k'(1) > 0$. This together with 32 imply that $G_k'(z)$ is initially positive and increasing on $z \in [1, \infty)$. Furthermore, there is a single stationary point since the equation $\left(\frac{k+1}{k}\right)^{(k+1)} = \frac{1}{z} e^z$ has only one solution for $z \geq 1$ which is given by the lower branch of the Lambert-W function:

$$z^* = -W\left(-\left(\frac{k}{k+1}\right)^{(k+1)}\right) \tag{39}$$

Thus, $G_k(z)$ is strictly positive at $z = 1$, increases for $z \leq z^*$ and then decreases for $z \geq z^*$ approaching 0. This implies that $G_k(z) > 0$ for $z \geq 1$. $\square$