Complete partial orders

An \((\omega\text{-}\text{chain-})\)complete partial order, \(\text{cpo}\):

\[
\mathcal{D} = \langle \mathcal{D}, \sqsubseteq, \bot \rangle
\]

- \(\sqsubseteq \subseteq \mathcal{D} \times \mathcal{D}\) is a partial order on \(\mathcal{D}\) such that each countable chain
  \[d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_i \sqsubseteq \ldots\]
  has the least upper bound \(\bigsqcup_{i>0} d_i\) in \(\mathcal{D}\)

- \(\bot \in \mathcal{D}\) is the least element w.r.t. \(\sqsubseteq\)

BTW: Equivalently: all countable \textit{directed} subsets of \(\mathcal{D}\) have lub’s in \(\mathcal{D}\).
(\(\Delta \subseteq \mathcal{D}\) is \textit{directed} if for every \(x, y \in \Delta\), there is \(d \in \Delta\) with \(x \sqsubseteq d\) and \(y \sqsubseteq d\).)

BTW: It is \textit{not} equivalent to require that \textit{all} chains have lub’s in \(\mathcal{D}\).
(\(C \subseteq \mathcal{D}\) is a \textit{chain} if for every \(x, y \in C\), \(x \sqsubseteq y\) or \(y \sqsubseteq x\).)
But it is equivalent to require that \textit{all} countable chains have lub’s in \(\mathcal{D}\).
<table>
<thead>
<tr>
<th>Examples</th>
<th>Non-examples</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle \mathcal{P}(X), \subseteq, \emptyset \rangle$</td>
<td>$\langle \mathcal{P}_{\text{fin}}(X), \subseteq, \emptyset \rangle$</td>
<td>$\mathcal{P}(X)$ is the set of all subsets, and $\mathcal{P}_{\text{fin}}(X)$ of all finite subsets of $X$</td>
</tr>
<tr>
<td>$\langle X \to Y, \subseteq, \emptyset \rangle$</td>
<td>$\langle X \to Y, \subseteq, ??? \rangle$</td>
<td>partial and total function spaces</td>
</tr>
<tr>
<td>$\langle \mathbb{N}^{\infty}, \leq, 0 \rangle$</td>
<td>$\langle \mathbb{N}, \leq, 0 \rangle$</td>
<td>$\mathbb{N}^{\infty} = \mathbb{N} \cup {\omega}$; $n \leq \omega$, for all $n \in \mathbb{N}$</td>
</tr>
<tr>
<td>$\langle (\mathbb{R}^+)^{\infty}, \leq, 0 \rangle$</td>
<td>$\langle (\mathbb{Q}^+)^{\infty}, \leq, 0 \rangle$</td>
<td>non-negative reals $\mathbb{R}^+$ and rationals $\mathbb{Q}^+$ with “infinity”</td>
</tr>
<tr>
<td>$\langle (\mathbb{R}^+)_{\leq a}, \leq, 0 \rangle$</td>
<td>$\langle (\mathbb{Q}^+)_{\leq a}, \leq, 0 \rangle$</td>
<td>their bounded versions</td>
</tr>
<tr>
<td>$\langle A^{\leq \omega}, \sqsubseteq, \varepsilon \rangle$</td>
<td>$\langle A^*, \sqsubseteq, \varepsilon \rangle$</td>
<td>$A^{\leq \omega} = A^* \cup A^\omega$ (finite and infinite strings of elements from $A$, including the empty string $\varepsilon$); $\sqsubseteq$ is the prefix ordering</td>
</tr>
</tbody>
</table>
Continuous functions

Given cpo's \( D = \langle D, \sqsubseteq, \bot \rangle \) and \( D' = \langle D', \sqsubseteq', \bot' \rangle \), a function \( f : D \to D' \) is

- **monotone** if it preserves the ordering, i.e., for all \( d_1, d_2 \in D \),
  \[
  d_1 \sqsubseteq d_2 \text{ implies } f(d_1) \sqsubseteq' f(d_2)
  \]

- **continuous** if it preserves lub's of all countable chains, i.e., for each chain \( d_0 \sqsubseteq d_1 \sqsubseteq \cdots \) in \( D \),
  \[
  f(\bigsqcup_{i \geq 0} d_i) = \bigsqcup_{i \geq 0} f(d_i)
  \]

- **strict** if it preserves the least element, i.e.,
  \[
  f(\bot) = \bot'
  \]

BTW: Continuous functions are monotone; in general they need not be strict.

BTW: Monotone functions in general need not be continuous.
Some intuition?

Topology

Given a cpo $D = \langle D, \sqsubseteq, \bot \rangle$, define a set $X \subseteq D$ to be open if
- if $d_1 \in X$ and $d_1 \sqsubseteq d_2$ then $d_2 \in X$
- if $d_0 \sqsubseteq d_1 \sqsubseteq \cdots$ is such that $\bigsqcup_{i \geq 0} d_i \in X$ then $d_i \in X$ for some $i \geq 0$.

This defines a topology on $D$:
- $\emptyset$ and $D$ are open
- intersection of two open sets is open
- union of any family of open sets is open

Given two cpo's $D = \langle D, \sqsubseteq, \bot \rangle$ and $D' = \langle D', \sqsubseteq', \bot' \rangle$, a function $f : D \to D'$ is continuous if and only if it is continuous in the topological sense, i.e., for $X' \subseteq D'$ open in $D'$, its co-image w.r.t. $f$, $f^{-1}(X') \subseteq D$ is open in $D$. 

Andrzej Tarlecki: Semantics & Verification
Think of a cpo $D = \langle D, \sqsubseteq, \bot \rangle$ as an “information space”.

- if $d_1 \sqsubseteq d_2$ then $d_2$ represents “more information” than $d_1$; $\bot$ is “no information”
- directed sets represent consistent sets of “information pieces”; their lub’s represent “information” that can be derived from the “informations” in the set
- a function is monotone if it yields more information when given more information
- a function is continuous if it deals with information “bit-by-bit” (very informal)

For a set of elements $X$, consider the cpo $\langle \mathcal{P}(X), \supseteq, X \rangle$ of “informations” about the elements in $X$ (a set $I \subseteq X$ represents the property — information — that holds for all the elements in $I$, and only for those elements).
Partial functions

\[ (X \to Y, \subseteq, \emptyset_{X \to Y}) \]

- \( \emptyset_{X \to Y} \) is nowhere defined
- given two partial functions \( f, g : X \to Y \), \( f \subseteq g \) if \( g \) is more defined than \( f \), but when \( f \) is defined, \( g \) yields the same result
- given a directed set of partial functions \( \mathcal{F} \subseteq X \to Y \), no two functions in \( \mathcal{F} \) yield different results for the same argument; then \( \bigsqcup \mathcal{F} = \bigcup \mathcal{F} \), which is a partial function in \( X \to Y \)
- a function \( F : (X \to Y) \to (X' \to Y') \) is continuous, if \( F(f)(x') \) (for \( f : X \to Y \) and \( x' \in X' \)) depends only on a finite number of applications of \( f \) to arguments in \( X \). Typical non-continuous functions:
  testing definedness, checking infinitely many values, ...
Fact: Given a cpo \( \mathbf{D} = \langle D, \sqsubseteq, \bot \rangle \) and a continuous function \( f : D \to D \), there exists the least fixed point \( \text{fix}(f) \in D \) of \( f \), i.e.,

- \( f(\text{fix}(f)) = \text{fix}(f) \)
- if \( f(d) = d \) for some \( d \in D \) then \( \text{fix}(f) \sqsubseteq d \)

Proof:

Define \( f^0(\bot) = \bot \), and \( f^{i+1}(\bot) = f(f^i(\bot)) \) for \( i \geq 0 \). This yields a chain:

\[
\begin{align*}
    f^0(\bot) \sqsubseteq f^1(\bot) \sqsubseteq \cdots \sqsubseteq f^i(\bot) \sqsubseteq f^{i+1}(\bot) \sqsubseteq \cdots
\end{align*}
\]

Put:

\[
\text{fix}(f) = \bigsqcup_{i \geq 0} f^i(\bot)
\]

- \( f(\text{fix}(f)) = f(\bigsqcup_{i \geq 0} f^i(\bot)) = \bot \sqcup \bigsqcup_{i \geq 0} f(f^i(\bot)) = \bigsqcup_{i \geq 0} f^i(\bot) = \text{fix}(f) \)
- Suppose \( f(d) = d \) for some \( d \in D \); then \( f^i(\bot) \sqsubseteq d \) for \( i \geq 0 \). Thus \( \text{fix}(f) = \bigsqcup_{i \geq 0} f^i(\bot) \sqsubseteq d \).
Given a cpo $D = \langle D, \sqsubseteq, \bot \rangle$ and a continuous function $f : D \to D$.

**Fact:** For any $d \in D$, if $f(d) \sqsubseteq d$ then $\text{fix}(f) \sqsubseteq d$.

---

**Fixed point induction**

A property $P \subseteq D$ is **admissible** if it is preserved by lub’s of all countable chains: for any chain $d_0 \sqsubseteq d_1 \sqsubseteq \cdots$, if $d_i \in P$ for all $i \geq 0$ then also $\bigcup_{i \geq 0} d_i \in P$, and $\bot \in P$.

**Fact:** For any admissible $P \subseteq D$ that is closed under $f$ (i.e., if $d \in P$ then $f(d) \in P$)

$\text{fix}(f) \in P$
Recall the (original direct) semantic clause for \textbf{while}:

\[ S[\text{while } b \text{ do } S] = \text{fix}(\Phi) \]

where \( \Phi: \text{STMT} \to \text{STMT} \) is given by \( \Phi(F) = \text{cond}(B[b], S[S]; F, id_{\text{State}}) \).

Is \text{STMT} a cpo?
Is \( \Phi \) continuous?

In this case we can easily check that indeed \( \langle \text{STMT}, \subseteq, \emptyset_{\text{State} \to \text{State}} \rangle \) is a cpo and \( \Phi: \text{STMT} \to \text{STMT} \) is continuous.

\textbf{BUT}: we do not want to have to check this each time we use a fixed point definition!
Domain constructors

Basic domains

For any set $X$, $X_\bot = \langle X_\bot, \sqsubseteq, \bot \rangle$ is a flat cpo, where $X_\bot = X \cup \{ \bot \}$, $\bot$ is a new element, $\bot \sqsubseteq a$ for all $x \in X$ and otherwise $\sqsubseteq$ is trivial.

Fact: Every monotone function defined on a flat cpo is continuous.
For any cpo’s \( D_1 = \langle D_1, \sqsubseteq_1, \bot_1 \rangle \) and \( D_2 = \langle D_2, \sqsubseteq_2, \bot_2 \rangle \):

Product of \( D_1 \) and \( D_2 \) is the following cpo:

\[
D_1 \times D_2 = \langle D_1 \times D_2, \sqsubseteq, \langle \bot_1, \bot_2 \rangle \rangle
\]

where for all \( d_1, d'_1 \in D_1 \) and \( d_2, d'_2 \in D_2 \), \( \langle d_1, d_2 \rangle \sqsubseteq \langle d'_1, d'_2 \rangle \) if \( d_1 \sqsubseteq_1 d'_1 \) and \( d_2 \sqsubseteq_2 d'_2 \).

Disjoint sum of \( D_1 \) and \( D_2 \) is the following cpo:

\[
D_1 + D_2 = \langle (D_1 \times \{1\}) \cup (D_2 \times \{2\}) \cup \{\bot\}, \sqsubseteq, \bot \rangle
\]

where for \( d_1, d'_1 \in D_1 \), \( \langle d_1, 1 \rangle \sqsubseteq \langle d'_1, 1 \rangle \) if \( d_1 \sqsubseteq_1 d'_1 \), for \( d_2, d'_2 \in D_2 \), \( \langle d_2, 2 \rangle \sqsubseteq \langle d'_2, 2 \rangle \) if \( d_2 \sqsubseteq_2 d'_2 \), and for \( d_1 \in D_1 \), \( d_2 \in D_2 \), \( \bot \sqsubseteq \langle d_1, 1 \rangle \) and \( \bot \sqsubseteq \langle d_2, 2 \rangle \).
To avoid proliferation of bottoms:

**Smashed product** of $D_1$ and $D_2$ is the following cpo:

$$D_1 \otimes D_2 = \langle (D_1 \setminus \{\perp_1\}) \times (D_2 \setminus \{\perp_2\}) \cup \{\perp\}, \sqsubseteq, \perp \rangle$$

where for all non-bottom $d_1, d'_1 \in D_1$ and $d_2, d'_2 \in D_2$, $\langle d_1, d_2 \rangle \sqsubseteq \langle d'_1, d'_2 \rangle$ if $d_1 \sqsubseteq_1 d'_1$ and $d_2 \sqsubseteq_2 d'_2$, and $\perp \sqsubseteq \langle d_1, d_2 \rangle$.

**Smashed sum** of $D_1$ and $D_2$ is the following cpo:

$$D_1 \oplus D_2 = \langle ((D_1 \setminus \{\perp_1\}) \times \{1\}) \cup ((D_2 \setminus \{\perp_2\}) \times \{2\}) \cup \{\perp\}, \sqsubseteq, \perp \rangle$$

where for all non-bottom $d_1, d'_1 \in D_1$, $\langle d_1, 1 \rangle \sqsubseteq \langle d'_1, 1 \rangle$ if $d_1 \sqsubseteq_1 d'_1$, for $d_2, d'_2 \in D_2$, $\langle d_2, 2 \rangle \sqsubseteq \langle d'_2, 2 \rangle$ if $d_2 \sqsubseteq_2 d'_2$, and $\perp \sqsubseteq \langle d_1, 1 \rangle$ and $\perp \sqsubseteq \langle d_2, 2 \rangle$. 

---

Andrzej Tarlecki: Semantics & Verification - 139 -
Function spaces

Continuous-function space from $D_1$ to $D_2$ is the following cpo:

$$[D_1 \to D_2] = \langle [D_1 \to D_2], \sqsubseteq, \bot \rangle$$

where

- $[D_1 \to D_2]$ is the set of all continuous functions from $D_1$ to $D_2$
- for functions $f, g: D_1 \to D_2$, $f \sqsubseteq g$ if for each $d_1 \in D_1$, $f(d_1) \sqsubseteq_2 g(d_1)$
- $\bot(d_1) = \bot_2$ for each $d_1 \in D_1$.

$\sqsubseteq$ does not depend on the ordering on $D_1$

For any set $X$, function space from $X$ to $D_2$ is the following cpo:

$$(X \to D_2) = \langle X \to D_2, \sqsubseteq, \bot \rangle$$

where $X \to D_2$ is the set of total functions from $X$ to $D_2$ ordered by $\sqsubseteq$ as above.
Domain isomorphism

Cpo’s $D_1$ and $D_2$ are *isomorphic* if there is a bijection between $D_1$ and $D_2$ which preserves and reflects the ordering.

$$D_1 \cong D_2$$

Examples:

$$\text{Bool}_{\bot} \cong \{\ast\}_{\bot} \oplus \{\ast\}_{\bot}$$

$$\langle X \rightarrow Y, \subseteq, \emptyset_X \rightarrow Y \rangle \cong \langle X \rightarrow Y_{\bot}, \subseteq, \bot \rangle$$

Consider semantic domains up to isomorphism only

So, we can forget (boolean values and) partial functions!

It is more difficult to forget natural numbers.
BTW:

Informally:

- $D \otimes D'$ admits only “strict” (defined) elements in the pairs
- $D \times D'$ admits both “strict” and “undefined” (“unknown”) elements in the pairs
- $D_\bot$ makes all elements in $D$ “strict”

Hence:

$$
(D \times D')_\bot \cong D_\bot \otimes D'_\bot \\
D + D' \cong D_\bot \oplus D'_\bot
$$

Recall also:

$$
D \otimes_L D' \cong D \otimes D'_\bot
$$

Define: $D \oplus_L D'$, $D \otimes_R D'$, $D \oplus_R D'$
Building continuous functions

- Every constant function is continuous
- Partial functions on sets, as used so far, can be replaced by (strict) continuous functions between flat domains; for instance, with a bit of abuse of notation:
  
  - $\text{ifte}_D \in [\text{Bool}_\bot \times D \times D \to D]$ is given by:
    
    $$\text{ifte}_D(c, d, d') = \begin{cases} 
    \text{ifte}_D(c, d, d') & \text{if } c \neq \bot \\
    \bot_D & \text{if } c = \bot
    \end{cases}$$

  - $\_ + \_ \in [\text{Int}_\bot \times \text{Int}_\bot \to \text{Int}_\bot]$ is given by:
    
    $$n + n' = \begin{cases} 
    n + n' & \text{if } n \neq \bot \text{ and } n' \neq \bot \\
    \bot & \text{if } n = \bot \text{ or } n' = \bot
    \end{cases}$$
function composition: \( \_ ; \_ \in [[D_1 \to D_2] \times [D_2 \to D_3] \to [D_1 \to D_3]], \) i.e.:
  - composition of continuous functions is continuous
  - the composition function is continuous

indexing:
\[
\text{lift}^I \in [[[D_1 \times \ldots \times D_n \to D] \to [[I \to D_1] \times \ldots \times [I \to D_n] \to [I \to D]]],
\]
  - indexing a continuous function yields a continuous function
  - the indexing function is continuous

Given a function \( f : D_1 \times \ldots \times D_n \to D, \) \( f \) is a continuous function from the product domain \( D_1 \times \ldots \times D_n \) to \( D \) if and only if it is continuous w.r.t. each argument separately
  - this justifies the use of lambda-notation to build continuous functions:
\[
\Lambda \in [[[D_0 \times D_1 \times \ldots \times D_n \to D] \to [D_1 \times \ldots \times D_n \to [D_0 \to D]]]
\]
• continuous-function application is continuous: \( \_\_ \in [D_1 \to D_2] \times D_1 \to D_2 \)

• projections: \( \pi_1 \in [D_1 \times D_2 \to D_1] \) and \( \pi_2 \in [D_1 \times D_2 \to D_2] \)

• (two-argument pairing, but how to write this sensibly?)

• injections: \( \iota_1 \in [D_1 \to D_1 + D_2] \) and \( \iota_2 \in [D_2 \to D_1 + D_2] \),

• domain checks: \( is\_\_in_1 \in [D_1 + D_2 \to \text{Bool}_\bot] \) and \( is\_\_in_2 \in [D_1 + D_2 \to \text{Bool}_\bot] \)

• function pairing: \( \langle \_\_\_ \rangle : [[D \to D_1] \times [D \to D_2] \to [D \to D_1 \times D_2]] \), where for \( f \in [D \to D_1] \) and \( g \in [D \to D_2] \), \( \langle f, g \rangle = \lambda d : D. \langle f(d), g(d) \rangle \).

• function sum: \( \_\_\_ \_ \_ \_ : [[D_1 \to D] \times [D_2 \to D] \to [D_1 + D_2 \to D]] \), where for \( f \in [D_1 \to D] \) and \( g \in [D_2 \to D] \), \( [f, g](d) = \text{ifte}_D(is\_\_in_1(d), f(d), g(d)) \)
the least fixed point operation \( \text{fix}(\_ \rightarrow \_) \in [[[D ightarrow D] \rightarrow D] \rightarrow D] \)

- for \( D = [D_1 \rightarrow D_2] \), it follows that the least fixed point of a continuous function on continuous functions is a continuous function. . .

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---
Elements of cpo's \( d_1 \in D_1, \ldots, d_n \in D_n \) can be defined by writing (sets of) fixed point equations

\[
\begin{align*}
  d_1 &= \Phi_1(d_1, \ldots, d_n) \\
  \vdots \\
  d_n &= \Phi_n(d_1, \ldots, d_n)
\end{align*}
\]

where \( \Phi_1 \in [D_1 \times \ldots \times D_n \rightarrow D_1], \ldots, \Phi_n \in [D_1 \times \ldots \times D_n \rightarrow D_n] \).

This defines \( \langle d_1, \ldots, d_n \rangle \) as the least fixed point of

\[
\langle \Phi_1, \ldots, \Phi_n \rangle \in [D_1 \times \ldots \times D_n \rightarrow D_1 \times \ldots \times D_n]
\]

The continuous functions used in such definitions may be build using the basic functions and the ways of their composition as discussed so far.
Domain equations

\[
\text{Int} = \{0, 1, -1, 2, -2, \ldots \}
\]
\[
\text{Bool} = \{\text{tt, ff}\}
\]
\[
\text{State} = \text{Var} \rightarrow \text{Int}
\]
\[
\text{EXP} = [\text{State} \rightarrow \text{Int}]
\]
\[
\text{BEXP} = [\text{State} \rightarrow \text{Bool}]
\]
\[
\text{STMT} = [\text{State} \rightarrow \text{State}]
\]

No problem!

Just use the operators to build cpo’s as discussed above.

If definitions of domains turn out to be recursive, use the successive approximation technique, as above for domain elements.
Recursive domain equations

\[ \text{Stream} = A\bot \times \text{Stream} \]

\[
\begin{align*}
\text{Stream}^0 &= \{ \bot \} \\
\text{Stream}^1 &= \{ \bot \sqsubseteq \langle a_1, \bot \rangle \} \\
\text{Stream}^2 &= \{ \bot \sqsubseteq \langle a_1, \bot \rangle \sqsubseteq \langle a_1, \langle a_2, \bot \rangle \} \\
& \quad \quad \ldots \\
\text{Stream}^n &= \{ \bot \sqsubseteq \langle a_1, \bot \rangle \sqsubseteq \langle a_1, \langle a_2, \bot \rangle \} \sqsubseteq \cdots \sqsubseteq \langle a_1, \langle a_2, \langle \ldots, \langle a_n, \bot \rangle \ldots \rangle \rangle \} \\
& \quad \quad \ldots \\
\text{Stream} &= \bigcup_{n \geq 0} \text{Stream}^n \\
&= \{ \bot \sqsubseteq \langle a_1, \bot \rangle \sqsubseteq \langle a_1, \langle a_2, \bot \rangle \} \sqsubseteq \cdots \sqsubseteq \langle a_1, \langle a_2, \langle \ldots, \langle a_n, \bot \rangle \ldots \rangle \rangle \} \\
& \quad \quad \sqsubseteq \cdots \langle a_1, \langle a_2, \langle \ldots, \langle a_n, \langle \ldots \rangle \rangle \ldots \rangle \rangle \} \\
\end{align*}
\]

where all \( a_1, a_2, \ldots, a_n, \ldots \in A \).
Recursive domain equations

\[ \text{Stream} = A_\perp \times \text{Stream} \]

\[
\begin{align*}
\text{Stream}^0 & = \{ \perp \} \\
\text{Stream}^1 & = \{ \perp \sqsubseteq \langle a_1, \perp \rangle \} \\
\text{Stream}^2 & = \{ \perp \sqsubseteq \langle a_1, \perp \rangle \sqsubseteq \langle a_1, a_2, \perp \rangle \} \\
& \quad \ldots \\
\text{Stream}^n & = \{ \perp \sqsubseteq \langle a_1, \perp \rangle \sqsubseteq \langle a_1, a_2, \perp \rangle \sqsubseteq \cdots \sqsubseteq \langle a_1, a_2, \ldots, a_n, \perp \rangle \} \\
& \quad \ldots \\
\text{Stream} & = \bigcup_{n \geq 0} \text{Stream}^n = \\
& \quad \{ \perp \sqsubseteq \langle a_1, \perp \rangle \sqsubseteq \langle a_1, a_2, \perp \rangle \sqsubseteq \cdots \sqsubseteq \langle a_1, a_2, \ldots, a_n, \perp \rangle \sqsubseteq \cdots \langle a_1, a_2, \ldots, a_n, \ldots \rangle \} \\
\text{where all } a_1, a_2, \ldots, a_n, \ldots & \in A.
\end{align*}
\]
If definitions of domains turn out to be recursive, use the successive approximation technique, as above for domain elements.

Suppose we want to add (parameterless) procedures, which are named statements to be stored in states and used in call statements:

\[
\begin{align*}
\text{State} &= \text{Var} \to \text{VAL} \\
\text{VAL} &= \text{Int} + \text{PROC} \\
\text{PROC} &= \left[ \text{State} \to \text{State} \right]
\end{align*}
\]
There is no (non-trivial) set that satisfies

\[ D = D \rightarrow D \]

Yet, any form of self-application (untyped procedure parameters, dynamic binding, etc) requires a semantic domain of this or similar form.

Models for \(\lambda\)-calculus

In particular, this is necessary to model \(\lambda\)-calculus, a formal untyped calculus where every term may be applied to an argument.

History: the semantics for Algol 60

Christopher Strachey, Dana Scott & many others
Good naive solution

Naive denotational semantics

- Use standard set-theoretic domain constructors
- Never use "heavy" recursion, as involved in the reflexive domain definition.
- Use naive set-theoretic approximations and set-theoretic unions to solve domain equations.
- This works for well-typed languages with a hierarchy of concepts and domains.
Solution

Scott-ery

- Limit the size of domains: require countable basis plus some technical conditions
- Use continuous functions only
- Define “domain of all domains” where all such domains can be interpreted
- Define continuous functions on this domain to interpret each of the domain constructors
- Write and solve domain equations as fixed point equations in this domain

Models: $P_\omega$, $T_\omega$, information systems, …