Universal algebra

Basics of universal algebra:

- signatures and algebras
- homomorphisms, subalgebras, congruences
- equations and varieties
- equational calculus
- equational specifications and initial algebras
- variations: partial algebras, first-order structures

Plus some hints on applications in

foundations of software semantics, verification, specification, development...
**Tiny data type**

Its signature $\Sigma$ (syntax):

- **sorts** $\text{Int, Bool}$
- **opns** $0, 1 : \text{Int}$
  - $\text{plus, times, minus} : \text{Int} \times \text{Int} \rightarrow \text{Int}$
  - $\text{false, true} : \text{Bool}$
  - $\text{lteq} : \text{Int} \times \text{Int} \rightarrow \text{Bool}$
  - $\text{not} : \text{Bool} \rightarrow \text{Bool}$
  - $\text{and} : \text{Bool} \times \text{Bool} \rightarrow \text{Bool}$

and $\Sigma$-algebra $\mathcal{A}$ (semantics):

- **carriers** $\mathcal{A}_{\text{Int}} = \text{Int}, \mathcal{A}_{\text{Bool}} = \text{Bool}$
- **operations**
  - $0_{\mathcal{A}} = 0, 1_{\mathcal{A}} = 1$
  - $\text{plus}_{\mathcal{A}}(n, m) = n + m, \text{times}_{\mathcal{A}}(n, m) = n \times m$
  - $\text{minus}_{\mathcal{A}}(n, m) = n - m$
  - $\text{false}_{\mathcal{A}} = \text{ff}, \text{true}_{\mathcal{A}} = \text{tt}$
  - $\text{lteq}_{\mathcal{A}}(n, m) = \text{tt} \text{ if } n \leq m \text{ else } \text{ff}$
  - $\text{not}_{\mathcal{A}}(b) = \text{tt} \text{ if } b = \text{ff} \text{ else } \text{ff}$
  - $\text{and}_{\mathcal{A}}(b, b') = \text{tt} \text{ if } b = b' = \text{tt} \text{ else } \text{ff}$

$\mathcal{A}_{\text{Int}} = \{ \text{min}, \ldots, -1, 0, 1, \ldots, \text{max} \} \ldots$

$\mathcal{A}_{\text{Int}}$ redefines (the semantics of) Tiny
Algebraic signature:

\[ \Sigma = (S, \Omega) \]

- **sort names**: \( S \)
- **operation names, classified by arities and result sorts**: \( \Omega = \langle \Omega_{w,s} \rangle_{w \in S^*, s \in S} \)

Alternatively:

\[ \Sigma = (S, \Omega, \text{arity}, \text{sort}) \]

with **sort names** \( S \), **operation names** \( \Omega \), and **arity and result sort functions**

\[
\text{arity}: \Omega \to S^* \quad \text{and} \quad \text{sort}: \Omega \to S.
\]

- \( f: s_1 \times \ldots \times s_n \to s \) stands for \( s_1, \ldots, s_n, s \in S \) and \( f \in \Omega_{s_1 \ldots s_n, s} \)

Compare the two notions
Fix a signature $\Sigma = (S, \Omega)$ for a while.

**Algebras**

- **$\Sigma$-algebra**:
  
  \[ A = (|A|, \langle f_A \rangle_{f \in \Omega}) \]

- **carrier sets**: $|A| = \langle |A|_s \rangle_{s \in S}$

- **operations**: $f_A : |A|_{s_1} \times \ldots \times |A|_{s_n} \rightarrow |A|_s$, for $f : s_1 \times \ldots \times s_n \rightarrow s$

- **the class of all $\Sigma$-algebras**: $\text{Alg}(\Sigma)$

**Questions**
Can $\text{Alg}(\Sigma)$ be empty? Finite?
Can $A \in \text{Alg}(\Sigma)$ have empty carriers?
Subalgebras

- for $A \in \text{Alg}(\Sigma)$, a $\Sigma$-subalgebra $A_{sub} \subseteq A$ is given by subset $|A_{sub}| \subseteq |A|$ closed under the operations:
  - for $f : s_1 \times \ldots \times s_n \rightarrow s$ and $a_1 \in |A_{sub}| s_1, \ldots, a_n \in |A_{sub}| s_n$,
    $$f_{A_{sub}}(a_1, \ldots, a_n) = f_A(a_1, \ldots, a_n)$$

- for $A \in \text{Alg}(\Sigma)$ and $X \subseteq |A|$, the subalgebra of $A$ generated by $X$, $\langle A \rangle_X$, is the least subalgebra of $A$ that contains $X$.

- $A \in \text{Alg}(\Sigma)$ is reachable if $\langle A \rangle_\emptyset$ coincides with $A$.

Fact: For any $A \in \text{Alg}(\Sigma)$ and $X \subseteq |A|$, $\langle A \rangle_X$ exists.

Proof (idea):
- generate the generated subalgebra from $X$ by closing it under operations in $A$; or
- the intersection of any family of subalgebras of $A$ is a subalgebra of $A$. 
Homomorphisms

- for $A, B \in \text{Alg}(\Sigma)$, a $\Sigma$-homomorphism $h: A \to B$ is a function $h: |A| \to |B|$ that preserves the operations:
  - for $f: s_1 \times \ldots \times s_n \to s$ and $a_1 \in |A|_{s_1}, \ldots, a_n \in |A|_{s_n}$,
    $$h_s(f_A(a_1, \ldots, a_n)) = f_B(h_{s_1}(a_1), \ldots, h_{s_n}(a_n))$$

**Fact:** Given a homomorphism $h: A \to B$ and subalgebras $A_{\text{sub}}$ of $A$ and $B_{\text{sub}}$ of $B$, the image of $A_{\text{sub}}$ under $h$, $h(A_{\text{sub}})$, is a subalgebra of $B$, and the coimage of $B_{\text{sub}}$ under $h$, $h^{-1}(B_{\text{sub}})$, is a subalgebra of $A$.

**Fact:** Given a homomorphism $h: A \to B$ and $X \subseteq |A|$, $h(\langle A \rangle_X) = \langle B \rangle_{h(X)}$.

**Fact:** Identity function on the carrier of $A \in \text{Alg}(\Sigma)$ is a homomorphism $id_A: A \to A$. Composition of homomorphisms $h: A \to B$ and $g: B \to C$ is a homomorphism $h;g: A \to C$. 
Isomorphisms

- for $A, B \in \text{Alg}(\Sigma)$, a $\Sigma$-isomorphism is any $\Sigma$-homomorphism $i : A \to B$ that has an inverse, i.e., a $\Sigma$-homomorphism $i^{-1} : B \to A$ such that $i \circ i^{-1} = id_A$ and $i^{-1} \circ i = id_B$.

- $\Sigma$-algebras are isomorphic if there exists an isomorphism between them.

**Fact:** A $\Sigma$-homomorphism is a $\Sigma$-isomorphism iff it is bijective ("1-1" and "onto").

**Fact:** Identities are isomorphisms, and any composition of isomorphisms is an isomorphism.
for \( A \in \text{Alg}(\Sigma) \), a \( \Sigma \)-\textit{congruence on} \( A \) is an equivalence \( \equiv \subseteq |A| \times |A| \) that is closed under the operations:

- for \( f : s_1 \times \ldots \times s_n \to s \) and \( a_1, a'_1 \in |A|_{s_1}, \ldots, a_n, a'_n \in |A|_{s_n} \),
  if \( a_1 \equiv_{s_1} a'_1, \ldots, a_n \equiv_{s_n} a'_n \) then \( f_A(a_1, \ldots, a_n) \equiv_s f_A(a'_1, \ldots, a'_n) \).

\textbf{Fact:} For any relation \( R \subseteq |A| \times |A| \) on the carrier of a \( \Sigma \)-algebra \( A \), there exists the least congruence on \( A \) that contains \( R \).

\textbf{Fact:} For any \( \Sigma \)-homomorphism \( h : A \to B \), the kernel of \( h \), \( K(h) \subseteq |A| \times |A| \), where \( a \in K(h) a' \) iff \( h(a) = h(a') \), is a \( \Sigma \)-congruence on \( A \).
Quotients

• for $A \in \text{Alg}(\Sigma)$ and $\Sigma$-congruence $\equiv \subseteq |A| \times |A|$ on $A$, the quotient algebra $A/\equiv$ is built in the natural way on the equivalence classes of $\equiv$:
  - for $s \in S$, $|A/\equiv|_s = \{[a]_\equiv \mid a \in |A|_s\}$, with $[a]_\equiv = \{a' \in |A|_s \mid a \equiv a'\}$
  - for $f: s_1 \times \ldots \times s_n \to s$ and $a_1 \in |A|_{s_1}, \ldots, a_n \in |A|_{s_n}$,
    $$f_{A/\equiv}([a_1]_\equiv, \ldots, [a_n]_\equiv) = [f_A(a_1, \ldots, a_n)]_\equiv$$

Fact: The above is well-defined; moreover, the natural map that assigns to every element its equivalence class is a $\Sigma$-homomorphisms $[\_]_\equiv: A \to A/\equiv$.

Fact: Given two $\Sigma$-congruences $\equiv$ and $\equiv'$ on $A$, $\equiv \subseteq \equiv'$ iff there exists a $\Sigma$-homomorphism $h: A/\equiv \to A/\equiv'$ such that $[\_]_\equiv; h = [\_]_{\equiv'}$.

Fact: For any $\Sigma$-homomorphism $h: A \to B$, $A/K(h)$ is isomorphic with $h(A)$. 
• for $A_i \in \text{Alg}(\Sigma), i \in \mathcal{I}$, the product of $\langle A_i \rangle_{i \in \mathcal{I}}$, $\prod_{i \in \mathcal{I}} A_i$ is built in the natural way on the Cartesian product of the carriers of $A_i, i \in \mathcal{I}$:
  
  - for $s \in S$, $\prod_{i \in \mathcal{I}} A_i|_s = \prod_{i \in \mathcal{I}} |A_i|_s$
  
  - for $f : s_1 \times \ldots \times s_n \to s$ and $a_1 \in |\prod_{i \in \mathcal{I}} A_i|_{s_1}, \ldots, a_n \in |\prod_{i \in \mathcal{I}} A_i|_{s_n}$, for $i \in \mathcal{I}$, $f\prod_{i \in \mathcal{I}} A_i(a_1, \ldots, a_n)(i) = f_{A_i}(a_1(i), \ldots, a_n(i))$

**Fact:** For any family $\langle A_i \rangle_{i \in \mathcal{I}}$ of $\Sigma$-algebras, projections $\pi_i(a) = a(i)$, where $i \in \mathcal{I}$ and $a \in \prod_{i \in \mathcal{I}} |A_i|$, are $\Sigma$-homomorphisms $\pi_i : \prod_{i \in \mathcal{I}} A_i \to A_i$.

Define the product of the empty family of $\Sigma$-algebras. When the projection $\pi_i$ is an isomorphism?
Consider an $S$-sorted set $X$ of variables.

- **terms** $t \in |T_\Sigma(X)|$ are built using variables $X$, constants and operations from $\Omega$ in the usual way: $|T_\Sigma(X)|$ is the least set such that
  - $X \subseteq |T_\Sigma(X)|$
  - for $f : s_1 \times \ldots \times s_n \to s$ and $t_1 \in |T_\Sigma(X)|_{s_1}, \ldots, t_n \in |T_\Sigma(X)|_{s_n}$,
    \[ f(t_1, \ldots, t_n) \in |T_\Sigma(X)|_s \]

- for any $\Sigma$-algebra $A$ and valuation $v : X \to |A|$, the value $t_A[v]$ of a term $t \in |T_\Sigma(X)|$ in $A$ under $v$ is determined inductively:
  - $x_A[v] = v_s(x)$, for $x \in X_s$, $s \in S$
  - $(f(t_1, \ldots, t_n))_A[v] = f_A((t_1)_A[v], \ldots, (t_n)_A[v])$, for $f : s_1 \times \ldots \times s_n \to s$ and $t_1 \in |T_\Sigma(X)|_{s_1}, \ldots, t_n \in |T_\Sigma(X)|_{s_n}$

Above and in the following: assuming unambiguous “parsing” of terms!
Consider an $S$-sorted set $X$ of variables.

- **The term algebra** $T_{\Sigma}(X)$ has the set of terms as the carrier and operations defined “syntactically”:

  $$f_{T_{\Sigma}(X)}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n).$$

**Fact:** For any $S$-sorted set $X$ of variables, $\Sigma$-algebra $A$ and valuation $v : X \rightarrow |A|$, there is a unique $\Sigma$-homomorphism $v^\# : T_{\Sigma}(X) \rightarrow A$ that extends $v$. Moreover, for $t \in |T_{\Sigma}(X)|$, $v^\#(t) = t_A[v]$. 

$$X \xrightarrow{id_X \lhd |T_{\Sigma}(X)|} |T_{\Sigma}(X)| \xrightarrow{|v^\#|} A \xrightarrow{\exists! v^\#} \text{Alg}(\Sigma)$$

Set$^S$
**Equations**

- **Equation:**
  \[ \forall X.t = t' \]
  where:
  - \( X \) is a set of variables, and
  - \( t, t' \in |T_\Sigma(X)|_s \) are terms of a common sort.

- **Satisfaction relation:** \( \Sigma \)-algebra \( A \) satisfies \( \forall X.t = t' \)
  \[ A \models \forall X.t = t' \]
  when for all \( v : X \rightarrow |A|, t_A[v] = t'_A[v] \).
Semantic entailment

$\Phi \models_{\Sigma} \varphi$

**Σ-equation** $\varphi$ is a semantic consequence of a set of **Σ-equations** $\Phi$ if $\varphi$ holds in every **Σ-algebra** that satisfies $\Phi$.

**BTW:**

- **Models** of a set of equations: $\text{Mod}(\Phi) = \{A \in \text{Alg}(\Sigma) \mid A \models \Phi\}$
- **Theory** of a class of algebras: $\text{Th}(C) = \{\varphi \mid C \models \varphi\}$
- $\Phi \models \varphi \iff \varphi \in \text{Th}(\text{Mod}(\Phi))$
- **Mod** and **Th** form a *Galois connection*
Equational calculus

\[
\begin{align*}
\forall X.t &= t' \quad \forall X.t' = t'' \\
\forall X.t &= t'' \\
\forall X.t_1 = t_1' \quad \ldots \quad \forall X.t_n = t_n' \\
\forall X.f(t_1 \ldots t_n) &= f(t_1' \ldots t_n') \\
\forall Y.t[\theta] &= t'[\theta]
\end{align*}
\]

for \( \theta : X \to |T_\Sigma(Y)| \)

Mind the variables!

\( a = b \) does not follow from \( a = f(x) \) and \( f(x) = b \), unless...
Proof-theoretic entailment

\[ \Phi \vdash_{\Sigma} \varphi \]

\( \Sigma \)-equation \( \varphi \) is a proof-theoretic consequence of a set of \( \Sigma \)-equations \( \Phi \) if \( \varphi \) can be derived from \( \Phi \) by the rules.

How to justify this?

Semantics!
Soundness & completeness

Fact: The equational calculus is sound and complete:

\[ \Phi \models \varphi \iff \Phi \vdash \varphi \]

- soundness: “all that can be proved, is true” (\( \Phi \models \varphi \iff \Phi \vdash \varphi \))
- completeness: “all that is true, can be proved” (\( \Phi \models \varphi \implies \Phi \vdash \varphi \))

Proof (idea):
- soundness: easy!
- completeness: not so easy!
One motivation

Software systems (data types, modules, programs, databases...):
sets of data with operations on them

• Disregarding: code, efficiency, robustness, reliability, ...

• Focusing on: CORRECTNESS

Universal algebra from rough analogy:
module interface \rightarrow \text{signature}
module \rightarrow \text{algebra}
module specification \rightarrow \text{class of algebras}
Equational specifications

\[ \langle \Sigma, \Phi \rangle \]

- signature \( \Sigma \), to determine the static module interface
- axioms (\( \Sigma \)-equations), to determine required module properties

BUT:

**Fact:** A class of \( \Sigma \)-algebras is equationally definable iff it is closed under subalgebras, products and homomorphic images.

Equational specifications typically admit a lot of undesirable “modules”
Example

\[
\text{spec } \text{NAIVE\textsc{Nat}} = \text{sort } \text{Nat} \\
\text{opns } 0: \text{Nat}; \\
\quad \text{succ: Nat } \rightarrow \text{Nat}; \\
\quad _ + _: \text{Nat } \times \text{Nat } \rightarrow \text{Nat} \\
\text{axioms } \forall n:\text{Nat}. n + 0 = n; \\
\quad \forall n, m:\text{Nat}. n + \text{succ}(m) = \text{succ}(n + m)
\]

Now:

\[
\text{NAIVE\textsc{Nat}} \not\models \forall n, m:\text{Nat}. n + m = m + n
\]

Perhaps worse:

There are models \( M \in \text{Mod(NAIVE\textsc{Nat})} \) such that \( M \models 0 = \text{succ}(0) \), or even:

\[
M \models \forall n, m:\text{Nat}. n = m
\]
How to fix this

- **Constraints**: 

  *initiality: “no junk” & “no confusion”*

  Also: *reachability ("no junk"), and their more general versions (freeness, generation).*

  **BTW**: Constraints can be thought of as special (higher-order) formulae.

- **Other (stronger) logical systems**: conditional equations, first-order logic, higher-order logics, other bells-and-whistles

  — more about this elsewhere... **Institutions!**

  *There has been a population explosion among logical systems.*
Initial models

Fact: Every equational specification $\langle \Sigma, \Phi \rangle$ has an initial model: there exists a $\Sigma$-algebra $I \in \text{Mod}(\Phi)$ such that for every $\Sigma$-algebra $M \in \text{Mod}(\Phi)$ there exists a unique $\Sigma$-homomorphism from $I$ to $M$.

Proof (idea):

- $I$ is the quotient of the algebra of ground $\Sigma$-terms by the congruence that glues together all ground terms $t, t'$ such that $\Phi \models \forall \emptyset. t = t'$.
- $I$ is the reachable subalgebra of the product of “all” (up to isomorphism) reachable algebras in $\text{Mod}(\Phi)$.

BTW: This can be generalised to the existence of a free model of $\langle \Sigma, \Phi \rangle$ over any (many-sorted) set of data.

BTW: Existence of initial (and free) models carries over to specifications with conditional equations, but not much further!
Example

```
spec NAT = initial {  sort Nat

    opns 0: Nat;

    succ: Nat \to Nat;

    _ + _ : Nat \times Nat \to Nat

    axioms \forall n:Nat. n + 0 = n;

    \forall n, m:Nat. n + succ(m) = succ(n + m)

}
```

Now:

\[ \text{NAT} \models \forall n, m: \text{Nat}. n + m = m + n \]
Try another example

\[ \textbf{spec} \ \text{NatPred} = \ \textbf{sort} \ Nat \]

\[ \text{opns} \ 0 : Nat; error : Nat; \]

\[ \text{succ} : Nat \rightarrow Nat; \]

\[ _+_- : Nat \times Nat \rightarrow Nat; \]

\[ \text{pred} : Nat \rightarrow Nat \]

\textbf{axioms} \ \forall n: Nat. n + 0 = n;

\[ \forall n, m: Nat. n + \text{succ}(m) = \text{succ}(n + m); \]

\[ \forall n: Nat. \text{pred}(\text{succ}(n)) = n; \]

\[ \text{pred}(0) = \text{error}; \]

\[ \text{pred}(\text{error}) = \text{error}; \text{succ}(\text{error}) = \text{error}; \]

\[ \forall n: Nat. \text{error} + n = \text{error}; \forall n: Nat. n + \text{error} = \text{error} \]

Looks okay. But try to add multiplication:

\[ 0 \ast n = 0; \text{succ}(m) \ast n = n + (m \ast n); \]

\[ \text{error} \ast n = \text{error}; n \ast \text{error} = \text{error} \]
Partial algebras

- **Algebraic signature** $\Sigma$: as before

- **Partial $\Sigma$-algebra**:
  
  \[ A = (|A|, \langle f_A \rangle_{f \in \Omega}) \]

  as before, but operations $f_A : |A|_{s_1} \times \ldots \times |A|_{s_n} \to |A|_s$, for $f : s_1 \times \ldots \times s_n \to s$, may now be **partial functions**.

  **BTW**: Constants may be undefined as well.

- **$\text{PAlg}(\Sigma)$** stands for the class of all partial $\Sigma$-algebras.
Fix a signature $\Sigma = (S, \Omega)$ for a while.

**Few further notions**

- **subalgebra** $A_{\text{sub}} \subseteq A$: given by subset $|A_{\text{sub}}| \subseteq |A|$ closed under the operations; (BTW: at least two other natural notions are possible)

- **homomorphism** $h: A \rightarrow B$: map $h: |A| \rightarrow |B|$ that preserves definedness and results of operations; it is *strong* if in addition it reflects definedness of operations; (strong) homomorphisms are closed under composition; (BTW: very interesting alternative: *partial* map $h: |A| \rightarrowtail |B|$ that preserves results of operations)

- **congruence** $\equiv$ on $A$: equivalence $\equiv \subseteq |A| \times |A|$ closed under the operations whenever they are defined; it is *strong* if in addition it reflects definedness of operations; (strong) congruences are kernels of (strong) homomorphisms;

- **quotient algebra** $A/\equiv$: built in the natural way on the equivalence classes of $\equiv$; the natural homomorphism from $A$ to $A/\equiv$ is strong if the congruence is strong.
**Formulae**

*(Strong) equation:*  
\[ \forall X. t \overset{s}{=} t' \]

as before  

*Definedness formula:*  
\[ \forall X. \text{def } t \]

where \( X \) is a set of variables, and \( t \in |T_\Sigma(X)|_s \) is a term

*Satisfaction relation*

partial \( \Sigma \)-algebra \( A \) **satisfies** \( \forall X. t \overset{s}{=} t' \)

\[ A \models \forall X. t \overset{s}{=} t' \]

when for all \( v: X \to |A| \), \( t_A[v] \) is defined iff \( t'_A[v] \) is defined, and then \( t_A[v] = t'_A[v] \)

partial \( \Sigma \)-algebra \( A \) **satisfies** \( \forall X. \text{def } t \)

\[ A \models \forall X. \text{def } t \]

when for all \( v: X \to |A| \), \( t_A[v] \) is defined
An alternative

- **(Existence) equation**: 
  \[
  \forall X. t \equiv^e t'
  \]
  where:
  - \( X \) is a set of variables, and
  - \( t, t' \in |T\Sigma(X)|_s \) are terms of a common sort.

- **Satisfaction relation**: \( \Sigma \)-algebra \( A \) satisfies \( \forall X. t \equiv^e t' \)
  
  \[
  A \models \forall X. t \equiv^e t'
  \]
  when for all \( v: X \to |A|, t_A[v] = t'_A[v] \) — both sides are defined and equal.

BTW:

- \( \forall X. t \equiv^e t' \) iff \( \forall X. (t \equiv^s t' \land \text{def } t) \)
- \( \forall X. t \equiv^s t' \) iff \( \forall X. (\text{def } t \iff \text{def } t') \land (\text{def } t \implies t \equiv^e t') \)
Example

\[
\text{spec } \text{NatPred} = \text{initial } \{ \text{ sort } \text{Nat} \\
\hspace{1cm} \text{opns } 0: \text{Nat}; \\
\hspace{2cm} \text{succ : Nat → Nat; } \\
\hspace{2cm} \_ + \_ : \text{Nat} \times \text{Nat} \rightarrow \text{Nat}; \\
\hspace{2cm} \text{pred : Nat →? Nat} \\
\hspace{1cm} \text{axioms } \forall n: \text{Nat}. n + 0 = n; \\
\hspace{2cm} \forall n, m: \text{Nat}. n + \text{succ}(m) = \text{succ}(n + m); \\
\hspace{2cm} \forall n: \text{Nat}. \text{pred}(\text{succ}(n)) = n \\
\}\n\]
First-order structures

- **First-order signature** \( \Sigma = (S, \Omega, \Pi) \): algebraic signature \((S, \Omega)\) plus *predicate names*, classified by arities: \( \Pi = \langle \Pi_w \rangle_{w \in S^*} \)

- **First-order \( \Sigma \)-structure**:

\[
A = (|A|, \langle f_A \rangle_{f \in \Omega}, \langle p_A \rangle_{p \in \Pi})
\]

consists of:

- \((S, \Omega)\)-algebra \((|A|, \langle f_A \rangle_{f \in \Omega})\)

- **predicates** (relations): \( p_A \subseteq |A|_{s_1} \times \ldots \times |A|_{s_n} \),
  for \( p: s_1 \times \ldots \times s_n \) (i.e., \( p \in \Pi_{s_1 \ldots s_n} \))

- **\( \text{Str}(\Sigma) \)** stands for the class of all first-order \( \Sigma \)-structures.
Fix a signature $\Sigma = (S, \Omega, \Pi)$ for a while.

**Few further notions**

- **substructure** $A_{sub} \subseteq A$: given by subset $|A_{sub}| \subseteq |A|$ closed under the operations and such that the inclusion preserves truth of predicates; the substructure is *closed* if the inclusion also preserves falsity of predicates;

- **homomorphism** $h: A \rightarrow B$: map $h: |A| \rightarrow |B|$ that preserves the results of operations and truth of predicates; it is *closed* if in addition it preserves falsity of predicates; (closed) homomorphisms are closed under composition;

- **congruence** $\equiv$ on $A$: equivalence $\equiv \subseteq |A| \times |A|$ closed under the operations; it is *closed* if in addition it preserves truth (and falsity) of predicates; (closed) congruences are kernels of (closed) homomorphisms;

- **quotient structures** $A/\equiv$: built in the natural way on the equivalence classes of $\equiv$ so that the natural map from $A$ to $A/\equiv$ is a homomorphism; it is closed if the congruence is closed.
• *atomic* \( \Sigma \)-*formulae* over set \( X \) of variables:
  
  - \( t = t' \), where \( t, t' \in |T_{(S,\Omega)}(X)|_s, s \in S \)
  
  - \( p(t_1, \ldots t_n) \), where \( p: s_1 \times \ldots \times s_n, t_1 \in |T_{(S,\Omega)}(X)|_{s_1}, \ldots t_n \in |T_{(S,\Omega)}(X)|_{s_n} \)

• \( \Sigma \)-*formulae* contain atomic formulae and are closed under logical connectives and quantification; \( \Sigma \)-*sentences* are \( \Sigma \)-formulae with no free variables

• *Satisfaction relation* defined as usual between \( \Sigma \)-*structures* \( A \) and \( \Sigma \)-*sentences* \( \varphi \)

\[
A \models \varphi
\]

As before, this yields the usual notions of the *class of models* for a set of sentences, the *semantic consequences* of a set of sentences, the *theory* of a class of models, etc.

*Initial* (and free) models exist for first-order specifications with universally quantified conditional atomic formulae, *but in general may fail to exist!*