Category Theory in Foundations of Computer Science
Exam assignment 2023/24

Concepts, terminology and notation:

We rely on the standard definitions of *algebraic signature* Σ, Σ-*algebra* and Σ-*homomorphism*, the category $\mathbf{Alg}(\Sigma)$ of Σ-algebras and their homomorphisms, and on the related notation, as introduced during the course.

A *bin-signature* $\Delta = \langle \Sigma, \delta \rangle$ consists of an algebraic signature $\Sigma = \langle S, \Omega \rangle$ and a family of functions $\delta = \langle \delta_f \rangle_{f \in \Omega}$, where for each $f : s_1 \times \ldots \times s_n \to s$ in Σ, $\delta_f : \{0,1\}^n \to \{0,1\}$ (the same n). A bin-signature $\Delta = \langle \Sigma, \delta \rangle$ is *monotone* if for each $f : s_1 \times \ldots \times s_n \to s$, $\delta_f : \{0,1\}^n \to \{0,1\}$ is monotone (w.r.t. the standard order on $\{0,1\}$, where $0 \leq 1$, and induced component-wise order on $\{0,1\}^n$).

Let, $\Delta = \langle \Sigma, \delta \rangle$, with $\Sigma = \langle S, \ldots \rangle$, be a bin-signature.

A Δ-*bin-algebra* $\mathcal{A} = \langle A, \alpha \rangle$ consists of a Σ-algebra $A \in |\mathbf{Alg}(\Sigma)|$ and a family of functions $\alpha = \langle \alpha_s : |A|_s \to \{0,1\} \rangle_{s \in S}$ (called the *bin-map* of $\mathcal{A}$) such that for all $f : s_1 \times \ldots \times s_n \to s$ in Σ and $a_1 \in |A|_{s_1}, \ldots, a_n \in |A|_{s_n}$, $\delta_f(\alpha_{s_1}(a_1), \ldots, \alpha_{s_n}(a_n)) \leq \alpha_s(f_A(a_1, \ldots, a_n))$. Such a Δ-bin-algebra $\mathcal{A} = \langle A, \alpha \rangle$ is *strict* if for each $f : s_1 \times \ldots \times s_n \to s$ in Σ and $a_1 \in |A|_{s_1}, \ldots, a_n \in |A|_{s_n}$, $\delta_f(\alpha_{s_1}(a_1), \ldots, \alpha_{s_n}(a_n)) = \alpha_s(f_A(a_1, \ldots, a_n))$.

Then, given Δ-bin-algebras $\mathcal{A} = \langle A, \alpha \rangle$ and $\mathcal{B} = \langle B, \beta \rangle$, a Δ-*bin-homomorphism* $h : \mathcal{A} \to \mathcal{B}$ is any Σ-homomorphism $h : A \to B$ such that for each $a \in |A|_s$, $s \in S$, $\alpha_s(a) \leq \beta_s(h_s(a))$. Such a Δ-*bin-homomorphism* $h : \mathcal{A} \to \mathcal{B}$ is *strict* if for each $a \in |A|_s$, $s \in S$, $\alpha_s(a) = \beta_s(h_s(a))$.

A Δ-*inequality* $\forall X.t \leq t'$ consists of an S-sorted set X (of variables) and two terms $t, t' \in |T_\Sigma(X)|_s$ of a common sort, $s \in S$. A Δ-bin-algebra $\mathcal{A} = \langle A, \alpha \rangle$ *satisfies* (or is a *model* of) such a Δ-inequality, written $\mathcal{A} \models \forall X.t \leq t'$, if for all valuations $v : X \to |A|$, $\alpha_s(t_A[v]) \leq \alpha_s(t'_A[v])$, where as usual $q_A[v] \in |A|_s$ is the value of term $q \in |T_\Sigma(X)|_s$, $s \in S$, in Σ-algebra A under valuation v.

With the usual composition of homomorphisms, this defines the following categories, for any bin-signature Δ and set Φ of Δ-inequalities:

- $\mathbf{BAlg}(\Delta, \Phi)$: the category of Δ-bin-algebras that satisfy all Δ-inequalities in Φ, with Δ-bin-homomorphisms as morphisms
- $\mathbf{BAlg}^{st}(\Delta, \Phi)$: the category of strict Δ-bin-algebras that satisfy all Δ-inequalities in Φ, with strict Δ-bin-homomorphisms as morphisms

Moreover, we have the following "forgetful" functors:

- $\mathbf{G}_{\Delta,\Phi} : \mathbf{BAlg}(\Delta, \Phi) \to \mathbf{Set}^S$
- $\mathbf{G}^{st}_{\Delta,\Phi} : \mathbf{BAlg}^{st}(\Delta, \Phi) \to \mathbf{Set}^S$

where $\mathbf{Set}^S$ is the category of S-sorted sets, as usual, and for any Δ-bin-algebra $\mathcal{A} = \langle A, \alpha \rangle$, $\mathbf{G}_{\Delta,\Phi}(\mathcal{A}) = |A|$, for any Δ-bin-homomorphism $h : \mathcal{A} \to \mathcal{B}$, where $\mathcal{A} = \langle A, \alpha \rangle$ and $\mathcal{B} = \langle B, \beta \rangle$, $\mathbf{G}_{\Delta,\Phi}(h) = h : |A| \to |B|$, and $\mathbf{G}^{st}_{\Delta,\Phi}$ is the restriction of $\mathbf{G}_{\Delta,\Phi}$ to the objects and morphisms in $\mathbf{BAlg}^{st}(\Delta, \Phi)$.

Finally, we put:

- $\mathbf{BAlg}(\Delta) = \mathbf{BAlg}(\Delta, \emptyset)$
- $\mathbf{G}_\Delta = \mathbf{G}_{\Delta,\emptyset} : \mathbf{BAlg}(\Delta) \to \mathbf{Set}^S$
- $\mathbf{BAlg}^{st}(\Delta) = \mathbf{BAlg}^{st}(\Delta, \emptyset)$
- $\mathbf{G}^{st}_\Delta = \mathbf{G}^{st}_{\Delta,\emptyset} : \mathbf{BAlg}^{st}(\Delta) \to \mathbf{Set}^S$

To do:

Prove a positive answer or give a counterexample to the following questions:

1. Consider categories:
   (a) $\mathbf{BAlg}(\Delta, \Phi)$
   (b) $\mathbf{BAlg}^{st}(\Delta, \Phi)$
   (c) $\mathbf{BAlg}(\Delta)$
   (d) $\mathbf{BAlg}^{st}(\Delta)$
   Which of the categories above is
   C. complete
   CC. cocomplete
   for all bin-signatures Δ and, where applicable, all sets Φ of Δ-inequalities?
2. Consider functors:
   (a) $\mathbf{G}_{\Delta,\Phi}\colon \mathbf{BAlg}(\Delta, \Phi) \to \mathbf{Set}^S$
   (b) $\mathbf{G}^{st}_{\Delta,\Phi}\colon \mathbf{BAlg}^{st}(\Delta, \Phi) \to \mathbf{Set}^S$
   (c) $\mathbf{G}_{\Delta}\colon \mathbf{BAlg}(\Delta) \to \mathbf{Set}^S$
   (d) $\mathbf{G}^{st}_{\Delta}\colon \mathbf{BAlg}^{st}(\Delta) \to \mathbf{Set}^S$
   Which of the functors above has a left adjoint for all bin-signatures Δ and, where applicable, all sets Φ of Δ-inequalities?
3. Again, consider categories:
   (a) $\mathbf{BAlg}(\Delta, \Phi)$
   (b) $\mathbf{BAlg}^{st}(\Delta, \Phi)$
   (c) $\mathbf{BAlg}(\Delta)$
   (d) $\mathbf{BAlg}^{st}(\Delta)$
   Which of the categories above is
   C. complete
   CC. cocomplete
   for all monotone bin-signatures Δ and, where applicable, all sets Φ of Δ-inequalities?
4. Consider functors:
   (a) $\mathbf{G}_{\Delta,\Phi}\colon \mathbf{BAlg}(\Delta, \Phi) \to \mathbf{Set}^S$
   (b) $\mathbf{G}^{st}_{\Delta,\Phi}\colon \mathbf{BAlg}^{st}(\Delta, \Phi) \to \mathbf{Set}^S$
   (c) $\mathbf{G}_{\Delta}\colon \mathbf{BAlg}(\Delta) \to \mathbf{Set}^S$
   (d) $\mathbf{G}^{st}_{\Delta}\colon \mathbf{BAlg}^{st}(\Delta) \to \mathbf{Set}^S$
   Which of the functors above has a left adjoint for all monotone bin-signatures Δ and, where applicable, all sets Φ of Δ-inequalities?

Notes:

- All constructions and facts presented during the course may be used without proofs. This applies in particular to the existence and constructions of limits and colimits in $\mathbf{Alg}(\Sigma)$.
- The answers to the questions above are not independent. For instance, a proof of **2.a** implies the positive answer to **2.c** as well, a counterexample to **1.d.CC** is a counterexample to **1.b.CC**, a proof for any of **1.{a,b,c,d}.{C,CC}** proves the corresponding **3.{a,b,c,d}.{C,CC}**, and a counterexample for any of **3.{a,b,c,d}.{C,CC}** is a counterexample for the corresponding **1.{a,b,c,d}.{C,CC}**, etc. No need to repeat detailed arguments in such cases, indicating the dependency is enough.
- Still, there are quite a few questions: deal with as many of them as you can…

Sketch of a solution:

The "strict" case:

Consider a bin-signature $\Delta = \langle \Sigma, \delta \rangle$, with $\Sigma = \langle S, \ldots \rangle$.

Let $\mathcal{BN} = \langle BN, id_{\{0,1\}} \rangle$ be a Δ-bin-algebra, with $|BN|_s = \{0,1\}$ for $s \in S$, and $f_{BN} = \delta_f \colon \{0,1\}^n \to \{0,1\}$ for $f \colon s_1 \times \ldots \times s_n \to s$ in Σ.

Then $\mathbf{BAlg}^{st}(\Delta)$ is the same as the slice category $\mathbf{Alg}(\Sigma){\downarrow}BN$ (the category of $\mathbf{Alg}(\Sigma)$-objects over BN). The slice category is complete (a limit of a diagram D in $\mathbf{BAlg}^{st}(\Delta)$ is the limit in $\mathbf{Alg}(\Sigma)$ of the obvious projection of the diagram D with an additional new node carrying BN and new edges from the nodes of D to this node carrying the bin-maps) and cocomplete (a colimit of a diagram D in $\mathbf{BAlg}^{st}(\Delta)$ is the colimit in $\mathbf{Alg}(\Sigma)$ of the projection of D with the bin-map induced by the colimit property). This directly gives:

YES: {1,3}.d.{C,CC}

Moreover, since the terminal object in $\mathbf{BAlg}^{st}(\Delta)$ (i.e., in $\mathbf{Alg}(\Sigma){\downarrow}BN$) is $\mathcal{BN}$, which shows that $\mathbf{G}_\Delta \colon \mathbf{BAlg}^{st}(\Delta) \to \mathbf{Set}^S$ is not continuous, we have:

NO: {2,4}.{b,d}

Consider a bin-signature $\Delta_1 = \langle \Sigma_1, \delta_1 \rangle$, where Σ_1 has a single sort s and two constants $a, b \colon s$ and $(\delta_1)_a = 1$, $(\delta_1)_b = 0$. Now, the inequality $a \leq b$ has no strict Δ_1-model, which shows:

NO: {1,3}.b.{C,CC} (and **{2,4}.b**)

The "lax" case:

Consider a bin-signature $\Delta = \langle \Sigma, \delta \rangle$, with $\Sigma = \langle S, \ldots \rangle$, and a set Φ of Δ-inequalities.

Completeness (monotone Δ): Let $\mathcal{A} = \langle A, \alpha \rangle$ and $\mathcal{B} = \langle B, \beta \rangle$ be Δ-bin-algebras that satisfy Φ, and let $h, h' \colon \mathcal{A} \to \mathcal{B}$ be bin-homomorphisms. Let then $e \colon E \to A$ be an equaliser of $h, h' \colon A \to B$ in $\mathbf{Alg}(\Sigma)$, and $\varepsilon = e;\alpha$. Given the construction of equalisers in $\mathbf{Alg}(\Sigma)$, it follows now that $e \colon \langle E, \varepsilon \rangle \to \mathcal{A}$ is an equaliser of $h, h' \colon \mathcal{A} \to \mathcal{B}$ in $\mathbf{BAlg}(\Delta, \Phi)$.
Let $\mathcal{A}_i = \langle A_i, \alpha_i \rangle$, $i \in \mathcal{J}$, be a family of Δ-bin-algebras that satisfy Φ. Let A with projections $\pi_i \colon A \to A_i$, $i \in \mathcal{J}$, be a product of $\langle A_i \rangle_{i \in \mathcal{J}}$ in $\mathbf{Alg}(\Sigma)$. For $s \in S$, define $\alpha_s \colon |A|_s \to \{0,1\}$ as follows: given $a \in |A|$, $\alpha_s(a) = 1$ iff for all $i \in \mathcal{J}$, $(\alpha_i)_s(\pi_i(a)) = 1$ (and so $\alpha_s(a) = 0$ iff for some $i \in \mathcal{J}$, $(\alpha_i)_s(\pi_i(a)) = 0$). This implies that $\alpha_s(a) \leq (\alpha_i)_s((\pi_i)_s(a))$. Then for $f \colon s_1 \times \ldots \times s_n \to s$ and $a_1 \in |A|_{s_1}, \ldots, a_n \in |A|_{s_n}$, we show $\delta_f(\alpha_{s_1}(a_1), \ldots, \alpha_{s_n}(a_n)) \leq \alpha_s(f_A(a_1, \ldots, a_n))$, i.e., if $\alpha_s(f_A(a_1, \ldots, a_n)) = 0$ then $\delta_f(\alpha_{s_1}(a_1), \ldots, \alpha_{s_n}(a_n)) = 0$ as well. Namely, $\alpha_s(f_A(a_1, \ldots, a_n)) = 0$ implies $(\alpha_i)_s(f_{A_i}((\pi_i)_{s_1}(a_1), \ldots, (\pi_i)_{s_s}(a_n))) = 0$ for some $i \in \mathcal{J}$. Now, *since Δ is monotone*, we get: $\delta_f(\alpha_{s_1}(a_1), \ldots, \alpha_{s_n}(a_n)) \leq \delta_f((\alpha_i)_{s_1}((\pi_i)_{s_1}(a_1)), \ldots, (\alpha_i)_{s_1}((\pi_i)_{s_s}(a_n))) = 0$. Consequently, $\mathcal{A} = \langle A, \alpha = \langle \alpha_s \rangle_{s \in S} \rangle$ is a Δ-bin-algebra. It is easy to check now that $\mathcal{A}$ is a model of Φ, and in fact is a product of $\mathcal{A}_i = \langle A_i, \alpha_i \rangle$, $i \in \mathcal{J}$, with projections $\pi_i \colon \mathcal{A} \to \mathcal{A}_i$, $i \in \mathcal{J}$, in $\mathbf{BAlg}(\Sigma, \Phi)$.
The above proves:

YES: 3.{a,c}.C

Counterexample (non-monotone Δ): Consider $\Delta_2 = \langle \Sigma_2, \delta_2 \rangle$ where Σ_2 has a single sort s, constant $a \colon s$ and operation $f \colon s \to s$, with $(\delta_2)_a = 0$ and $(\delta_2)_f(0) = 1$, $(\delta_2)_f(1) = 0$. Consider now two Δ_2-bin-algebras, $\mathcal{A} = \langle T_{\Sigma_2}, \alpha \rangle$ and $\mathcal{B} = \langle T_{\Sigma_2}, \beta \rangle$, where T_{Σ_2} is the usual algebra of ground Σ_2-terms of the form $f^n(a)$, $n \geq 0$, and:

$$\alpha_s(f^n(a)) = \begin{cases} 0 & \text{for even } n \\ 1 & \text{for odd } n \end{cases} \qquad \beta_s(f^n(a)) = \begin{cases} 1 & \text{for even } n \\ 0 & \text{for odd } n \end{cases}$$

Suppose now there is a Δ_2-bin-algebra $\mathcal{C} = \langle C, \gamma \rangle$ with Δ_2-bin-homomorphisms $h_A: \mathcal{C} \to \mathcal{A}$ and $h_B: \mathcal{C} \to \mathcal{B}$. Since $(h_A)_s(a_C) = a$, $\gamma_s(a_C) \leq \alpha_s(a) = 0$. Then $\gamma_s(f_C(a_c)) \geq (\delta_2)_f(0) = 1$. But $(h_B)_s(f_C(a_c)) = f(a)$, with $\beta_s(f(a)) = 0$, and so h_B is not a bin-homomorphism. This contradiction shows that there is no product of $\mathcal{A}$ and $\mathcal{B}$ in $\mathbf{BAlg}(\Delta)$, and that there is no initial Δ_2-bin-algebra, which proves

NO: 1.{a,c}.{C,CC}

Moreover, since left adjoints preserve initial objects, there is no free Δ_2-bin-algebra w.r.t. $\mathbf{G}_{\Delta_2}$ over the empty set, and so:

NO: 2.{a,c}

Construction of the minimal bin-map: Consider a Σ-algebra $A \in \mathbf{Alg}(\Sigma)$. Given a family of Δ-bin-algebras $\mathcal{A}_i = \langle A_i, \alpha_i \rangle$ with Σ-homomorphisms $h_i: A_i \to A$, $i \in \mathcal{J}$, there is the least (w.r.t. the order on bin-maps induced by the standard order on $\{0,1\}$) bin-map $\alpha = \langle \alpha_s: |A|_s \to \{0,1\} \rangle_{s \in S}$ such that

- $\mathcal{A} = \langle A, \alpha \rangle$ is a Δ-bin-algebra
- $\mathcal{A} = \langle A, \alpha \rangle \models \Phi$
- all $h_i: \mathcal{A}_i \to \mathcal{A}$, $i \in \mathcal{J}$, are Δ-bin-homomorphisms

More explicitly, for all $s \in S$, $a \in |A|_s$, define $\alpha_s(a) = \bigsqcup\{\alpha_s^k(a) \mid k \geq 0\}$ (the least upper bound w.r.t. the standard order on $\{0,1\}$ of $\alpha_s^k(a)$, $k \geq 0$), where $\alpha^k = \langle \alpha_s^k: |A| \to \{0,1\} \rangle_{s \in S}$, are defined inductively:

- for $s \in S$, $a \in |A|_s$, $\alpha_s^0(a) = \bigsqcup\{(\alpha_i)_s(a_i) \mid i \in \mathcal{J}, (h_i)_s(a_i) = a\}$.
- for $k \geq 0$, for $s \in S$, $a \in |A|_s$, $\alpha_s^{k+1}(a)$ is the least upper bound of the following elements:
  - $\alpha_s^k(a)$
  - $\delta_f(\alpha_{s_1}^k(a_1), \ldots, \alpha_{s_n}^k(a_n))$ for all $f: s_1 \times \ldots \times s_n \to s$ in Σ and $a_1 \in |A|_{s_1}, \ldots, a_n \in |A|_{s_n}$ such that $f_A(a_1, \ldots, a_n) = a$
  - $\alpha_s^k(t_A[v])$ for all inequalities $\forall X.t \leq t'$ in Φ and valuations $v: X \to |A|$ such that $t'_A[v] = a$.

As usual, the least upper bound of the empty set is 0. The required properties of the so defined bin-map α are now easy to check, since for $s \in S$, $a \in |A|_s$, for some $m \geq 0$ we have $\alpha_s(a) = \alpha_s^k(a)$ for all $k \geq m$.

Moreover, *if Δ is monotone*, we get:

- given any $\mathcal{B} = \langle B, \beta \rangle \in |\mathbf{BAlg}(\Delta, \Phi)|$ and Σ-homomorphism $h: A \to B$, if all $h_i;h: \mathcal{A}_i \to \mathcal{B}$, $i \in \mathcal{J}$, are Δ-bin-homomorphisms then so is $h: \mathcal{A} \to \mathcal{B}$.

To see this, it is enough to notice that for all $s \in S$, $a \in |A|_s$, $\alpha_s^k(a) \leq \beta(h_s(a))$ for all $k \geq 0$ — easy proof by induction follows:

- $\alpha_s^0(a) = \bigsqcup\{(\alpha_i)_s(a_i) \mid i \in \mathcal{J}, (h_i)_s(a_i) = a\} \leq \beta_s(h_s(a))$, since for $i \in \mathcal{J}, a_i \in |A_i|_s$, $(\alpha_i)_s(a_i) \leq \beta_s(h_s((h_i)_s(a_i)))$.
- for $k \geq 0$, if for all $s \in S$, $a \in |A|_s$:
  - for $f: s_1 \times \ldots \times s_n \to s$ in Σ and $a_1 \in |A|_{s_1}, \ldots, a_n \in |A|_{s_n}$ with $f_A(a_1, \ldots, a_n) = a$, by the inductive hypothesis $\alpha_{s_1}^k(a_1) \leq \beta_{s_1}(h_{s_1}(a_1))$, …, $\alpha_{s_n}^k(a_n) \leq \beta_{s_n}(h_{s_n}(a_n))$. Then, since Δ is monotone:

$$\begin{aligned} \delta_f(\alpha_{s_1}^k(a_1), \ldots, \alpha_{s_n}^k(a_n)) &\leq \delta_f(\beta_{s_1}(h_{s_1}(a_1)), \ldots, \beta_{s_n}(h_{s_n}(a_n)) \\ &\leq \beta_s(f_B(h_{s_1}(a_1), \ldots, h_{s_n}(a_n))) \\ &= \beta_s(h_s(a)). \end{aligned}$$

– for all inequalities $\forall X.t \leq t'$ in Φ and valuations $v\colon X \to |A|$ such that $t'_A[v] = a$, by the inductive hypothesis and since $\mathcal{B} \models \Phi$: $\alpha^k_s(t_A[v]) \leq \beta_s(h_s(t_A[v])) = \beta_s(t_A[v;h])) \leq \beta_s(t'_A[v;h])) = \beta_s(h_s(t_A[v]) = \beta_s(h_s(a))$.

Hence, $\alpha^{k+1}_s(a) \leq \beta_s(h_s(a))$.

Cocompleteness (monotone Δ): Consider now any diagram $\mathcal{D}$ in $\mathbf{BAlg}(\Delta, \Phi)$ with nodes $n \in N$ and edges $e \in E$, i.e., for each node $n \in N$ we have a Δ-bin-algebra satisfying Φ, $\mathcal{D}_n = \langle A_n, \alpha_n \rangle \in |\mathbf{BAlg}(\Delta, \Phi)|$, and for each edge $e\colon n \to m$ in E we have Δ-bin-homomorphism $\mathcal{D}_e\colon \mathcal{D}_n \to \mathcal{D}_m$. Let now D be the projection of $\mathcal{D}$ to $\mathbf{Alg}(\Sigma)$, i.e., D is the diagram of the same shape as $\mathcal{D}$ and for all nodes $n \in N$, $D_n = A_n \in \mathbf{Alg}(\Sigma)$, and for all edges $e\colon n \to m$ in E, $D_n = \mathcal{D}_n\colon A_n \to A_m$. Let A with injections $\iota_n\colon A_n \to A$ be a colimit of D in $\mathbf{Alg}(\Sigma)$. Given the construction above, we can now equip A with the least bin-map $\alpha = \langle \alpha_s\colon |A|_s \to \{0,1\} \rangle_{s \in S}$ such that

- $\mathcal{A} = \langle A, \alpha \rangle$ is a Δ-bin-algebra
- $\mathcal{A} = \langle A, \alpha \rangle \models \Phi$
- all $\iota_i\colon \mathcal{A}_i \to \mathcal{A}$, $i \in \mathcal{J}$, are Δ-bin-homomorphisms

and *since* Δ *is monotone*

- given any $\mathcal{B} = \langle B, \beta \rangle \in |\mathbf{BAlg}(\Delta, \Phi)|$ and Σ-homomorphism $h\colon A \to B$, if all $\iota_i;h\colon \mathcal{A}_i \to \mathcal{B}$, $i \in \mathcal{J}$, are Δ-bin-homomorphisms then so is $h\colon \mathcal{A} \to \mathcal{B}$.

It is easy to check now that $\mathcal{A} = \langle A, \alpha \rangle$ with injections $\iota_n\colon \mathcal{A}_n \to \mathcal{A}$ is a colimit of $\mathcal{D}$ in $\mathbf{BAlg}(\Delta, \Phi)$. This proves:

YES: {3}.{a,c}.CC

Left adjoints (monotone Δ): Given an S-sorted set X, equip the usual Σ-algebra of terms, $T_\Sigma(X)$, with the least bin-map $\alpha = \langle \alpha_s\colon |T_\Sigma(X)| \to \{0,1\} \rangle_{s \in S}$ induced by the empty family (of Δ-bin-algebras with Σ-homomorphisms) and the set of Δ-inequalities Φ. *Since* Δ *is monotone*, it follows now that $\langle T_\Sigma(X), \alpha \rangle$ with the usual injection $\eta_X\colon X \to |T_\Sigma(X)|$ is free over X w.r.t. $\mathbf{G}_{\Delta,\Phi}\colon \mathbf{BAlg}(\Delta, \Phi) \to \mathbf{Set}^S$, which proves:

YES: 4.{a,c}

Summing up:

	$\mathbf{BAlg}(\Delta, \Phi)$ **_.a._**	$\mathbf{BAlg}^{st}(\Delta, \Phi)$ **_.b._**	$\mathbf{BAlg}(\Delta)$ **_.c._**	$\mathbf{BAlg}^{st}(\Delta)$ **_.d._**
1._.C	**NO**	**NO**	**NO**	**YES**
1._.CC	**NO**	**NO**	**NO**	**YES**
monotone: **3._.C**	**YES**	**NO**	**YES**	**YES**
monotone: **3._.CC**	**YES**	**NO**	**YES**	**YES**
left adjoint to $\mathbf{G}^{(\_)}_{(\_)}$: **2._**	**NO**	**NO**	**NO**	**NO**
monotone, left adjoint to $\mathbf{G}^{(\_)}_{(\_)}$: **4._**	**YES**	**NO**	**YES**	**NO**