Category theory for computer science

- generality
- abstraction
- convenience
- constructiveness

Overall idea

look at all objects exclusively through relationships between them

capture relationships between objects as appropriate morphisms between them
(Cartesian) product

- **Cartesian product** of two sets $A$ and $B$, is the set
  
  $$A \times B = \{ \langle a, b \rangle \mid a \in A, b \in B \}$$
  
  with projections $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ given by $\pi_1(\langle a, b \rangle) = a$ and $\pi_2(\langle a, b \rangle) = b$.

- A **product** of two sets $A$ and $B$, is any set $P$ with projections $\pi_1 : P \to A$ and $\pi_2 : P \to B$ such that for any set $C$ with functions $f_1 : C \to A$ and $f_2 : C \to B$ there exists a unique function $h : C \to P$ such that $h;\pi_1 = f_1$ and $h;\pi_2 = f_2$.

**Fact:** Cartesian product (of sets $A$ and $B$) is a product (of $A$ and $B$).

Recall the definition of (Cartesian) product of $\Sigma$-algebras.
Define product of $\Sigma$-algebras as above. **What have you changed?**
Given a function \( f : A \to B \), the following conditions are equivalent:

- \( f \) is a **surjection**: \( \forall b \in B \cdot \exists a \in A \cdot f(a) = b \).
- \( f \) is an **epimorphism**: for all \( h_1, h_2 : B \to C \), if \( f; h_1 = f; h_2 \) then \( h_1 = h_2 \).
- \( f \) is a **retraction**: there exists \( g : B \to A \) such that \( g; f = id_B \).

**BUT**: Given a \( \Sigma \)-homomorphism \( f : A \to B \) for \( A, B \in \text{Alg}(\Sigma) \):

\[
\text{\( f \) is retraction} \implies \text{\( f \) is surjection} \iff \text{\( f \) is epimorphism}
\]

**BUT**: Given a (weak) \( \Sigma \)-homomorphism \( f : A \to B \) for \( A, B \in \text{PAlg}(\Sigma) \):

\[
\text{\( f \) is retraction} \implies \text{\( f \) is surjection} \implies \text{\( f \) is epimorphism}
\]
Definition: Category $K$ consists of:

- a collection of objects: $|K|$
- mutually disjoint collections of morphisms: $K(A, B)$, for all $A, B \in |K|$; $m : A \to B$ stands for $m \in K(A, B)$
- morphism composition: for $m : A \to B$ and $m' : B \to C$, we have $m; m' : A \to C$;
  - the composition is associative: for $m_1 : A_0 \to A_1$, $m_2 : A_1 \to A_2$ and $m_3 : A_2 \to A_3$, $(m_1;m_2);m_3 = m_1;(m_2;m_3)$
  - the composition has identities: for $A \in |K|$, there is $id_A : A \to A$ such that for all $m_1 : A_1 \to A$, $m_1;id_A = m_1$, and $m_2 : A \to A_2$, $id_A;m_2 = m_2$.

BTW: “collection” means “set”, “class”, etc, as appropriate.

$K$ is locally small if for all $A, B \in |K|$, $K(A, B)$ is a set. $K$ is small if in addition $|K|$ is a set.
Presenting finite categories

0:

1:

2:

3:

4:

\ldots

(identities omitted)
**Generic examples**

**Discrete categories:** A category $\mathbf{K}$ is *discrete* if all $\mathbf{K}(A, B)$ are empty, for distinct $A, B \in \mathbf{|K|}$, and $\mathbf{K}(A, A) = \{id_A\}$ for all $A \in \mathbf{|K|}$.

**Preorders:** A category $\mathbf{K}$ is *thin* if for all $A, B \in \mathbf{|K|}$, $\mathbf{K}(A, B)$ contains at most one element.

Every preorder $\leq \subseteq X \times X$ determines a thin category $\mathbf{K}_\leq$ with $\mathbf{|K}_\leq| = X$ and for $x, y \in \mathbf{|K}_\leq|$, $\mathbf{K}_\leq(x, y)$ is nonempty iff $x \leq y$.

Every (small) category $\mathbf{K}$ determines a preorder $\leq_K \subseteq \mathbf{|K|} \times \mathbf{|K|}$, where for $A, B \in \mathbf{|K|}$, $A \leq_K B$ iff $\mathbf{K}(A, B)$ is nonempty.

**Monoids:** A category $\mathbf{K}$ is a *monoid* if $\mathbf{|K|}$ is a singleton.

Every monoid $\mathcal{X} = \langle X, \cdot, id \rangle$, where $\cdot : X \times X \to X$ and $id \in X$, determines a (monoid) category $\mathbf{K}_\mathcal{X}$ with $\mathbf{|K}_\leq| = \{\ast\}$, $\mathbf{K}(\ast, \ast) = X$ and the composition given by the monoid operation.
Examples

- Sets (as objects) and functions between them (as morphisms) with the usual composition form the category $\textbf{Set}$.

  Functions have to be considered with their sources and targets

- For any set $S$, $S$-sorted sets (as objects) and $S$-functions between them (as morphisms) with the usual composition form the category $\textbf{Set}^S$.

- For any signature $\Sigma$, $\Sigma$-algebras (as objects) and their homomorphisms (as morphisms) form the category $\textbf{Alg}(\Sigma)$.

- For any signature $\Sigma$, partial $\Sigma$-algebras (as objects) and their weak homomorphisms (as morphisms) form the category $\textbf{PAlg}(\Sigma)$.

- For any signature $\Sigma$, partial $\Sigma$-algebras (as objects) and their strong homomorphisms (as morphisms) form the category $\textbf{PAlg}_s(\Sigma)$.

- Algebraic signatures (as objects) and their morphisms (as morphisms) with the composition defined in the obvious way form the category $\textbf{AlgSig}$.
Substitutions

For any signature $\Sigma = (S, \Omega)$, the category of $\Sigma$-substitutions $\text{Subst}_\Sigma$ is defined as follows:

- objects of $\text{Subst}_\Sigma$ are $S$-sorted sets (of variables);
- morphisms in $\text{Subst}_\Sigma(X, Y)$ are substitutions $\theta : X \to |T_\Sigma(Y)|$,
- composition is defined in the obvious way:
  for $\theta_1 : X \to Y$ and $\theta_2 : Y \to Z$, that is functions $\theta_1 : X \to |T_\Sigma(Y)|$ and $\theta_2 : Y \to |T_\Sigma(Z)|$, their composition $\theta_1;\theta_2 : X \to Z$ in $\text{Subst}_\Sigma$ is the function $\theta_1;\theta_2 : X \to |T_\Sigma(Z)|$ such that for each $x \in X$, $(\theta_1;\theta_2)(x) = \theta_2^\#(\theta_1(x))$. 
Given a category \( K \), a subcategory of \( K \) is any category \( K' \) such that

- \( |K'| \subseteq |K| \),
- \( K'(A, B) \subseteq K(A, B) \), for all \( A, B \in |K'| \),
- composition in \( K' \) coincides with the composition in \( K \) on morphisms in \( K' \), and
- identities in \( K' \) coincide with identities in \( K \) on objects in \( |K'| \).

A subcategory \( K' \) of \( K \) is full if \( K'(A, B) = K(A, B) \) for all \( A, B \in |K'| \).

Any collection \( X \subseteq |K| \) gives the full subcategory \( K|_X \) of \( K \) by \( |K|_X = X \).

- The category \( \text{FinSet} \) of finite sets is a full subcategory of \( \text{Set} \).
- The discrete category of sets is a subcategory of sets with inclusions as morphisms, which is a subcategory of sets with injective functions as morphisms, which is a subcategory of \( \text{Set} \).
- The category of single-sorted signatures is a full subcategory of \( \text{AlgSig} \).
Reversing arrows

Given a category $\mathbf{K}$, its **opposite category** $\mathbf{K}^{op}$ is defined as follows:

- **objects:** $|\mathbf{K}^{op}| = |\mathbf{K}|$

- **morphisms:** $\mathbf{K}^{op}(A, B) = \mathbf{K}(B, A)$ for all $A, B \in |\mathbf{K}^{op}| = |\mathbf{K}|$

- **composition:** given $m_1 : A \to B$ and $m_2 : B \to C$ in $\mathbf{K}^{op}$, that is, $m_1 : B \to A$ and $m_2 : C \to B$ in $\mathbf{K}$, their composition in $\mathbf{K}^{op}$, $m_1;m_2 : A \to C$, is set to be their composition $m_2;m_1 : C \to A$ in $\mathbf{K}$.

**Fact:** The identities in $\mathbf{K}^{op}$ coincide with the identities in $\mathbf{K}$.

**Fact:** Every category is opposite to some category:

$$(\mathbf{K}^{op})^{op} = \mathbf{K}$$
Duality principle

If $W$ is a categorical concept (notion, property, statement, . . .) then its dual, $co-W$, is obtained by reversing all the morphisms in $W$.

Example:

$P(X)$: “for any object $Y$ there exists a morphism $f: X \to Y$”

$co-P(X)$: “for any object $Y$ there exists a morphism $f: Y \to X$”

NOTE: $co-P(X)$ in $K$ coincides with $P(X)$ in $K^{op}$.

Fact: If a property $W$ holds for all categories then $co-W$ holds for all categories as well.
Given categories $\mathbf{K}$ and $\mathbf{K}'$, their \textit{product} $\mathbf{K} \times \mathbf{K}'$ is the category defined as follows:

- **objects:** $|\mathbf{K} \times \mathbf{K}'| = |\mathbf{K}| \times |\mathbf{K}'|$

- **morphisms:** $(\mathbf{K} \times \mathbf{K}')((\langle A, A'\rangle, \langle B, B'\rangle)) = \mathbf{K}(A, B) \times \mathbf{K}'(A', B')$ for all $A, B \in |\mathbf{K}|$ and $A', B' \in |\mathbf{K}'|$

- **composition:** for $\langle m_1, m'_1\rangle: \langle A, A'\rangle \to \langle B, B'\rangle$ and $\langle m_2, m'_2\rangle: \langle B, B'\rangle \to \langle C, C'\rangle$ in $\mathbf{K} \times \mathbf{K}'$, their composition in $\mathbf{K} \times \mathbf{K}'$ is

\[
\langle m_1, m'_1\rangle;\langle m_2, m'_2\rangle = \langle m_1;m_2, m'_1;m'_2\rangle
\]

Define $\mathbf{K}^n$, where $\mathbf{K}$ is a category and $n \geq 1$. Extend this definition to $n = 0$. 

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Given a category $\mathbf{K}$, its \textit{morphism category} $\mathbf{K} \rightarrow$ is the category defined as follows:

- objects: $|\mathbf{K} \rightarrow|$ is the collection of all morphisms in $\mathbf{K}$

- morphisms: for $f: A \to A'$ and $g: B \to B'$ in $\mathbf{K}$, $\mathbf{K} \rightarrow (f, g)$ consists of all $\langle k, k' \rangle$, where $k: A \to B$ and $k': A' \to B'$ are such that $k; g = f; k'$ in $\mathbf{K}$

- composition: for $\langle k, k' \rangle: (f: A \to A') \to (g: B \to B')$ and $\langle j, j' \rangle: (g: B \to B') \to (h: C \to C')$ in $\mathbf{K} \rightarrow$, their composition in $\mathbf{K} \rightarrow$ is $\langle k, k' \rangle; \langle j, j' \rangle = \langle k; j, k'; j' \rangle$.

Check that the composition is well-defined.
Slice categories

Given a category $K$ and an object $A \in |K|$, the category of $K$-objects over $A$, $K\downarrow A$, is the category defined as follows:

- objects: $K\downarrow A$ is the collection of all morphisms into $A$ in $K$
- morphisms: for $f : B \to A$ and $g : B' \to A$ in $K$, $(K\downarrow A)(f, g)$ consists of all morphisms $k : B \to B'$ such that $k \circ g = f$ in $K$
- composition: the composition in $K\downarrow A$ is the same as in $K$

Check that the composition is well-defined.

View $K\downarrow A$ as a subcategory of $K\to$.

Define $K\uparrow A$, the category of $K$-objects under $A$. 
Fix a category $\mathbf{K}$ for a while.

**Simple categorical definitions**

- $f : A \rightarrow B$ is an **epimorphism** (is *epi*):
  
  for all $g, h : B \rightarrow C$, $f;g = f;h$ implies $g = h$

  ![Diagram of epimorphism]

  *In $\mathbf{Set}$, a function is epi iff it is surjective*

- $f : A \rightarrow B$ is a **monomorphism** (is *mono*):
  
  for all $g, h : C \rightarrow A$, $g;f = h;f$ implies $g = h$

  ![Diagram of monomorphism]

  *In $\mathbf{Set}$, a function is mono iff it is injective*
Simple facts

- If \( f : A \to B \) and \( g : B \to C \) are mono then \( f;g : A \to C \) is mono as well.
- If \( f;g : A \to C \) is mono then \( f : A \to B \) is mono as well.

Prove, and then dualise the above facts.

**NOTE**: A morphism \( f \) is mono in \( \mathbf{K} \) iff \( f \) is epi in \( \mathbf{K}^{op} \).

**mono = co-epi**

Give “natural” examples of categories where epis need not be “surjective”.
Give “natural” examples of categories where monos need not be “injective”.

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**Isomorphisms**

\[ f : A \to B \] is an *isomorphism* (is *iso*)

if there is \( g : B \to A \) such that \( f \circ g = \text{id}_A \) and \( g \circ f = \text{id}_B \).

Then \( g \) is the (unique) *inverse of* \( f \), \( g = f^{-1} \).

In \( \textbf{Set} \), a function is iso iff it is both epi and mono.

**Fact:** If \( f \) is iso then it is both epi and mono. Give counterexamples to show that the opposite implication fails.

**Fact:** \( f : A \to B \) is iso iff

- \( f \) is a *retraction*, i.e., there is \( g_1 : B \to A \) such that \( g_1 \circ f = \text{id}_B \), and
- \( f \) is a *coretraction*, i.e., there is \( g_2 : B \to A \) such that \( f \circ g_2 = \text{id}_A \).

**Fact:** A morphism is iso iff it is an epi coretraction.

**Fact:** Composition of isomorphisms is an isomorphism.