Category theory

for computer science

• *generality* • *abstraction* • *convenience* • *constructiveness* •

Overall idea

look at all objects exclusively through relationships between them

capture relationships between objects as appropriate morphisms between them

(Cartesian) product

- *Cartesian product* of two sets A and B, is the set $A \times B = \{\langle a, b\rangle \mid a \in A, b \in B\}$ with projections $\pi_1\colon A \times B \to A$ and $\pi_2\colon A \times B \to B$ given by $\pi_1(\langle a, b\rangle) = a$ and $\pi_2(\langle a, b\rangle) = b$.
- A *product* of two sets A and B, is any set P with projections $\pi_1\colon P \to A$ and $\pi_2\colon P \to B$ such that for any set C with functions $f_1\colon C \to A$ and $f_2\colon C \to B$ there exists a unique function $h\colon C \to P$ such that $h;\pi_1 = f_1$ and $h;\pi_2 = f_2$.

$$
\begin{tikzcd}
A & P \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& C \arrow[ul, "f_1"] \arrow[u, "\exists!\ h"] \arrow[ur, "f_2"'] &
\end{tikzcd}
$$

Fact: *Cartesian product (of sets A and B) is a product (of A and B).*

Recall the definition of (Cartesian) product of Σ-algebras.
Define product of Σ-algebras as above. *What have you changed?*

Pitfalls of generalization

the same concrete definition $\rightsquigarrow$ *distinct abstract generalizations*

Given a function $f\colon A \to B$, the following conditions are equivalent:

- f is a *surjection*: $\forall b \in B \cdot \exists a \in A \cdot f(a) = b$.
- f is an *epimorphism*: for all $h_1, h_2\colon B \to C$, if $f;h_1 = f;h_2$ then $h_1 = h_2$.
- f is a *retraction*: there exists $g\colon B \to A$ such that $g;f = id_B$.

BUT: Given a Σ-homomorphism $f\colon A \to B$ for $A, B \in \mathbf{Alg}(\Sigma)$:

f *is retraction* $\Longrightarrow$ f *is surjection* $\Longleftrightarrow$ f *is epimorphism*

BUT: Given a (weak) Σ-homomorphism $f\colon A \to B$ for $A, B \in \mathbf{PAlg}(\Sigma)$:

f *is retraction* $\Longrightarrow$ f *is surjection* $\Longrightarrow$ f *is epimorphism*

Categories

Definition: *Category* $\mathbf{K}$ *consists of:*

- *a collection of objects:* $|\mathbf{K}|$
- *mutually disjoint collections of morphisms:* $\mathbf{K}(A, B)$, *for all* $A, B \in |\mathbf{K}|$;
  $m \colon A \to B$ *stands for* $m \in \mathbf{K}(A, B)$
- *morphism composition: for* $m \colon A \to B$ *and* $m' \colon B \to C$, *we have* $m;m' \colon A \to C$;
  - *the composition is associative: for* $m_1 \colon A_0 \to A_1$, $m_2 \colon A_1 \to A_2$ *and* $m_3 \colon A_2 \to A_3$, $(m_1;m_2);m_3 = m_1;(m_2;m_3)$
  - *the composition has identities: for* $A \in |\mathbf{K}|$, *there is* $id_A \colon A \to A$ *such that for all* $m_1 \colon A_1 \to A$, $m_1;id_A = m_1$, *and* $m_2 \colon A \to A_2$, $id_A;m_2 = m_2$.

BTW: *"collection"* means *"set"*, *"class"*, etc, as appropriate.

$\mathbf{K}$ is *locally small* if for all $A, B \in |\mathbf{K}|$, $\mathbf{K}(A, B)$ is a set.
$\mathbf{K}$ is *small* if in addition $|K|$ is a set.

Presenting finite categories

0:

1:
$$\bullet$$

2:
$$\begin{tikzcd} \bullet \arrow[r] & \bullet \end{tikzcd}$$

3:
$$\begin{tikzcd} \bullet \arrow[r] \arrow[rr, bend left] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

4:
$$\begin{tikzcd} \bullet \arrow[r] \arrow[rr, bend left] \arrow[rrr, bend left] & \bullet \arrow[r] \arrow[rr, bend right] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

...

(identities omitted)

Generic examples

Discrete categories: A category $\mathbf{K}$ is *discrete* if all $\mathbf{K}(A, B)$ are empty, for distinct $A, B \in |\mathbf{K}|$, and $\mathbf{K}(A, A) = \{id_A\}$ for all $A \in |\mathbf{K}|$.

Preorders: A category $\mathbf{K}$ is *thin* if for all $A, B \in |\mathbf{K}|$, $\mathbf{K}(A, B)$ contains at most one element.

Every preorder $\leq \subseteq X \times X$ determines a thin category $\mathbf{K}_\leq$ with $|\mathbf{K}_\leq| = X$ and for $x, y \in |\mathbf{K}_\leq|$, $\mathbf{K}_\leq(x, y)$ is nonempty iff $x \leq y$.

Every (small) category $\mathbf{K}$ determines a preorder $\leq_\mathbf{K} \subseteq |\mathbf{K}| \times |\mathbf{K}|$, where for $A, B \in |\mathbf{K}|$, $A \leq_\mathbf{K} B$ iff $\mathbf{K}(A, B)$ is nonempty.

Monoids: A category $\mathbf{K}$ is a *monoid* if $|\mathbf{K}|$ is a singleton.

Every monoid $\mathcal{X} = \langle X, ;, id\rangle$, where $\_;\_ : X \times X \to X$ and $id \in X$, determines a (monoid) category $\mathbf{K}_\mathcal{X}$ with $|\mathbf{K}_\leq| = \{*\}$, $\mathbf{K}(*, *) = X$ and the composition given by the monoid operation.

Examples

- Sets (as objects) and functions between them (as morphisms) with the usual composition form the category $\mathbf{Set}$.

  Functions have to be considered with their sources and targets

- For any set S, S-sorted sets (as objects) and S-functions between them (as morphisms) with the usual composition form the category $\mathbf{Set}^S$.
- For any signature Σ, Σ-algebras (as objects) and their homomorphisms (as morphisms) form the category $\mathbf{Alg}(\Sigma)$.
- For any signature Σ, partial Σ-algebras (as objects) and their weak homomorphisms (as morphisms) form the category $\mathbf{PAlg}(\Sigma)$.
- For any signature Σ, partial Σ-algebras (as objects) and their strong homomorphisms (as morphisms) form the category $\mathbf{PAlg_s}(\Sigma)$.
- Algebraic signatures (as objects) and their morphisms (as morphisms) with the composition defined in the obvious way form the category $\mathbf{AlgSig}$.

Substitutions

For any signature $\Sigma = (S, \Omega)$, the category of Σ-substitutions $\mathbf{Subst}_\Sigma$ is defined as follows:

- objects of $\mathbf{Subst}_\Sigma$ are S-sorted sets (of variables);
- morphisms in $\mathbf{Subst}_\Sigma(X, Y)$ are substitutions $\theta\colon X \to |T_\Sigma(Y)|$,
- composition is defined in the obvious way:
  for $\theta_1\colon X \to Y$ and $\theta_2\colon Y \to Z$, that is functions $\theta_1\colon X \to |T_\Sigma(Y)|$ and $\theta_2\colon Y \to |T_\Sigma(Z)|$, their composition $\theta_1;\theta_2\colon X \to Z$ in $\mathbf{Subst}_\Sigma$ is the function $\theta_1;\theta_2\colon X \to |T_\Sigma(Z)|$ such that for each $x \in X$, $(\theta_1;\theta_2)(x) = \theta_2^\#(\theta_1(x))$.

Subcategories

Given a category $\mathbf{K}$, a *subcategory* of $\mathbf{K}$ is any category $\mathbf{K}'$ such that

- $|\mathbf{K}'| \subseteq |\mathbf{K}|$,
- $\mathbf{K}'(A, B) \subseteq \mathbf{K}(A, B)$, for all $A, B \in |\mathbf{K}'|$,
- composition in $\mathbf{K}'$ coincides with the composition in K on morphisms in $\mathbf{K}'$, and
- identities in $\mathbf{K}'$ coincide with identities in $\mathbf{K}$ on objects in $|\mathbf{K}'|$.

A subcategory $\mathbf{K}'$ of $\mathbf{K}$ is *full* if $\mathbf{K}'(A, B) = \mathbf{K}(A, B)$ for all $A, B \in |\mathbf{K}'|$.

Any collection $X \subseteq |\mathbf{K}|$ gives the full subcategory $\mathbf{K}|_X$ of $\mathbf{K}$ by $|\mathbf{K}|_X| = X$.

- The category $\mathbf{FinSet}$ of finite sets is a full subcategory of $\mathbf{Set}$.
- The discrete category of sets is a subcategory of sets with inclusions as morphisms, which is a subcategory of sets with injective functions as morphisms, which is a subcategory of $\mathbf{Set}$.
- The category of single-sorted signatures is a full subcategory of $\mathbf{AlgSig}$.

Reversing arrows

Given a category $\mathbf{K}$, its *opposite category* $\mathbf{K}^{op}$ is defined as follows:

- objects: $|\mathbf{K}^{op}| = |\mathbf{K}|$
- morphisms: $\mathbf{K}^{op}(A, B) = \mathbf{K}(B, A)$ for all $A, B \in |\mathbf{K}^{op}| = |\mathbf{K}|$
- composition: given $m_1 : A \to B$ and $m_2 : B \to C$ in $\mathbf{K}^{op}$, that is, $m_1 : B \to A$ and $m_2 : C \to B$ in $\mathbf{K}$, their composition in $\mathbf{K}^{op}$, $m_1;m_2 : A \to C$, is set to be their composition $m_2;m_1 : C \to A$ in $\mathbf{K}$.

Fact: *The identities in $\mathbf{K}^{op}$ coincide with the identities in $\mathbf{K}$.*

Fact: *Every category is opposite to some category:*

$$\boxed{(\mathbf{K}^{op})^{op} = \mathbf{K}}$$

Duality principle

If W is a categorical concept (notion, property, statement, ...)
then its *dual*, co-W, is obtained by reversing all the morphisms in W.

Example:

$P(X)$**:** "for any object Y there exists a morphism $f: X \to Y$"

co**-**$P(X)$**:** "for any object Y there exists a morphism $f: Y \to X$"

NOTE: co-$P(X)$ in $\mathbf{K}$ coincides with $P(X)$ in $\mathbf{K}^{op}$.

Fact: *If a property W holds for all categories then co-W holds for all categories as well.*

Product categories

Given categories $\mathbf{K}$ and $\mathbf{K}'$, their *product* $\mathbf{K} \times \mathbf{K}'$ is the category defined as follows:

- objects: $|\mathbf{K} \times \mathbf{K}'| = |\mathbf{K}| \times |\mathbf{K}'|$
- morphisms: $(\mathbf{K} \times \mathbf{K}')(\langle A, A' \rangle, \langle B, B' \rangle) = \mathbf{K}(A, B) \times \mathbf{K}'(A', B')$ for all $A, B \in |\mathbf{K}|$ and $A', B' \in |\mathbf{K}'|$
- composition: for $\langle m_1, m_1' \rangle \colon \langle A, A' \rangle \to \langle B, B' \rangle$ and $\langle m_2, m_2' \rangle \colon \langle B, B' \rangle \to \langle C, C' \rangle$ in $\mathbf{K} \times \mathbf{K}'$, their composition in $\mathbf{K} \times \mathbf{K}'$ is

$$\boxed{\langle m_1, m_1' \rangle ; \langle m_2, m_2' \rangle = \langle m_1 ; m_2, m_1' ; m_2' \rangle}$$

$$\begin{tikzcd}
A \arrow[r, "m_1"] \arrow[rr, bend left, "m_1;m_2"] & B \arrow[r, "m_2"] & C \\
A' \arrow[r, "m_1'"] \arrow[rr, bend right, "m_1';m_2'"'] & B' \arrow[r, "m_2'"] & C'
\end{tikzcd}$$

Define $\mathbf{K}^n$, where $\mathbf{K}$ is a category and $n \geq 1$. Extend this definition to $n = 0$.

Morphism categories

Given a category $\mathbf{K}$, its *morphism category* $\mathbf{K}^{\rightarrow}$ is the category defined as follows:

- objects: $|\mathbf{K}^{\rightarrow}|$ is the collection of all morphisms in $\mathbf{K}$
- morphisms: for $f\colon A \to A'$ and $g\colon B \to B'$ in $\mathbf{K}$, $\mathbf{K}^{\rightarrow}(f,g)$ consists of all $\langle k, k'\rangle$, where $k\colon A \to B$ and $k'\colon A' \to B'$ are such that $k;g = f;k'$ in $\mathbf{K}$
- composition: for $\langle k, k'\rangle\colon (f\colon A \to A') \to (g\colon B \to B')$ and $\langle j, j'\rangle\colon (g\colon B \to B') \to (h\colon C \to C')$ in $\mathbf{K}^{\rightarrow}$, their composition in $\mathbf{K}^{\rightarrow}$ is $\langle k, k'\rangle;\langle j, j'\rangle = \langle k;j, k';j'\rangle$.

$$
\begin{tikzcd}
A \arrow[r, "k"] \arrow[d, "f"'] \arrow[rr, bend left, "k;j"] & B \arrow[r, "j"] \arrow[d, "g"'] & C \arrow[d, "h"] \\
A' \arrow[r, "k'"] \arrow[rr, bend right, "k';j'"'] & B' \arrow[r, "j'"] & C'
\end{tikzcd}
$$

Check that the composition is well-defined.

Slice categories

Given a category $\mathbf{K}$ and an object $A \in |K|$, the category of $\mathbf{K}$-*objects over* A, $\mathbf{K}{\downarrow}A$, is the category defined as follows:

- objects: $\mathbf{K}{\downarrow}A$ is the collection of all morphisms into A in $\mathbf{K}$
- morphisms: for $f : B \to A$ and $g : B' \to A$ in $\mathbf{K}$, $(\mathbf{K}{\downarrow}A)(f, g)$ consists of all morphisms $k : B \to B'$ such that $k;g = f$ in $\mathbf{K}$
- composition: the composition in $\mathbf{K}{\downarrow}A$ is the same as in $\mathbf{K}$

$$\begin{tikzcd}
B \arrow[r, "k"] \arrow[rr, bend left, "k;j"] \arrow[dr, "f"'] & B' \arrow[r, "j"] \arrow[d, "g"] & B'' \arrow[dl, "h"] \\
 & A &
\end{tikzcd}$$

Check that the composition is well-defined.

View $\mathbf{K}{\downarrow}A$ as a subcategory of $\mathbf{K}^{\to}$.

Define $\mathbf{K}{\uparrow}A$, the category of $\mathbf{K}$-objects under A.

Simple categorical definitions

- $f\colon A \to B$ is an *epimorphism* (is *epi*):
  for all $g, h\colon B \to C$, $f;g = f;h$ implies $g = h$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left, "f;g"] \arrow[rr, bend right, "f;h"'] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C \end{tikzcd}$$

> *In* **Set**, *a function is epi iff it is surjective*

- $f\colon A \to B$ is a *monomorphism* (is *mono*):
  for all $g, h\colon C \to A$, $g;f = h;f$ implies $g = h$

$$\begin{tikzcd} C \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] \arrow[rr, bend left, "g;f"] \arrow[rr, bend right, "h;f"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

> *In* **Set**, *a function is mono iff it is injective*

Simple facts

- If $f\colon A \to B$ and $g\colon B \to C$ are mono then $f;g\colon A \to C$ is mono as well.
- If $f;g\colon A \to C$ is mono then $f\colon A \to B$ is mono as well.

Prove, and then dualise the above facts.

NOTE: A morphism f is mono in $\mathbf{K}$ iff f is epi in $\mathbf{K}^{op}$.

$$\boxed{\text{mono} = co\text{-epi}}$$

Give "natural" examples of categories where epis need not be "surjective".
Give "natural" examples of categories where monos need not be "injective".

Isomorphisms

$f: A \to B$ is an *isomorphism* (is *iso*)
if there is $g: B \to A$ such that $f;g = id_A$ and $g;f = id_B$.

Then g is the (unique) *inverse of* f, $g = f^{-1}$.

In **Set**, a function is iso iff it is both epi and mono

Fact: *If f is iso then it is both epi and mono. Give counterexamples to show that the opposite implication fails.*

Fact: *$f: A \to B$ is iso iff*

- *f is a retraction, i.e., there is $g_1: B \to A$ such that $g_1;f = id_B$, and*
- *f is a coretraction, i.e., there is $g_2: B \to A$ such that $f;g_2 = id_A$.*

Fact: *A morphism is iso iff it is an epi coretraction.*

Fact: *Composition of isomorphisms is an isomorphism.*

Dualise!