Category theory for computer science

• *generality* • *abstraction* • *convenience* • *constructiveness* •

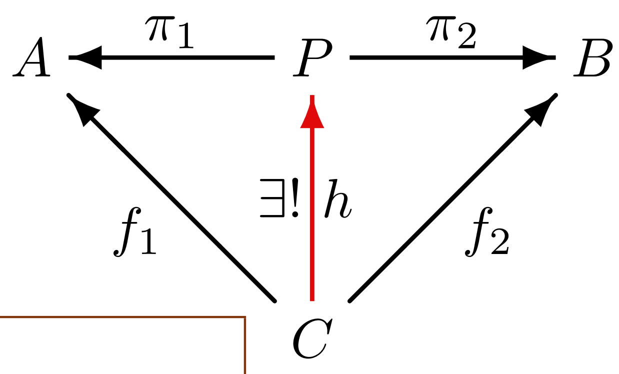
Overall idea

look at all objects exclusively through relationships between them

capture relationships between objects as appropriate morphisms between them

(Cartesian) product

- *Cartesian product* of two sets A and B, is the set $A \times B = \{\langle a, b\rangle \mid a \in A, b \in B\}$ with projections $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ given by $\pi_1(\langle a, b\rangle) = a$ and $\pi_1(\langle a, b\rangle) = b$.
- A *product* of two sets A and B, is any set P with projections $\pi_1 : P \to A$ and $\pi_2 : P \to B$ such that for any set C with functions $f_1 : P \to A$ and $f_2 : P \to B$ there exists a unique function $h : C \to P$ such that $h;\pi_1 = f_1$ and $h;\pi_2 = f_2$.



Fact: *Cartesian product (of sets A and B) is a product (of A and B).*

Recall the definition of (Cartesian) product of Σ-algebras.
Define product of Σ-algebras as above. *What have you changed?*

Pitfalls of generalization

the same concrete definition $\rightsquigarrow$ *distinct abstract generalizations*

Given a function $f : A \to B$, the following conditions are equivalent:

- f is a *surjection*: $\forall a \in A \cdot \exists b \in B \cdot f(a) = b$.
- f is an *epimorphism*: for all $h_1, h_2 : B \to C$, if $f;h_1 = f;h_2$ then $h_1 = h_2$.
- f is a *retraction*: there exists $g : B \to A$ such that $g;f = id_B$.

BUT: Given a Σ-homomorphism $f : A \to B$ for $A, B \in \mathbf{Alg}(\Sigma)$:

f *is retraction* $\Longrightarrow$ f *is surjection* $\Longleftrightarrow$ f *is epimorphism*

BUT: Given a (weak) Σ-homomorphism $f : A \to B$ for $A, B \in \mathbf{PAlg}(\Sigma)$:

f *is retraction* $\Longrightarrow$ f *is surjection* $\Longrightarrow$ f *is epimorphism*

Categories

Definition: *Category* $\mathbf{K}$ *consists of:*

- *a collection of objects:* $|\mathbf{K}|$
- *mutually disjoint collections of morphisms:* $\mathbf{K}(A, B)$*, for all* $A, B \in |\mathbf{K}|$*;*
  $m \colon A \to B$ *stands for* $m \in \mathbf{K}(A, B)$
- *morphism composition: for* $m \colon A \to B$ *and* $m' \colon B \to C$*, we have* $m;m' \colon A \to C$*;*
  - *the composition is associative: for* $m_1 : A_0 \to A_1$*,* $m_2 : A_1 \to A_2$ *and* $m_3 : A_2 \to A_3$*,* $(m_1;m_2);m_3 = m_1;(m_2;m_3)$
  - *the composition has identities: for* $A \in |\mathbf{K}|$*, there is* $id_A : A \to A$ *such that for all* $m_1 : A_1 \to A$*,* $m_1;id_A = m_1$*, and* $m_2 : A \to A_2$*,* $id_A;m_2 = m_2$*.*

BTW: *"collection"* means *"set"*, *"class"*, etc, as appropriate.

$\mathbf{K}$ is *locally small* if for all $A, B \in |\mathbf{K}|$, $\mathbf{K}(A, B)$ is a set.
$\mathbf{K}$ is *small* if in addition $|K|$ is a set.

Presenting finite categories

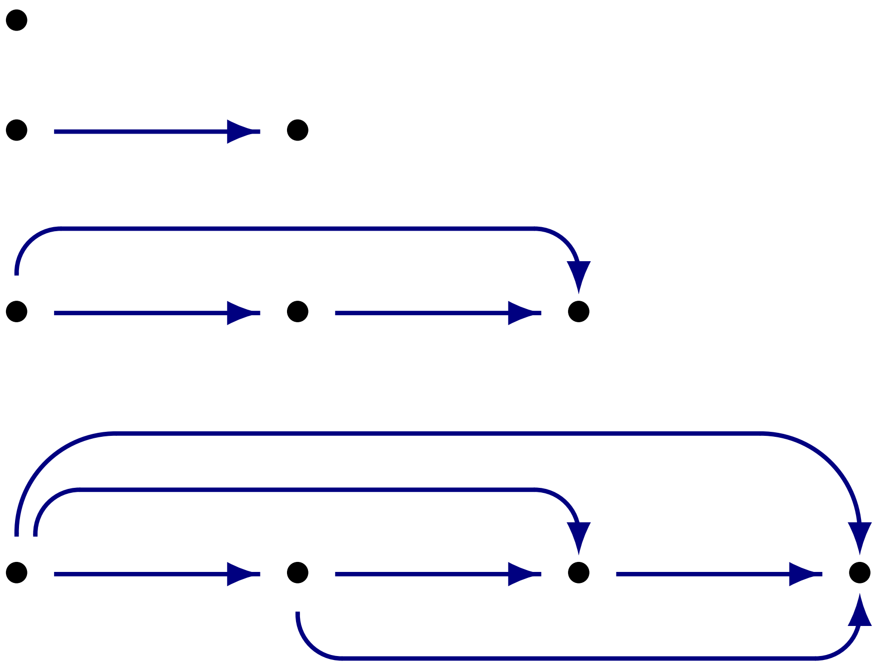
0:

1:

2:

3:

4:



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(identities omitted)

Generic examples

Discrete categories: A category $\mathbf{K}$ is *discrete* if all $\mathbf{K}(A, B)$ are empty, for distinct $A, B \in |\mathbf{K}|$, and $\mathbf{K}(A, A) = \{id_A\}$ for all $A \in |\mathbf{K}|$.

Preorders: A category $\mathbf{K}$ is *thin* if for all $A, B \in |\mathbf{K}|$, $\mathbf{K}(A, B)$ contains at most one element.

Every preorder $\leq \subseteq X \times X$ determines a thin category $\mathbf{K}_{\leq}$ with $|\mathbf{K}_{\leq}| = X$ and for $x, y \in |\mathbf{K}_{\leq}|$, $\mathbf{K}_{\leq}(x, y)$ is nonempty iff $x \leq y$.

Every (small) category $\mathbf{K}$ determines a preorder $\leq_{\mathbf{K}} \subseteq |\mathbf{K}| \times |\mathbf{K}|$, where for $A, B \in |\mathbf{K}|$, $A \leq_{\mathbf{K}} B$ iff $\mathbf{K}(A, B)$ is nonempty.

Monoids: A category $\mathbf{K}$ is a *monoid* if $|\mathbf{K}|$ is a singleton.

Every monoid $\mathcal{X} = \langle X, ;, id \rangle$, where $\_;\_ : X \times X \to X$ and $id \in X$, determines a (monoid) category $\mathbf{K}_{\mathcal{X}}$ with $|\mathbf{K}_{\leq}| = \{*\}$, $\mathbf{K}(*, *) = X$ and the composition given by the monoid operation.

Examples

- Sets (as objects) and functions between them (as morphisms) with the usual composition form the category $\mathbf{Set}$.

Functions have to be considered with their sources and targets

- For any set S, S-sorted sets (as objects) and S-functions between them (as morphisms) with the usual composition form the category $\mathbf{Set}^S$.
- For any signature Σ, Σ-algebras (as objects) and their homomorphisms (as morphisms) form the category $\mathbf{Alg}(\Sigma)$.
- For any signature Σ, partial Σ-algebras (as objects) and their weak homomorphisms (as morphisms) form the category $\mathbf{PAlg}(\Sigma)$.
- For any signature Σ, partial Σ-algebras (as objects) and their strong homomorphisms (as morphisms) form the category $\mathbf{PAlg_s}(\Sigma)$.
- Algebraic signatures (as objects) and their morphisms (as morphisms) with the composition defined in the obvious way form the category $\mathbf{AlgSig}$.

Substitutions

For any signature $\Sigma = (S, \Omega)$, the category of Σ-substitutions $\mathbf{Subst}_\Sigma$ is defined as follows:

- objects of $\mathbf{Subst}_\Sigma$ are S-sorted sets (of variables);
- morphisms in $\mathbf{Subst}_\Sigma(X, Y)$ are substitutions $\theta : X \to |T_\Sigma(Y)|$,
- composition is defined in the obvious way:
  for $\theta_1 : X \to Y$ and $\theta_2 : Y \to Z$, that is functions $\theta_1 : X \to |T_\Sigma(Y)|$ and $\theta_2 : Y \to |T_\Sigma(Z)|$, their composition $\theta_1;\theta_2 : X \to Z$ in $\mathbf{Subst}_\Sigma$ is the function $\theta_1;\theta_2 : X \to |T_\Sigma(Z)|$ such that for each $x \in X$, $(\theta_1;\theta_2)(x) = \theta_2^{\#}(\theta_1(x))$.

Subcategories

Given a category $\mathbf{K}$, a *subcategory* of $\mathbf{K}$ is any category $\mathbf{K}'$ such that

- $|\mathbf{K}'| \subseteq |\mathbf{K}|$,
- $\mathbf{K}'(A,B) \subseteq \mathbf{K}(A,B)$, for all $A, B \in |\mathbf{K}'|$,
- composition in $\mathbf{K}'$ coincides with the composition in K on morphisms in $\mathbf{K}'$, and
- identities in $\mathbf{K}'$ coincide with identities in $\mathbf{K}$ on objects in $|\mathbf{K}'|$.

A subcategory $\mathbf{K}'$ of $\mathbf{K}$ is *full* if $\mathbf{K}'(A,B) = \mathbf{K}(A,B)$ for all $A, B \in |\mathbf{K}'|$.

Any collection $X \subseteq |\mathbf{K}|$ gives the full subcategory $\mathbf{K}\big|_X$ of $\mathbf{K}$ by $|\mathbf{K}\big|_X| = X$.

- The category $\mathbf{FinSet}$ of finite sets is a full subcategory of $\mathbf{Set}$.
- The discrete category of sets is a subcategory of sets with inclusions as morphisms, which is a subcategory of sets with injective functions as morphisms, which is a subcategory of $\mathbf{Set}$.
- The category of single-sorted signatures is a full subcategory of $\mathbf{AlgSig}$.

Reversing arrows

Given a category $\mathbf{K}$, its *opposite category* $\mathbf{K}^{op}$ is defined as follows:

- objects: $|\mathbf{K}^{op}| = |\mathbf{K}|$
- morphisms: $\mathbf{K}^{op}(A, B) = \mathbf{K}(B, A)$ for all $A, B \in |\mathbf{K}^{op}| = |\mathbf{K}|$
- composition: given $m_1 : A \to B$ and $m_2 : B \to C$ in $\mathbf{K}^{op}$, that is, $m_1 : B \to A$ and $m_2 : C \to B$ in $\mathbf{K}$, their composition in $\mathbf{K}^{op}$, $m_1;m_2 : A \to C$, is set to be their composition $m_2;m_1 : C \to A$ in $\mathbf{K}$.

Fact: *The identities in $\mathbf{K}^{op}$ coincide with the identities in $\mathbf{K}$.*

Fact: *Every category is opposite to some category:*

$$(\mathbf{K}^{op})^{op} = \mathbf{K}$$

Duality principle

If W is a categorical concept (notion, property, statement, ...) then its *dual*, co-W, is obtained by reversing all the morphisms in W.

Example:

$P(X)$**:** "for any object Y there exists a morphism $f : X \to Y$"

co**-**$P(X)$**:** "for any object Y there exists a morphism $f : Y \to X$"

NOTE: co-$P(X)$ in $\mathbf{K}$ coincides with $P(X)$ in $\mathbf{K}^{op}$.

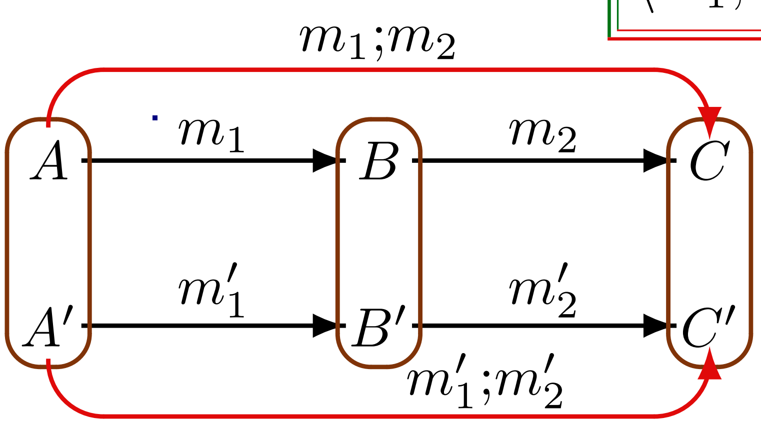
Fact: *If a property W holds for all categories then co-W holds for all categories as well.*

Product categories

Given categories $\mathbf{K}$ and $\mathbf{K}'$, their *product* $\mathbf{K} \times \mathbf{K}'$ is the category defined as follows:

- objects: $|\mathbf{K} \times \mathbf{K}'| = |K| \times |\mathbf{K}'|$
- morphisms: $(\mathbf{K} \times \mathbf{K}')(\langle A, A' \rangle, \langle B, B' \rangle) = \mathbf{K}(A, B) \times \mathbf{K}'(A', B')$ for all $A, B \in |\mathbf{K}|$ and $A', B' \in |\mathbf{K}'|$
- composition: for $\langle m_1, m_1' \rangle : \langle A, A' \rangle \to \langle B, B' \rangle$ and $\langle m_2, m_2' \rangle : \langle B, B' \rangle \to \langle C, C' \rangle$ in $\mathbf{K} \times \mathbf{K}'$, their composition in $\mathbf{K} \times \mathbf{K}'$ is

$$\langle m_1, m_1' \rangle ; \langle m_2, m_2' \rangle = \langle m_1 ; m_2, m_1' ; m_2' \rangle$$



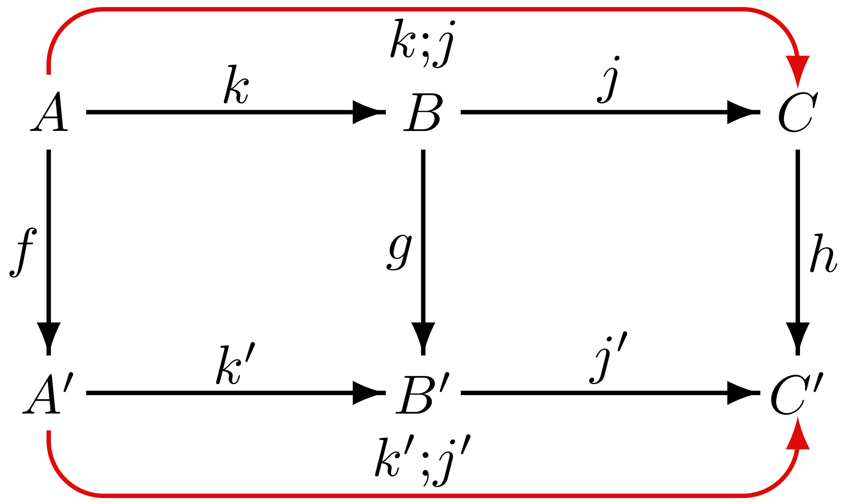
Define $\mathbf{K}^n$, where $\mathbf{K}$ is a category and $n \geq 1$. Extend this definition to $n = 0$.

Morphism categories

Given a category $\mathbf{K}$, its *morphism category* $\mathbf{K}^{\rightarrow}$ is the category defined as follows:

- objects: $|\mathbf{K}^{\rightarrow}|$ is the collection of all morphisms in $\mathbf{K}$
- morphisms: for $f : A \rightarrow A'$ and $g : B \rightarrow B'$ in $\mathbf{K}$, $\mathbf{K}^{\rightarrow}(f, g)$ consists of all $\langle k, k' \rangle$, where $k : A \rightarrow B$ and $k' : A' \rightarrow B'$ are such that $k;g = f;k'$ in $\mathbf{K}$
- composition: for $\langle k, k' \rangle : (f : A \rightarrow A') \rightarrow (g : B \rightarrow B')$ and $\langle j, j' \rangle : (g : B \rightarrow B') \rightarrow (h : C \rightarrow C')$ in $\mathbf{K}^{\rightarrow}$, their composition in $\mathbf{K}^{\rightarrow}$ is $\langle k, k' \rangle;\langle j, j' \rangle = \langle k;j, k';j' \rangle$.

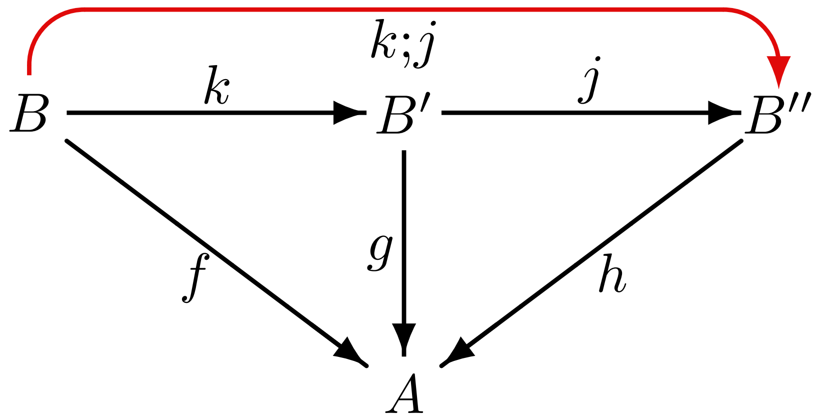
Check that the composition is well-defined.



Slice categories

Given a category $\mathbf{K}$ and an object $A \in |K|$, the category of $\mathbf{K}$-*objects over* A, $\mathbf{K}{\downarrow}A$, is the category defined as follows:

- objects: $\mathbf{K}{\downarrow}A$ is the collection of all morphisms into A in $\mathbf{K}$
- morphisms: for $f : B \to A$ and $g : B' \to A$ in $\mathbf{K}$, $(\mathbf{K}{\downarrow}A)(f, g)$ consists of all morphisms $k : B \to B'$ such that $k;g = f$ in $\mathbf{K}$
- composition: the composition in $\mathbf{K}{\downarrow}A$ is the same as in $\mathbf{K}$



Check that the composition is well-defined.

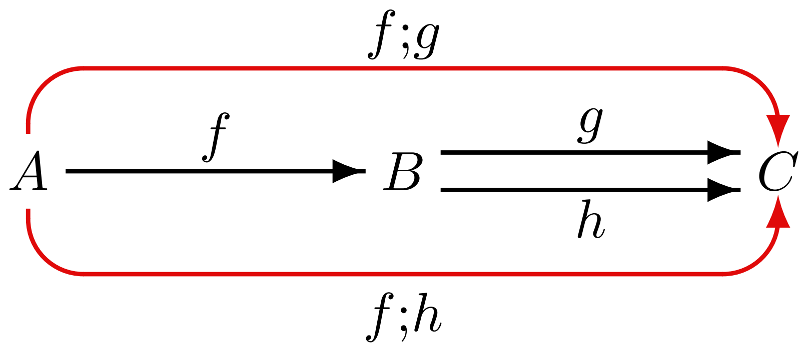
View $\mathbf{K}{\downarrow}A$ as a subcategory of $\mathbf{K}^{\to}$.

Define $\mathbf{K}{\uparrow}A$, the category of $\mathbf{K}$-objects under A.

Fix a category $\mathbf{K}$ for a while.

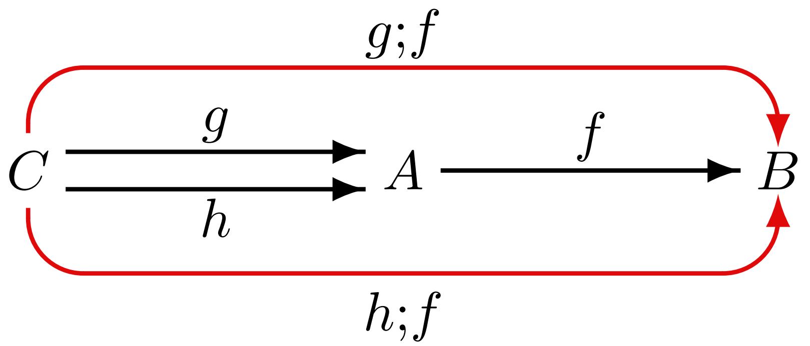
Simple categorical definitions

- $f : A \to B$ is an *epimorphism* (is *epi*):
  for all $g, h : B \to C$, $f;g = f;h$ implies $g = h$



In $\mathbf{Set}$*, a function is epi iff it is surjective*

- $f : A \to B$ is a *monomorphism* (is *mono*):
  for all $g, h : C \to A$, $g;f = h;f$ implies $g = h$



In $\mathbf{Set}$*, a function is mono iff it is injective*

Simple facts

- If $f : A \to B$ and $g : B \to C$ are mono then $f;g : A \to C$ is mono as well.
- If $f;g : A \to C$ is mono then $f : A \to B$ is mono as well.

Prove, and then dualise the above facts.

NOTE: A morphism f is mono in $\mathbf{K}$ iff f is epi in $\mathbf{K}^{op}$.

mono = *co*-epi

Give "natural" examples of categories where epis need not be "surjective".
Give "natural" examples of categories where monos need not be "injective".

Isomorphisms

$f : A \to B$ is an *isomorphism* (is *iso*)
if there is $g : B \to A$ such that $f;g = id_A$ and $g;f = id_B$.

Then g is the (unique) *inverse of* f, $g = f^{-1}$.

In **Set**, a function is iso iff it is both epi and mono

Fact: *If f is iso then it is both epi and mono. Give counterexamples to show that the opposite implication fails.*

Fact: *$f : A \to B$ is iso iff*

- *f is a retraction, i.e., there is $g_1 : B \to A$ such that $g_1;f = id_B$, and*
- *f is a coretraction, i.e., there is $g_2 : B \to A$ such that $f;g_2 = id_A$.*

Fact: *A morphism is iso iff it is an epi coretraction.*

Fact: *Composition of isomorphisms is an isomorphism.*

Dualise!

Universal constructions: limits and colimits

Consider and arbitrary but fixed category $\mathbf{K}$ for a while.

Initial and terminal objects

An object $I \in |\mathbf{K}|$ is *initial* in $\mathbf{K}$ if for each object $A \in |\mathbf{K}|$ there is exactly one morphism from I to A.

Examples:

- $\emptyset$ is initial in $\mathbf{Set}$.
- For any signature $\Sigma \in |\mathbf{AlgSig}|$, T_Σ is initial in $\mathbf{Alg}(\Sigma)$.
- For any signature $\Sigma \in |\mathbf{AlgSig}|$ and set of Σ-equations Φ, the initial model of $\langle \Sigma, \Phi \rangle$ is initial in $\mathbf{Mod}(\Sigma, \Phi)$, the full subcategory of $\mathbf{Alg}(\Sigma)$ determined by the class $Mod(\Sigma, \Phi)$ of all models of Φ.

Look for initial objects in other categories.

Fact: *Initial objects, if exist, are unique up to isomorphism:*

- *Any two initial objects in* $\mathbf{K}$ *are isomorphic.*
- *If* I *is initial in* $\mathbf{K}$ *and* I' *is isomorphic to* I *in* $\mathbf{K}$ *then* I' *is initial in* $\mathbf{K}$ *as well.*

Terminal objects

An object $I \in |\mathbf{K}|$ is *terminal* in $\mathbf{K}$ if for each object $A \in |\mathbf{K}|$ there is exactly one morphism from A to I.

terminal = *co*-initial

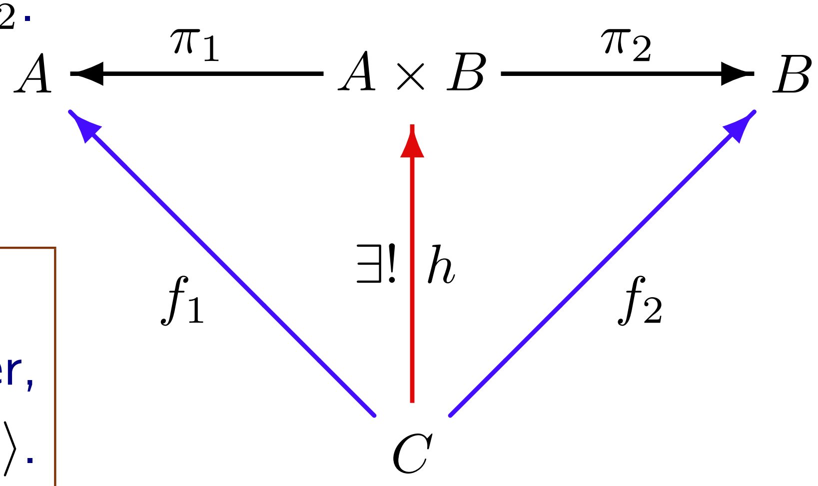
Dualise those for initial objects.

Exercises:

- Look for terminal objects in standard categories.
- Show that terminal objects are unique to within an isomorphism.
- Look for categories where there is an object which is both initial and terminal.

Products

A *product* of two objects $A, B \in |\mathbf{K}|$, is any object $A \times B \in |\mathbf{K}|$ with two morphisms (*product projections*) $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ such that for any object $C \in |\mathbf{K}|$ with morphisms $f_1 : C \to A$ and $f_2 : C \to B$ there exists a unique morphism $h : C \to A \times B$ such that $h;\pi_1 = f_1$ and $h;\pi_2 = f_2$.



In $\mathbf{Set}$, *Cartesian product is a product*

We write $\langle f_1, f_2 \rangle$ for h defined as above. Then: $\langle f_1, f_2 \rangle;\pi_1 = f_1$ and $\langle f_1, f_2 \rangle;\pi_2 = f_2$. Moreover, for any h into the product $A \times B$: $h = \langle h;\pi_1, h;\pi_2 \rangle$.

Essentially, this equationally defines a product!

Fact: *Products are defined to within an isomorphism (which commutes with projections).*

Exercises

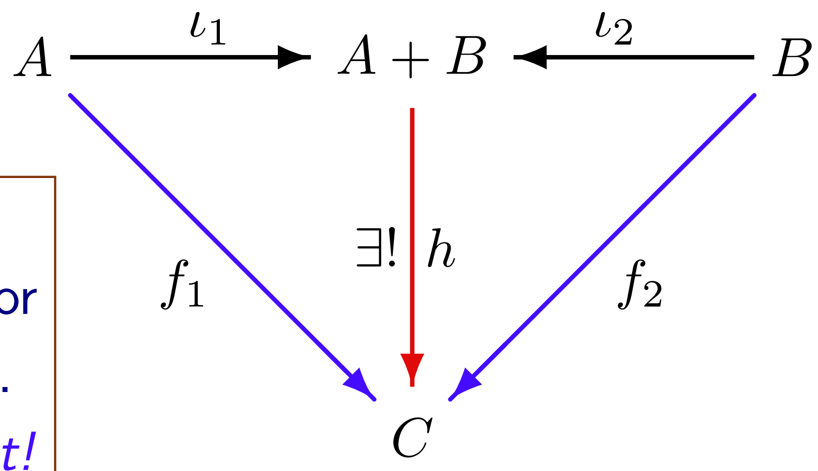
- Product commutes (up to isomorphism): $A \times B \cong B \times A$
- Product is associative (up to isomorphism): $(A \times B) \times C \cong A \times (B \times C)$
- What is a product of two objects in a preorder category?
- Define the product of any family of objects. What is the product of the empty family?
- For any algebraic signature $\Sigma \in |\mathbf{AlgSig}|$, try to define products in $\mathbf{Alg}(\Sigma)$, $\mathbf{PAlg_s}(\Sigma)$, $\mathbf{PAlg}(\Sigma)$. Expect troubles in the two latter cases...
- Define products in the *category of partial functions*, $\mathbf{Pfn}$, with sets (as objects) and partial functions as morphisms between them.
- Define products in the *category of relations*, $\mathbf{Rel}$, with sets (as objects) and binary relations as morphisms between them.
  - BTW: What about products in $\mathbf{Rel}^{op}$?

Coproducts

coproduct = *co*-product

A *coproduct* of two objects $A, B \in |\mathbf{K}|$, is any object $A + B \in |\mathbf{K}|$ with two morphisms (*coproduct injections*) $\iota_1 : A \rightarrow A + B$ and $\iota_2 : B \rightarrow A + B$ such that for any object $C \in |\mathbf{K}|$ with morphisms $f_1 : A \rightarrow C$ and $f_2 : B \rightarrow C$ there exists a unique morphism $h : A + B \rightarrow C$ such that $h;\iota_1 = f_1$ and $h;\iota_2 = f_2$.

In $\mathbf{Set}$, *disjoint union is a coproduct*



We write $[f_1, f_2]$ for h defined as above. Then: $\iota_1;[f_1, f_2] = f_1$ and $\iota_2;[f_1, f_2] = f_2$. Moreover, for any h from the coproduct $A + B$: $h = [h;\iota_1, h;\iota_2]$.

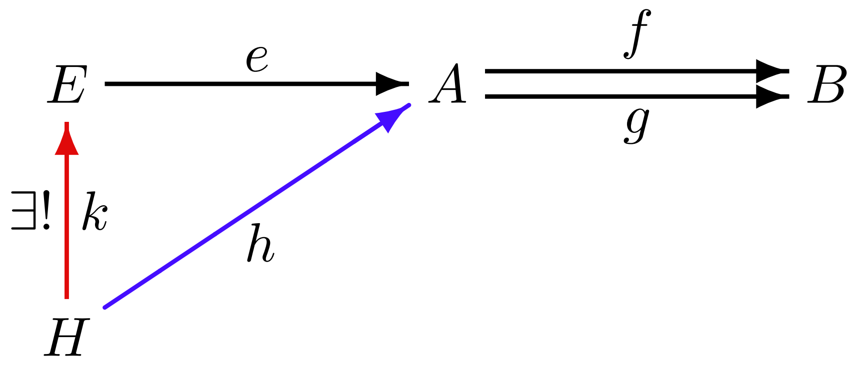
Essentially, this equationally defines a product!

Fact: *Coproducts are defined to within an isomorphism (which commutes with injections).*

Exercises: Dualise!

Equalisers

An *equaliser* of two “parallel” morphisms $f, g : A \to B$ is a morphism $e : E \to A$ such that $e;f = e;g$, and such that for all $h : H \to A$, if $h;f = h;g$ then for a unique morphism $k : H \to E$, $k;e = h$.



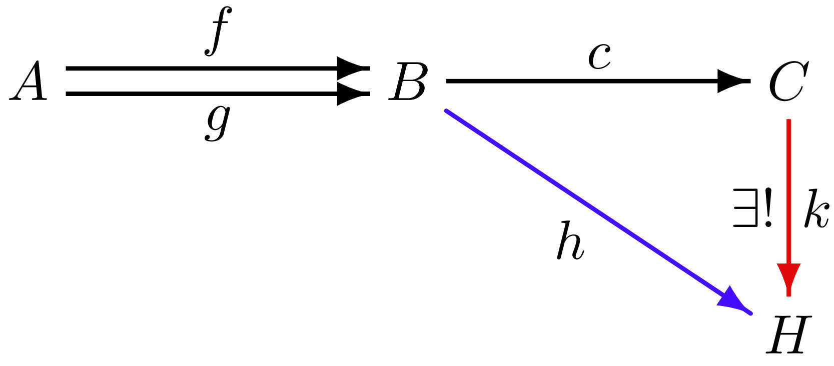
- Equalisers are unique up to isomorphism.
- Every equaliser is mono.
- Every epi equaliser is iso.

In $\mathbf{Set}$, given functions $f, g : A \to B$, define $E = \{a \in A \mid f(a) = g(a)\}$
The inclusion $e : E \hookrightarrow A$ is an equaliser of f and g.

Define equalisers in $\mathbf{Alg}(\Sigma)$.
Try also in: $\mathbf{PAlg_s}(\Sigma)$, $\mathbf{PAlg}(\Sigma)$, $\mathbf{Pfn}$, $\mathbf{Rel}$, ...

Coequalisers

A *coequaliser* of two "parallel" morphisms $f, g : A \to B$ is a morphism $c : B \to C$ such that $f;c = g;c$, and such that for all $h : B \to H$, if $f;h = g;h$ then for a unique morphism $k : C \to H$, $c;k = h$.



- Coequalisers are unique up to isomorphism.
- Every coequaliser is epi.
- Every mono coequaliser is iso.

In $\mathbf{Set}$, given functions $f, g : A \to B$,
let $\equiv \subseteq B \times B$ be the least equivalence such that $f(a) \equiv g(a)$ for all $a \in A$
The quotient function $[\_]_\equiv : B \to B/\equiv$ is a coequaliser of f and g.

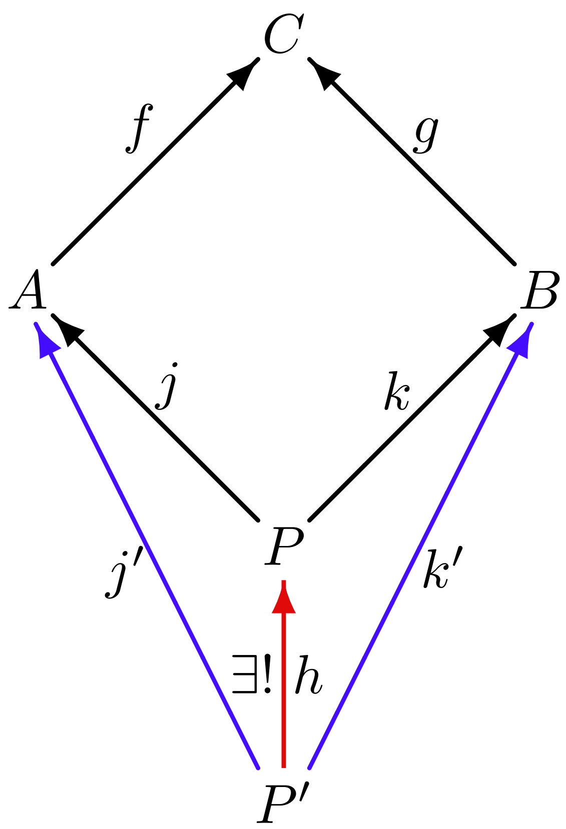
Define coequalisers in $\mathbf{Alg}(\Sigma)$.
Try also in: $\mathbf{PAlg_s}(\Sigma)$, $\mathbf{PAlg}(\Sigma)$, $\mathbf{Pfn}$, $\mathbf{Rel}$, ...

Most general unifiers are coequalisers in $\mathbf{Subst}_\Sigma$

Pullbacks

A *pullback* of two morphisms with common target $f : A \to C$ and $g : B \to C$ is an object $P \in |\mathbf{K}|$ with morphisms $j : P \to A$ and $k : P \to B$ such that $j;f = k;g$, and such that for all $P' \in |\mathbf{K}|$ with morphisms $j' : P' \to A$ and $k' : P' \to B$, if $j';f = k';g$ then for a unique morphism $h : P' \to P$, $h;j = j'$ and $h;k = k'$.

In $\mathbf{Set}$, given functions $f : A \to C$ and $f : B \to C$,
define $P = \{\langle a, b\rangle \in A \times B \mid f(a) = g(b)\}$
Then P with obvious projections on A and B, respectively, is a pullback of f and g.

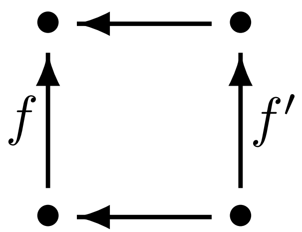


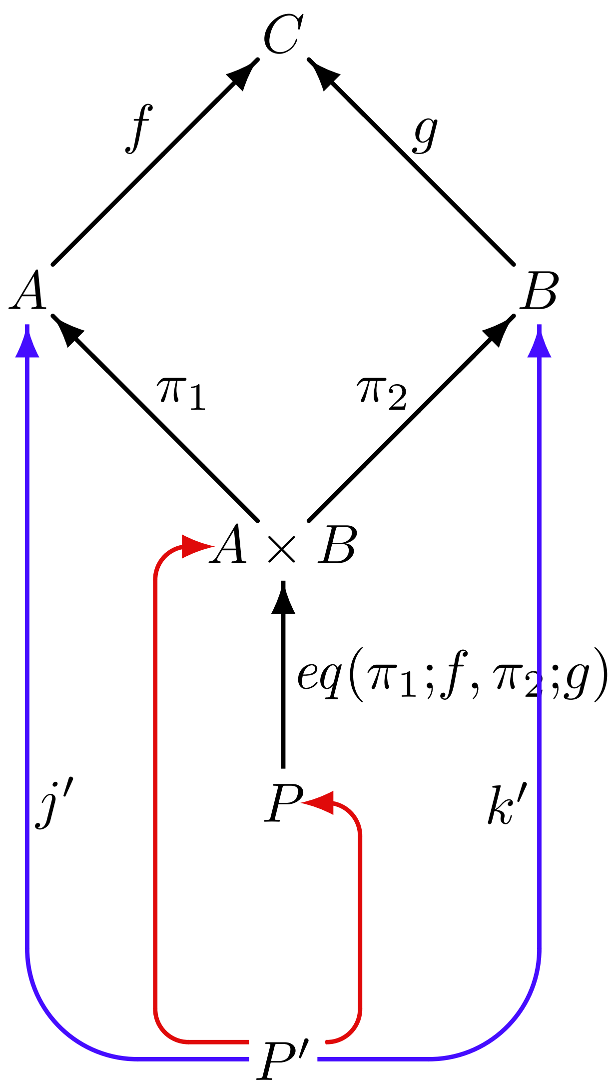
Define pullbacks in $\mathbf{Alg}(\Sigma)$.
Try also in: $\mathbf{PAlg_s}(\Sigma)$, $\mathbf{PAlg}(\Sigma)$, $\mathbf{Pfn}$, $\mathbf{Rel}$, ...

Wait for a hint to come...

Few facts

- Pullbacks are unique up to isomorphism.
- If $\mathbf{K}$ has all products (of pairs of objects) and all equalisers (of pairs of parallel morphisms) then it has all pullbacks (of pairs of morphisms with common target).
- If $\mathbf{K}$ has all pullbacks and a terminal object then it has all binary products and equalisers. HINT: to build an equaliser of $f, g : A \to B$, consider a pullback of $\langle id_A, f\rangle, \langle id_A, g\rangle : A \to A \times B$.
- Pullbacks translate monos to monos: if the following is a pullback square and f is mono then f' is mono as well.



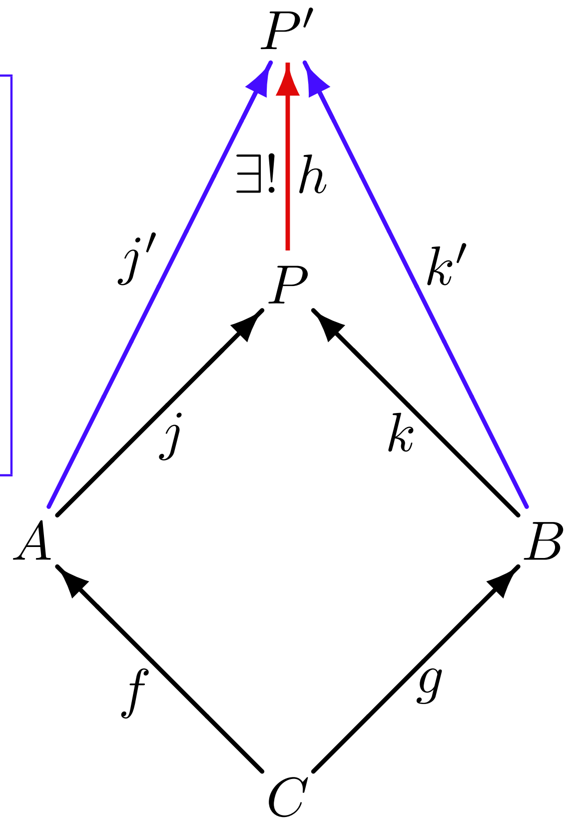


Pushouts

pushout = *co*-pullback

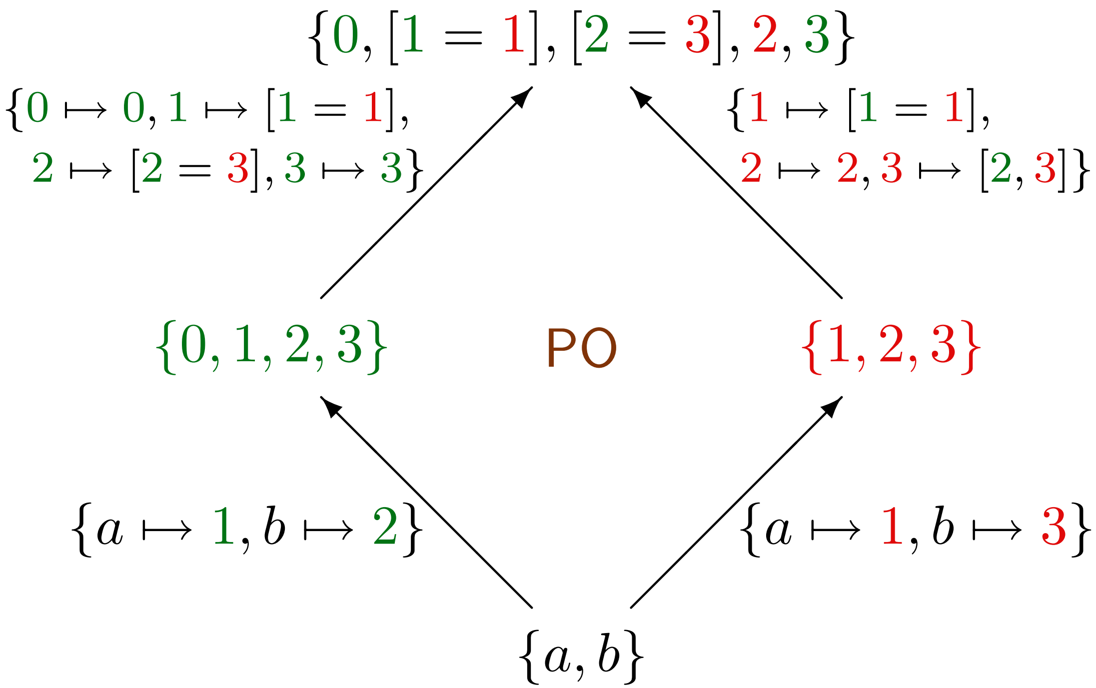
A *pushout* of two morphisms with common source $f : C \to A$ and $g : C \to A$ is an object $P \in |\mathbf{K}|$ with morphisms $j : A \to P$ and $k : B \to P$ such that $f;j = g;k$, and such that for all $P' \in |\mathbf{K}|$ with morphisms $j' : A \to P'$ and $k' : B \to P'$, if $f;j' = g;k'$ then for a unique morphism $h : P \to P'$, $j;h = j'$ and $k;h = k'$.

In $\mathbf{Set}$, given functions $f : A \to C$ and $f : B \to C$, define the least equivalence $\equiv$ on $A \uplus B$ such that $f(c) \equiv g(c)$ for all $c \in C$ The quotient $(A \uplus B)/\equiv$ with compositions of injections and the quotient function is a pushout of f and g.



Dualise facts for pullbacks!

Example



Pushouts put objects together taking account of the indicated sharing

Example in **AlgSig**

sort $String$
ops $a, \ldots, z : String;$
$\_\, \hat{}\, \_ : String \times String \longrightarrow String$

$\longrightarrow$

sorts $String, Nat, Array[String]$
ops $a, \ldots, z : String;$
$\_\, \hat{}\, \_ : String \times String \longrightarrow String;$
$empty : Array[String];$
$put : Nat \times String \times Array[String] \longrightarrow Array[String];$
$get : Nat \times Array[String] \longrightarrow String$

PO

sort $Elem$

$\longrightarrow$

sorts $Elem, Nat, Array[Elem]$
ops $empty : Array[Elem];$
$put : Nat \times Elem \times Array[Elem] \longrightarrow Array[Elem];$
$get : Nat \times Array[Elem] \longrightarrow Elem$

Graphs

A graph consists of sets of nodes and edges,
and indicate source and target nodes for each edge

$$\Sigma_{Graph} = \textbf{sorts } nodes, edges$$
$$\textbf{opns } source : edges \to nodes$$
$$target : edges \to nodes$$

Graph is any Σ_{Graph}-algebra.

The category of graphs:

$$\mathbf{Graph} = \mathbf{Alg}(\Sigma_{Graph})$$

For any small category $\mathbf{K}$, define its *graph*, $G(\mathbf{K})$

For any graph $G \in |\mathbf{Graph}|$, define *the category of paths in* G, $\mathbf{Path}(G)$:

- objects: $|G|_{nodes}$
- morphisms: *paths* in G, i.e., sequences $n_0 e_1 n_1 \ldots n_{k-1} e_k n_k$ of nodes $n_0, \ldots, n_k \in |G|_{nodes}$ and edges $e_1, \ldots, e_k \in |G|_{edges}$ such that $source(e_i) = n_{i-1}$ and $target(e_i) = n_i$ for $i = 1, \ldots, k$.

Diagrams

A diagram in $\mathbf{K}$ is a graph with nodes labelled with $\mathbf{K}$-objects and edges labelled with $\mathbf{K}$-morphisms with appropriate sources and targets.

A *diagram* D consists of:

– a graph $G(D)$,
– an object $D_n \in |\mathbf{K}|$ for each node $n \in |G(D)|_{nodes}$,
– a morphism $D_e : D_{source(e)} \to D_{target(e)}$ for each edge $e \in |G(D)|_{edges}$.

For any small category $\mathbf{K}$, define its *diagram*, $D(\mathbf{K})$, with graph $G(D(\mathbf{K})) = G(\mathbf{K})$

BTW: A diagram D *commutes* (or is *commutative*) if for any two paths in $G(D)$ with common source and target, the compositions of morphisms that label the edges of each of them coincide.

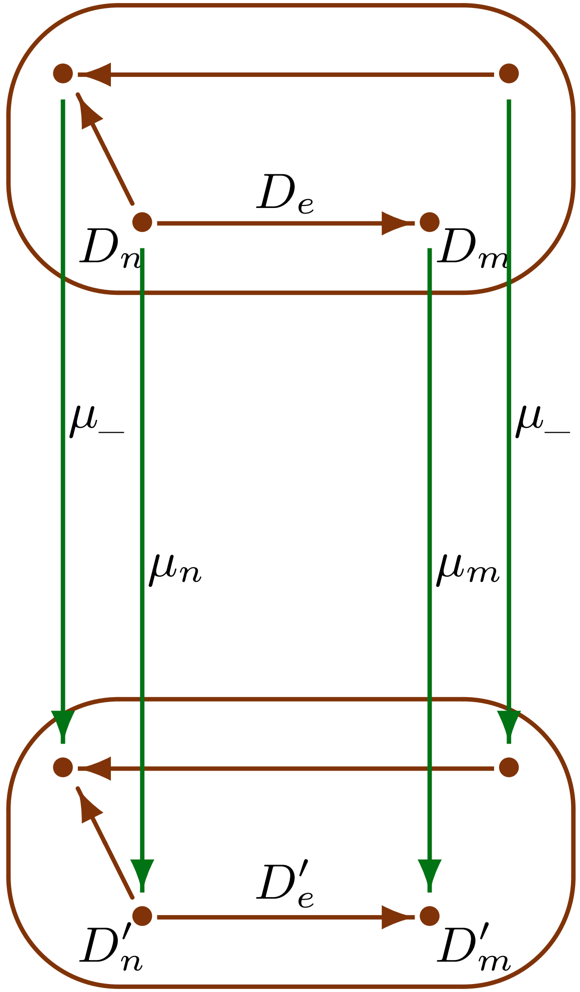
Diagram categories

Given a graph G with nodes $N = |G|_{nodes}$ and edges $E = |G|_{edges}$, the *category of diagrams of shape G in* $\mathbf{K}$, $\mathbf{Diag}^{G}_{\mathbf{K}}$, is defined as follows:

- objects: all diagrams D in $\mathbf{K}$ with $G(D) = G$
- morphisms: for any two diagrams D and D' in $\mathbf{K}$ of shape G, a morphism $\mu : D \to D'$ is any family $\mu = \langle \mu_n : D_n \to D'_n \rangle_{n \in N}$ of morphisms in $\mathbf{K}$ such that for each edge $e \in E$ with $source_{G(D)}(e) = n$ and $target_{G(D)}(e) = m$,

$$\mu_n ; D'_e = D_e ; \mu_m$$

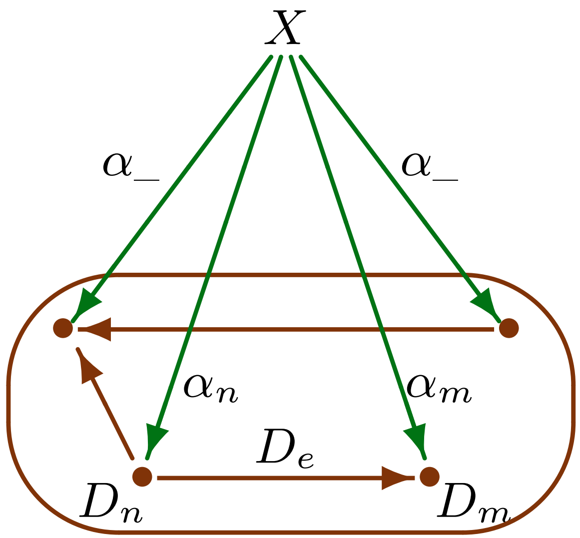
.

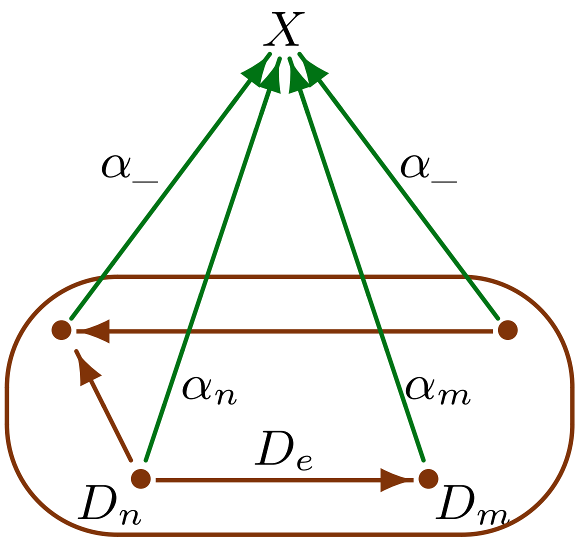


Let D be a diagram over $G(D)$ with nodes $N = |G(D)|_{nodes}$ and edges $E = |G(D)|_{edges}$.

Cones and cocones

A *cone* on D (in $\mathbf{K}$) is an object $X \in |\mathbf{K}|$ together with a family of morphisms $\langle \alpha_n : X \to D_n \rangle_{n \in N}$ such that for each edge $e \in E$ with $source_{G(D)}(e) = n$ and $target_{G(D)}(e) = m$, $\alpha_n;D_e = \alpha_m$.

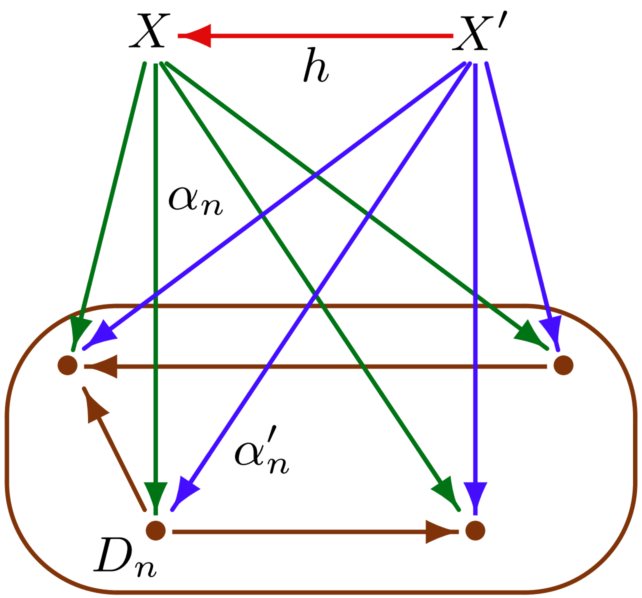


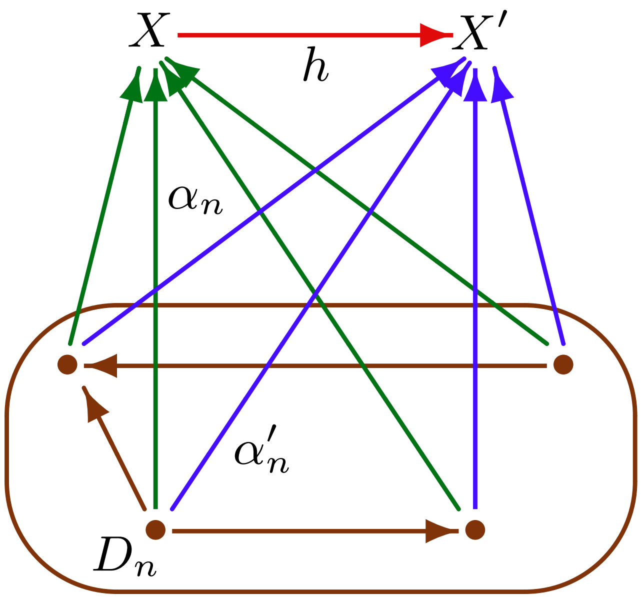


A *cocone* on D (in $\mathbf{K}$) is an object $X \in |\mathbf{K}|$ together with a family of morphisms $\langle \alpha_n : D_n \to X \rangle_{n \in N}$ such that for each edge $e \in E$ with $source_{G(D)}(e) = n$ and $target_{G(D)}(e) = m$, $\alpha_n = D_e;\alpha_m$.

Limits and colimits

A *limit* of D (in $\mathbf{K}$) is a cone $\langle \alpha_n : X \to D_n \rangle_{n \in N}$ on D such that for all cones $\langle \alpha'_n : X' \to D_n \rangle_{n \in N}$ on D, for a unique morphism $h : X' \to X$, $h;\alpha_n = \alpha'_n$ for all $n \in N$.





A *colimit* of D (in $\mathbf{K}$) is a cocone $\langle \alpha_n : D_n \to X \rangle_{n \in N}$ on D such that for all cocones $\langle \alpha'_n : D_n \to X' \rangle_{n \in N}$ on D, for a unique morphism $h : X \to X'$, $\alpha_n;h = \alpha'_n$ for all $n \in N$.

Some limits

diagram	limit	in **Set**
(empty)	*terminal object*	$\{*\}$
$A \qquad B$	*product*	$A \times B$
$A \overset{f}{\underset{g}{\rightrightarrows}} B$	*equaliser*	$\{a \in A \mid f(a) = g(a)\} \hookrightarrow A$
$A \xrightarrow{f} C \xleftarrow{g} B$	*pullback*	$\{(a, b) \in A \times B \mid f(a) = g(b)\}$

. . . & colimits

diagram	colimit	in $\mathbf{Set}$
(empty)	*initial object*	$\emptyset$
$A \qquad B$	*coproduct*	$A \uplus B$
$A \overset{f}{\underset{g}{\rightrightarrows}} B$	*coequaliser*	$B \longrightarrow B/{\equiv}$ where $f(a) \equiv g(a)$ for all $a \in A$
$A \xleftarrow{f} C \xrightarrow{g} B$	*pushout*	$(A \uplus B)/{\equiv}$ where $f(c) \equiv g(c)$ for all $c \in C$

Exercises

- For any diagram D, define the *category of cones over* D, $\mathbf{Cone}(D)$:
  - objects: all cones over D
  - morphisms: a morphism from $\langle \alpha_n : X \to D_n \rangle_{n \in N}$ to $\langle \alpha'_n : X' \to D_n \rangle_{n \in N}$ is any $\mathbf{K}$-morphism $h : X \to X'$ such that $h;\alpha'_n = \alpha_n$ for all $n \in N$.
- Show that limits of D are terminal objects in $\mathbf{Cone}(D)$. Conclude that limits are defined uniquely up to isomorphism (which commutes with limit projections).
- Construct a limit in $\mathbf{Set}$ of the following diagram:

$$A_0 \xleftarrow{f_0} A_1 \xleftarrow{f_1} A_2 \xleftarrow{f_2} \cdots$$

- Show that limiting cones are *jointly mono*, i.e., if $\langle \alpha_n : X \to D_n \rangle_{n \in N}$ is a limit of D then for all $f, g : A \to X$, $f = g$ whenever $f;\alpha_n = g;\alpha_n$ for all $n \in N$.

Dualise all the exercises above!

Completeness and cocompleteness

A category $\mathbf{K}$ *is (finitely) complete if any (finite) diagram in* $\mathbf{K}$ *has a limit.*

A category $\mathbf{K}$ *is (finitely) cocomplete if any (finite) diagram in* $\mathbf{K}$ *has a colimit.*

- If $\mathbf{K}$ has a terminal object, binary products (of all pairs of objects) and equalisers (of all pairs of parallel morphisms) then it is finitely complete.
- If $\mathbf{K}$ has products of all families of objects and equalisers (of all pairs of parallel morphisms) then it is complete.

Prove completeness of $\mathbf{Set}$, $\mathbf{Alg}(\Sigma)$, $\mathbf{AlgSig}$, $\mathbf{Pfn}$, ...

When a preorder category is complete?

BTW: If a small category is complete then it is a preorder.

Dualise the above!

Functors

and

natural transformations

functors	$\rightsquigarrow$	*category morphisms*
natural transformations	$\rightsquigarrow$	*functor morphisms*

Functors

A *functor* $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ from a category $\mathbf{K}$ to a category $\mathbf{K}'$ consists of:

- a function $\mathbf{F} : |\mathbf{K}| \to |\mathbf{K}'|$, and
- for all $A, B \in |\mathbf{K}|$, a function $\mathbf{F} : \mathbf{K}(A, B) \to \mathbf{K}'(\mathbf{F}(A), \mathbf{F}(B))$

such that:

Make explicit categories in which we work at various places here

- $\mathbf{F}$ preserves identities, i.e.,

$$\mathbf{F}(id_A) = id_{\mathbf{F}(A)}$$

  for all $A \in |\mathbf{K}|$, and

- $\mathbf{F}$ preserves composition, i.e.,

$$\mathbf{F}(f;g) = \mathbf{F}(f);\mathbf{F}(g)$$

  for all $f : A \to B$ and $g : B \to C$ in $\mathbf{K}$.

We really should differentiate between various components of F

Examples

- *identity functors*: $\mathbf{Id_K} : \mathbf{K} \to \mathbf{K}$, for any category $\mathbf{K}$
- *inclusions*: $\mathbf{I}_{\mathbf{K} \hookrightarrow \mathbf{K}'} : \mathbf{K} \to \mathbf{K}'$, for any subcategory $\mathbf{K}$ of $\mathbf{K}'$
- *constant functors*: $\mathbf{C}_A : \mathbf{K} \to \mathbf{K}'$, for any categories $\mathbf{K}, \mathbf{K}'$ and $A \in |\mathbf{K}'|$, with $\mathbf{C}_A(f) = id_A$ for all morphisms f in $\mathbf{K}$
- *powerset functor*: $\mathbf{P} : \mathbf{Set} \to \mathbf{Set}$ given by
  - $\mathbf{P}(X) = \{Y \mid Y \subseteq X\}$, for all $X \in |\mathbf{Set}|$
  - $\mathbf{P}(f) : \mathbf{P}(X) \to \mathbf{P}(X')$ for all $f : X \to X'$ in $\mathbf{Set}$, $\mathbf{P}(f)(Y) = \{f(y) \mid y \in Y\}$ for all $Y \subseteq X$
- *contravariant powerset functor*: $\mathbf{P}_{-1} : \mathbf{Set}^{op} \to \mathbf{Set}$ given by
  - $\mathbf{P}_{-1}(X) = \{Y \mid Y \subseteq X\}$, for all $X \in |\mathbf{Set}|$
  - $\mathbf{P}_{-1}(f) : \mathbf{P}(X') \to \mathbf{P}(X)$ for all $f : X \to X'$ in $\mathbf{Set}$, $\mathbf{P}_{-1}(f)(Y') = \{x \in X \mid f(x) \in Y'\}$ for all $Y' \subseteq X'$

Examples, cont'd.

- *projection functors*: $\pi_1 : \mathbf{K} \times \mathbf{K}' \to \mathbf{K}$, $\pi_2 : \mathbf{K} \times \mathbf{K}' \to \mathbf{K}'$
- *list functor*: $\mathbf{List} : \mathbf{Set} \to \mathbf{Monoid}$, where $\mathbf{Monoid}$ is the category of monoids (as objects) with monoid homomorphisms as morphisms:
  - $\mathbf{List}(X) = \langle X^*, {}^\frown, \epsilon \rangle$, for all $X \in |\mathbf{Set}|$, where X^* is the set of all finite lists of elements from X, ${}^\frown$ is the list concatenation, and ϵ is the empty list.
  - $\mathbf{List}(f) : \mathbf{List}(X) \to \mathbf{List}(X')$ for $f : X \to X'$ in $\mathbf{Set}$, $\mathbf{List}(f)(\langle x_1, \ldots, x_n \rangle) = \langle f(x_1), \ldots, f(x_n) \rangle$ for all $x_1, \ldots, x_n \in X$
- *totalisation functor*: $\mathbf{Tot} : \mathbf{Pfn} \to \mathbf{Set}_*$, where $\mathbf{Set}_*$ is the subcategory of $\mathbf{Set}$ of sets with a distinguished element $*$ and $*$-preserving functions
  - $\mathbf{Tot}(X) = X \uplus \{*\}$
  - $\mathbf{Tot}(f)(x) = \begin{cases} f(x) & \text{if it is defined} \\ * & \text{otherwise} \end{cases}$

Define $\mathbf{Set}_*$ as the category of algebras

Examples, cont'd.

- *carrier set functors*: $|\_| : \mathbf{Alg}(\Sigma) \to \mathbf{Set}^S$, for any algebraic signature $\Sigma = \langle S, \Omega \rangle$, yielding the algebra carriers and homomorphisms as functions between them
- *reduct functors*: $\_|_\sigma : \mathbf{Alg}(\Sigma') \to \mathbf{Alg}(\Sigma')$, for any signature morphism $\sigma : \Sigma \to \Sigma'$, as defined earlier
- *term algebra functors*: $\mathbf{T}_\Sigma : \mathbf{Set} \to \mathbf{Alg}(\Sigma)$ for all (single-sorted) algebraic signatures $\Sigma \in |\mathbf{AlgSig}|$

  Generalise to many-sorted signatures

  - $\mathbf{T}_\Sigma(X) = T_\Sigma(X)$ for all $X \in |\mathbf{Set}|$
  - $\mathbf{T}_\Sigma(f) = f^\# : T_\Sigma(X) \to T_\Sigma(X')$ for all functions $f : X \to X'$
- *diagonal functors*: $\Delta^G_\mathbf{K} : \mathbf{K} \to \mathbf{Diag}^G_\mathbf{K}$ for any graph G with nodes $N = |G|_{nodes}$ and edges $E = |G|_{edges}$, and category $\mathbf{K}$
  - $\Delta^G_\mathbf{K}(A) = D^A$, where D^A is the "constant" diagram, with $D^A_n = A$ for all $n \in N$ and $D^A_e = id_A$ for all $e \in E$
  - $\Delta^G_\mathbf{K}(f) = \mu^f : D^A \to D^B$, for all $f : A \to B$, where $\mu^f_n = f$ for all $n \in N$

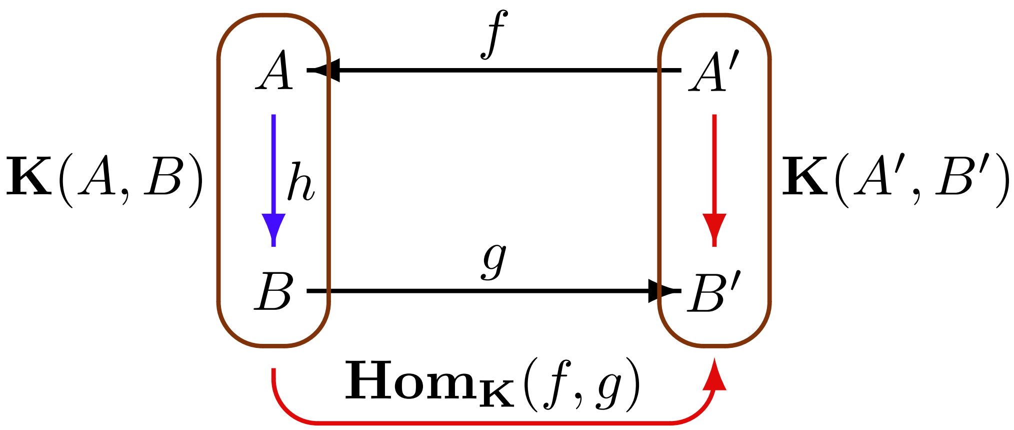
Hom-functors

Given a *locally small* category $\mathbf{K}$, define

$$\mathbf{Hom}_{\mathbf{K}} : \mathbf{K}^{op} \times \mathbf{K} \to \mathbf{Set}$$

a binary *hom-functor*, contravariant on the first argument and covariant on the second argument, as follows:

- $\mathbf{Hom}_{\mathbf{K}}(\langle A, B \rangle) = \mathbf{K}(A, B)$, for all $\langle A, B \rangle \in |\mathbf{K}^{op} \times \mathbf{K}|$, i.e., $A, B \in |\mathbf{K}|$
- $\mathbf{Hom}_{\mathbf{K}}(\langle f, g \rangle) : \mathbf{K}(A, B) \to \mathbf{K}(A', B')$, for $\langle f, g \rangle : \langle A, B \rangle \to \langle A', B' \rangle$ in $\mathbf{K}^{op} \times \mathbf{K}$, i.e., $f : A' \to A$ and $g : B \to B'$ in $\mathbf{K}$, as a function given by $\mathbf{Hom}_{\mathbf{K}}(\langle f, g \rangle)(h) = f;g;h$.



Also: $\mathbf{Hom}_{\mathbf{K}}(A, \_) : \mathbf{K} \to \mathbf{Set}$
$\mathbf{Hom}_{\mathbf{K}}(\_, B) : \mathbf{K}^{op} \to \mathbf{Set}$

Functors preserve. . .

- Check whether functors preserve:
  - monomorphisms
  - epimorphisms
  - (co)retractions
  - isomorphisms
  - (co)cones
  - (co)limits
  - . . .
- A functor is (finitely) continuous if it preserves all existing (finite) limits.
  Which of the above functors are (finitely) continuous?

Dualise!

Functors compose...

Given two functors $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ and $\mathbf{G} : \mathbf{K}' \to \mathbf{K}''$, their *composition* $\mathbf{F};\mathbf{G} : \mathbf{K} \to \mathbf{K}''$ is defined as expected:

- $(\mathbf{F};\mathbf{G})(A) = \mathbf{G}(\mathbf{F}(A))$ for all $A \in |\mathbf{K}|$
- $(\mathbf{F};\mathbf{G})(f) = \mathbf{G}(\mathbf{F}(f))$ for all $f : A \to B$ in $\mathbf{K}$

Cat, *the category of (sm)all categories*

- objects: (sm)all categories
- morphisms: functors between them
- composition: as above

Characterise isomorphisms in **Cat**

Define products, terminal objects, equalisers and pullback in **Cat**

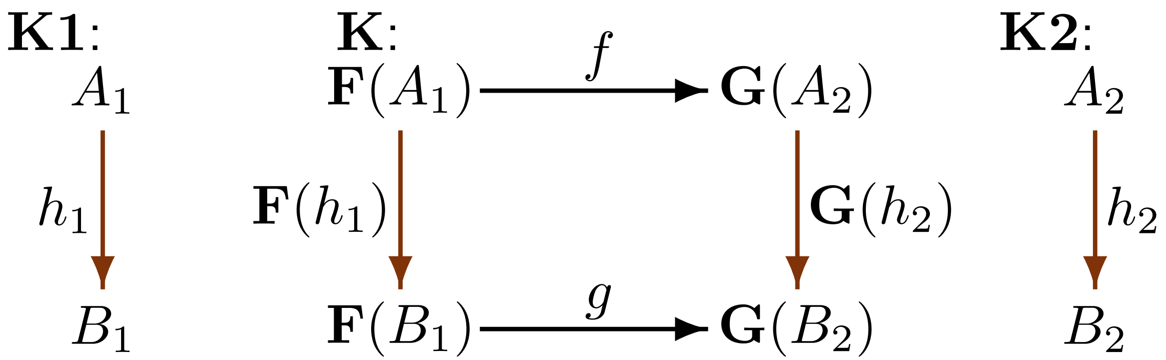
Try to define their duals

Comma categories

Given two functors with a common target, $\mathbf{F} : \mathbf{K1} \to \mathbf{K}$ and $\mathbf{G} : \mathbf{K2} \to \mathbf{K}$, define their *comma category*

$$(\mathbf{F}, \mathbf{G})$$

- objects: triples $\langle A_1, f : \mathbf{F}(A_1) \to \mathbf{G}(A_2), A_2 \rangle$, where $A_1 \in |\mathbf{K1}|$, $A_2 \in |\mathbf{K2}|$, and $f : \mathbf{F}(A_1) \to \mathbf{G}(A_2)$ in $\mathbf{K}$
- morphisms: a morphism in $(\mathbf{F}, \mathbf{G})$ is any pair $\langle h_1, h_2 \rangle : \langle A_1, f : \mathbf{F}(A_1) \to \mathbf{G}(A_2), A_2 \rangle \to \langle B_1, g : \mathbf{F}(B_1) \to \mathbf{G}(B_2), B_2 \rangle$, where $h_1 : A_1 \to B_1$ in $\mathbf{K1}$, $h_2 : A_2 \to B_2$ in $\mathbf{K2}$, and $\mathbf{F}(h_1);g = f;\mathbf{G}(h_2)$ in $\mathbf{K}$.
- composition: component-wise



Examples

- The category of graphs as a comma category:

$$\mathbf{Graph} = (\mathbf{Id}_{\mathbf{Set}}, \mathbf{CP})$$

where $\mathbf{CP} : \mathbf{Set} \to \mathbf{Set}$ is the (Cartesian) product functor ($\mathbf{CP}(X) = X \times X$ and $\mathbf{CP}(f)(\langle x, x' \rangle) = \langle f(x), f(x') \rangle$). Hint: write objects of this category as $\langle E, \langle source, target \rangle : E \to N \times N, N \rangle$

- The category of algebraic signatures as a comma category:

$$\mathbf{AlgSig} = (\mathbf{Id}_{\mathbf{Set}}, (\_)^+)$$

where $(\_)^+ : \mathbf{Set} \to \mathbf{Set}$ is the non-empty list functor ($(X)^+$ is the set of all non-empty lists of elements from X, $(f)^+(\langle x_1, \ldots, x_n \rangle) = \langle f(x_1), \ldots, f(x_n) \rangle$). Hint: write objects of this category as $\langle \Omega, \langle arity, sort \rangle : \Omega \to S^+, S \rangle$

Define $\mathbf{K}^{\to}$, $\mathbf{K}{\downarrow}A$ as comma categories. The same for $\mathbf{Alg}(\Sigma)$.

Cocompleteness of comma categories

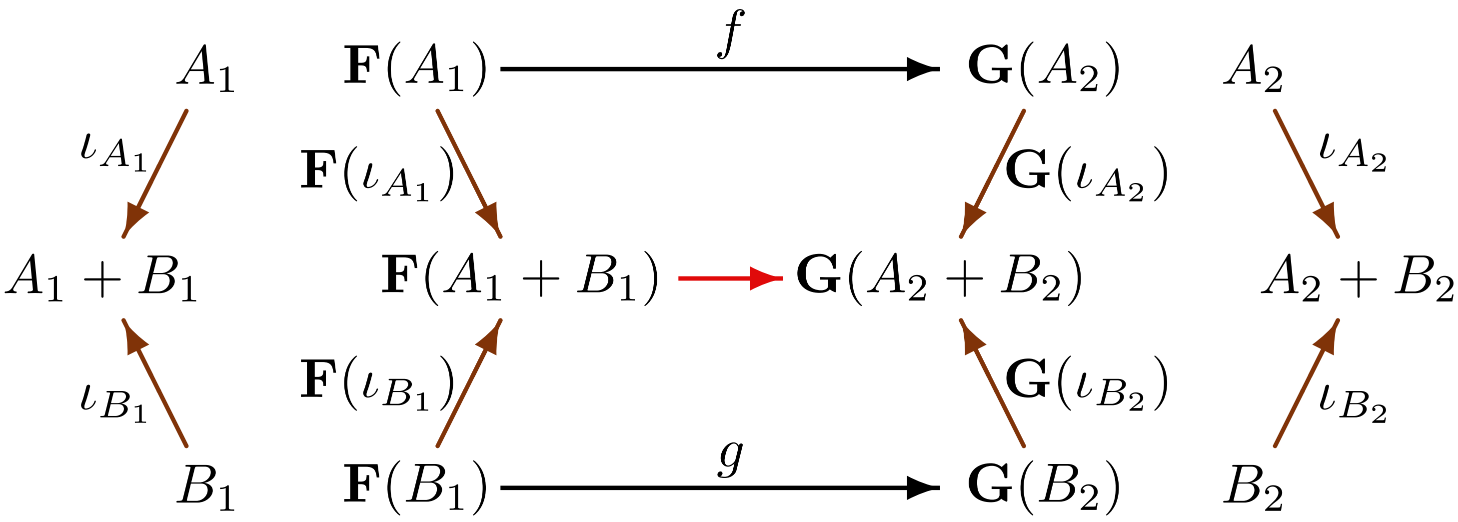
Fact: *If* $\mathbf{K1}$ *and* $\mathbf{K2}$ *are (finitely) cocomplete categories,* $\mathbf{F} : \mathbf{K1} \to \mathbf{K}$ *is a (finitely) cocontinuous functor, and* $\mathbf{G} : \mathbf{K2} \to \mathbf{K}$ *is a functor then the comma category* $(\mathbf{F}, \mathbf{G})$ *is (finitely) cocomplete.*

Proof (idea):

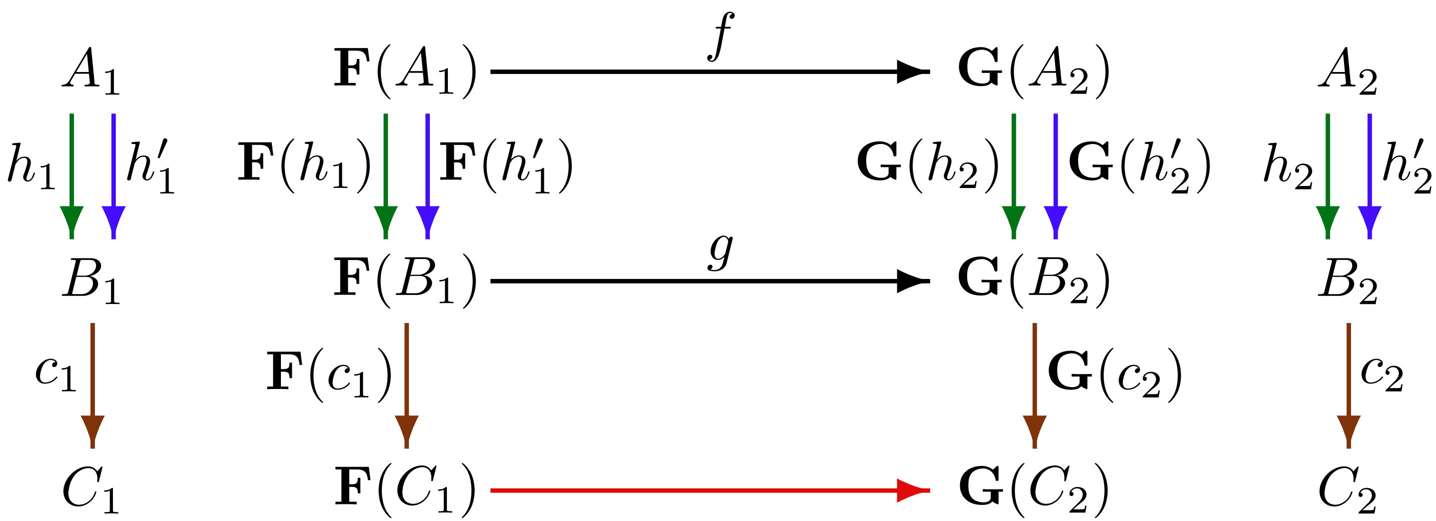
Construct coproducts and coequalisers in $(\mathbf{F}, \mathbf{G})$, using the corresponding constructions in $\mathbf{K1}$ and $\mathbf{K2}$, and cocontinuity of $\mathbf{F}$.

State and prove the dual fact,
concerning completeness of comma categories

Coproducts:



Coequalisers:



Indexed categories

An *indexed category* is a functor $\mathcal{C} : \mathbf{Ind}^{op} \to \mathbf{Cat}$.

Standard example: $\mathbf{Alg} : \mathbf{AlgSig}^{op} \to \mathbf{Cat}$

The Grothendieck construction: Given $\mathcal{C} : \mathbf{Ind}^{op} \to \mathbf{Cat}$, define a category $\mathbf{Flat}(\mathcal{C})$:

- objects: $\langle i, A \rangle$ for all $i \in |\mathbf{Ind}|$, $A \in |\mathcal{C}(i)|$
- morphisms: a morphism from $\langle i, A \rangle$ to $\langle j, B \rangle$, $\langle \sigma, f \rangle : \langle i, A \rangle \to \langle j, B \rangle$, consists of a morphism $\sigma : i \to j$ in $\mathbf{Ind}$ and a morphism $f : A \to \mathcal{C}(\sigma)$ in $\mathcal{C}(i)$
- composition: given $\langle \sigma, f \rangle : \langle i, A \rangle \to \langle i', A' \rangle$ and $\langle \sigma', f' \rangle : \langle i', A' \rangle \to \langle i'', A'' \rangle$, their composition in $\mathbf{Flat}(\mathcal{C})$, $\langle \sigma, f \rangle ; \langle \sigma', f' \rangle : \langle i, A \rangle \to \langle i'', A'' \rangle$, is given by

$$\langle \sigma, f \rangle ; \langle \sigma', f' \rangle = \langle \sigma ; \sigma', f ; \mathcal{C}(\sigma)(f') \rangle$$

Fact: *If $\mathbf{Ind}$ is complete, $\mathcal{C}(i)$ are complete for all $i \in |\mathbf{Ind}|$, and $\mathcal{C}(\sigma)$ are continuous for all $\sigma : i \to j$ in $\mathbf{Ind}$, then $\mathbf{Flat}(\mathcal{C})$ is complete.*

Try to formulate and prove a theorem concerning cocompleteness of $\mathbf{Flat}(\mathcal{C})$

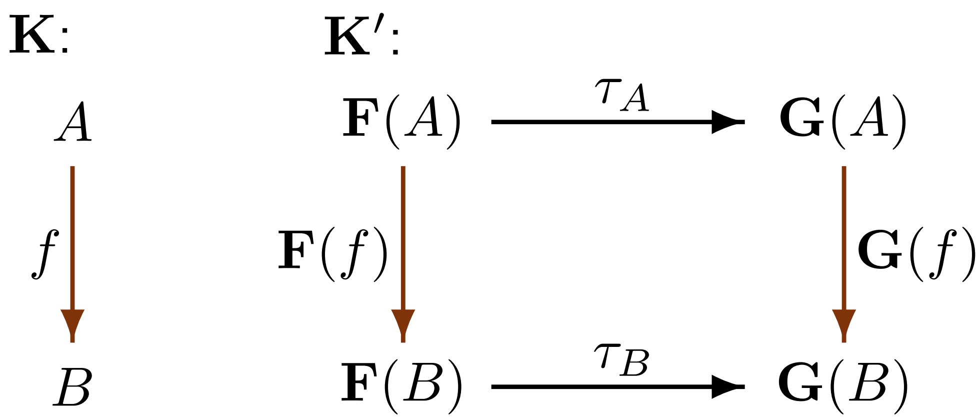
Natural transformations

Given two parallel functors $\mathbf{F}, \mathbf{G} : \mathbf{K} \to \mathbf{K}'$, a *natural transformation* from $\mathbf{F}$ to $\mathbf{G}$

$$\boxed{\tau : \mathbf{F} \to \mathbf{G}}$$

is a family $\tau = \langle \tau_a : \mathbf{F}(A) \to \mathbf{G}(A) \rangle_{A \in |\mathbf{K}|}$ of $\mathbf{K}'$-morphisms such that for all $f : A \to B$ in $\mathbf{K}$ (with $A, B \in |\mathbf{K}|$), $\boxed{\tau_A;\mathbf{G}(f) = \mathbf{F}(f);\tau_B}$

Then, τ is a *natural isomorphism* if for all $A \in |\mathbf{K}|$, τ_A is an isomorphism.



Examples

- *identity transformations*: $id_{\mathbf{F}} : \mathbf{F} \to \mathbf{F}$, where $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$, for all objects $A \in |\mathbf{K}|$, $(id_{\mathbf{F}})_A = id_A : \mathbf{F}(A) \to \mathbf{F}(A)$
- *singleton functions*: $sing : \mathbf{Id_{Set}} \to \mathbf{P}$ $(: \mathbf{Set} \to \mathbf{Set})$, where for all $X \in |\mathbf{Set}|$, $sing_X : X \to \mathbf{P}(X)$ is a function defined by $sing_X(x) = \{x\}$ for $x \in X$
- *singleton-list functions*: $sing^{\mathbf{List}} : \mathbf{Id_{Set}} \to |\mathbf{List}|$ $(: \mathbf{Set} \to \mathbf{Set})$, where $|\mathbf{List}| = \mathbf{List};|\_| : \mathbf{Set} \to \mathbf{Monoid} \to \mathbf{Set}$, and for all $X \in |\mathbf{Set}|$, $sing^{\mathbf{List}}_X : X \to X^*$ is a function defined by $sing^{\mathbf{List}}_X(x) = \langle x \rangle$ for $x \in X$
- *append functions*: $append : |\mathbf{List}|;\mathbf{CP} \to |\mathbf{List}|$ $(: \mathbf{Set} \to \mathbf{Set})$, where for all $X \in |\mathbf{Set}|$, $append_X : (X^* \times X^*) \to X^*$ is the usual append function (list concatenation) polymorphic functions between algebraic types

Polymorphic functions

Work out the following generalisation of the last two examples:

- for each algebraic type scheme $\forall\alpha_1 \ldots \alpha_n \cdot T$, built in SML using at least products and algebraic data types (no function types though), define the corresponding functor $[\![T]\!] : \mathbf{Set}^n \to \mathbf{Set}$
- argue that in a representative subset of SML, for each polymorphic expression $E : \forall\alpha_1 \ldots \alpha_n \cdot T \to T'$ its semantics is a natural transformation $[\![E]\!] : [\![T]\!] \to [\![T']\!]$

Theorems for free!
(see Wadler 89)

Yoneda lemma

Given a locally small category $\mathbf{K}$, functor $\mathbf{F} : \mathbf{K} \longrightarrow \mathbf{Set}$ and object $A \in |\mathbf{K}|$:

$$Nat(\mathbf{Hom_K}(A, \_), \mathbf{F}) \cong \mathbf{F}(A)$$

natural transformations from $\mathbf{Hom_K}(A, \_)$ to $\mathbf{F}$, between functors from $\mathbf{K}$ to $\mathbf{Set}$, are given exactly by the elements of the set $\mathbf{F}(A)$

EXERCISES:

- Dualise: for $\mathbf{G} : \mathbf{K}^{op} \longrightarrow \mathbf{Set}$,

$$Nat(\mathbf{Hom_K}(\_, A), \mathbf{G}) \cong \mathbf{G}(A)$$

.

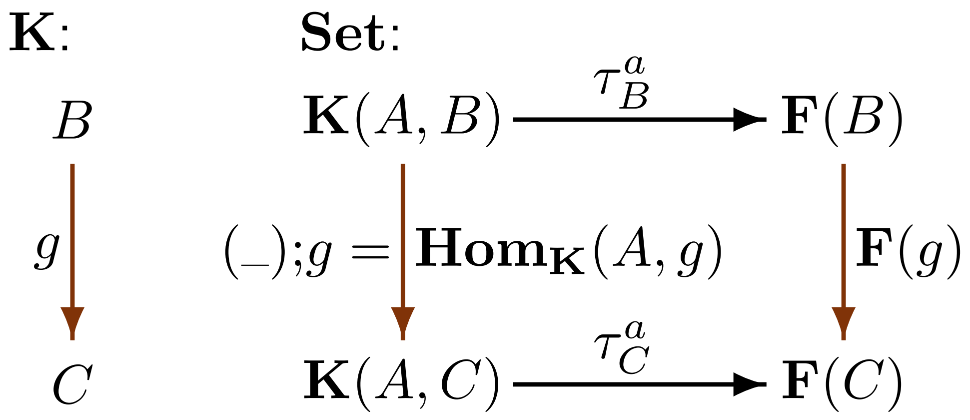
- Characterise all natural transformations from $\mathbf{Hom_K}(A, \_)$ to $\mathbf{Hom_K}(B, \_)$, for all objects $A, B \in |\mathbf{K}|$.

Proof

- For $a \in \mathbf{F}(A)$, define $\tau^a : \mathbf{Hom_K}(A, \_) \to \mathbf{F}$, as the family of functions $\tau^a_B : \mathbf{K}(A,B) \to \mathbf{F}(B)$ given by $\tau^a_B(f) = \mathbf{F}(f)(a)$ for $f : A \to B$ in $\mathbf{K}$.

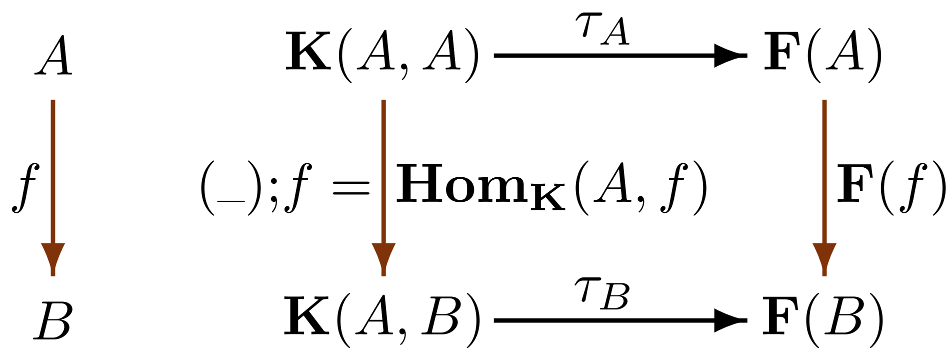
  This is a natural transformation, since for $g : B \to C$ and then $f : A \to B$,

  $$\mathbf{F}(g)(\tau^a_B(f)) = \mathbf{F}(g)(\mathbf{F}(f)(a))$$
  $$= \mathbf{F}(f;g)(a) = \tau^a_C(f;g)$$
  $$= \tau^a_C(\mathbf{Hom_K}(A,g)(f))$$

  Then $\tau^a_A(id_A) = a$, and so for distinct $a, a' \in \mathbf{F}(A)$, τ^a and $\tau^{a'}$ differ.

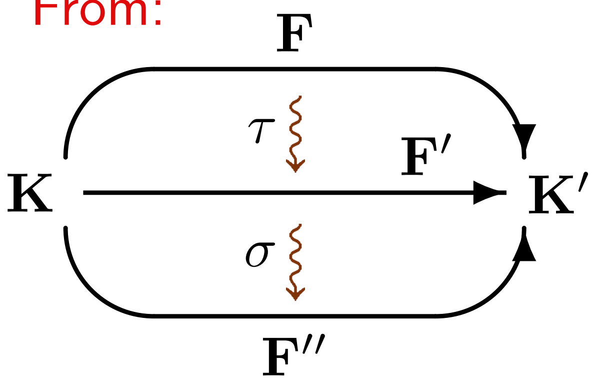
- If $\tau : \mathbf{Hom_K}(A, \_) \to \mathbf{F}$ is a natural transformation then $\tau = \tau^a$, where we put $a = \tau_A(id_A)$, since for $B \in |\mathbf{K}|$ and $f : A \to B$, $\tau_B(f) = \mathbf{F}(f)(\tau_A(id_A))$ by naturality of τ:

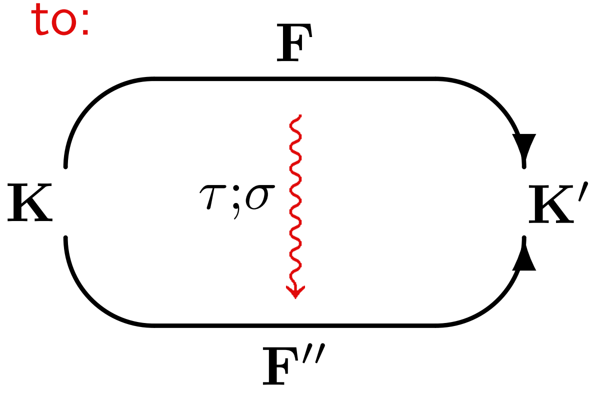
Compositions

vertical composition:

From:

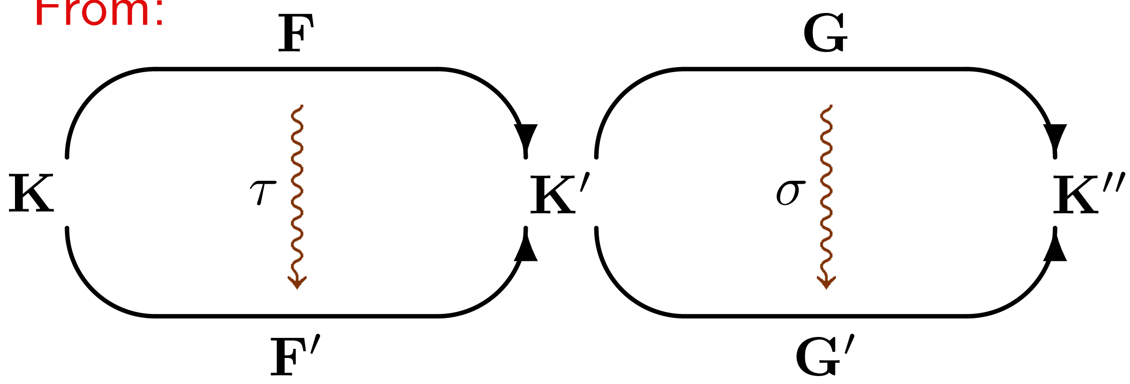


to:

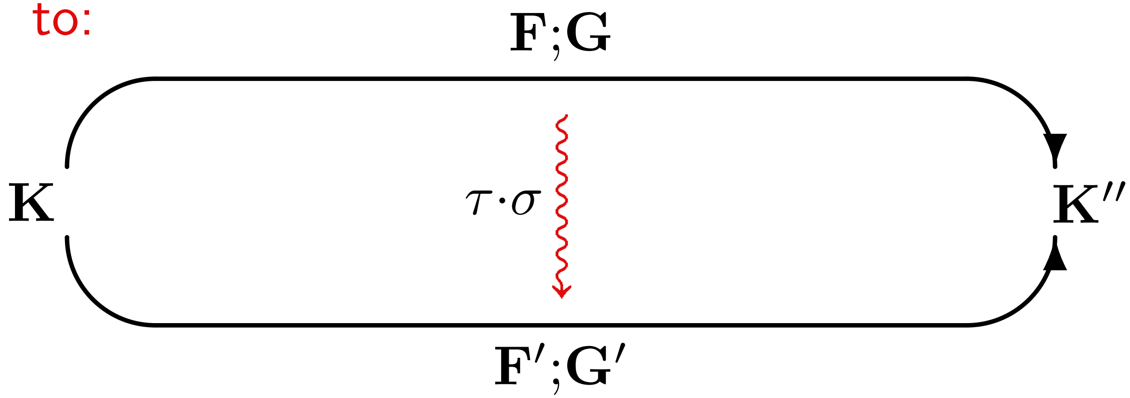


horizontal composition:

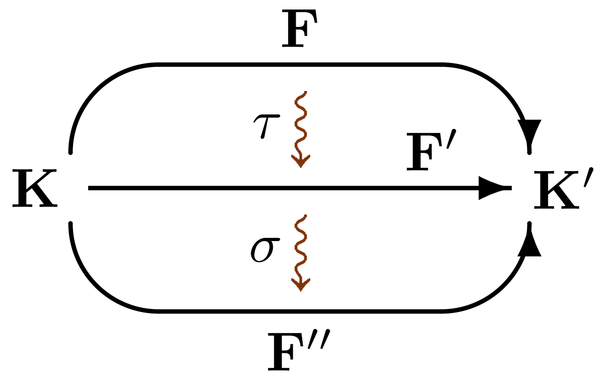
From:



to:



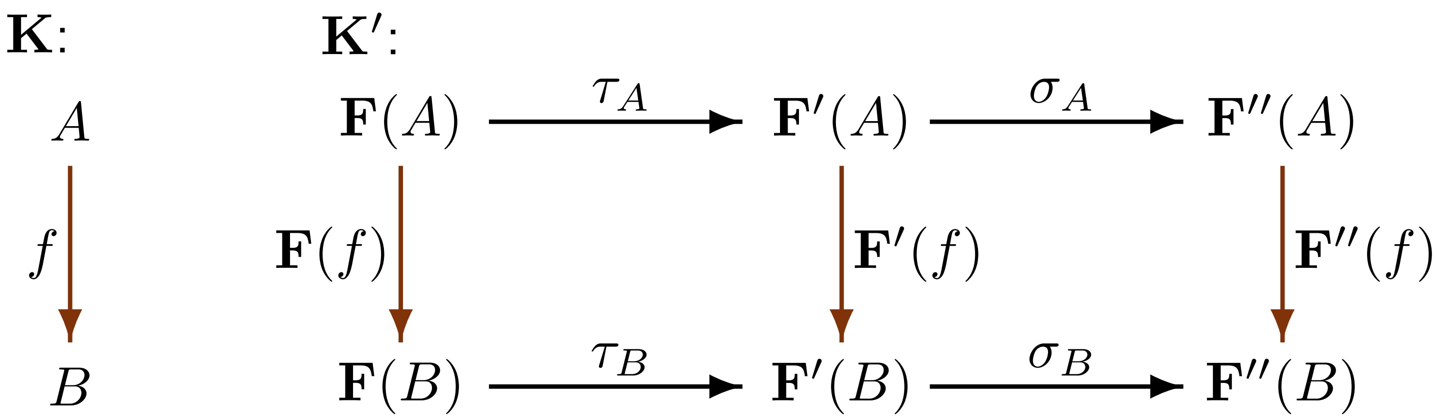
Vertical composition



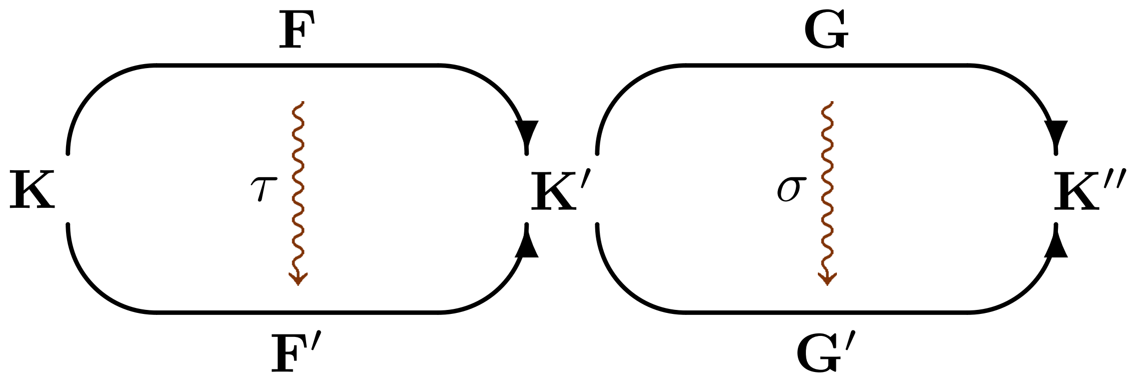
The *vertical composition* of natural transformations $\tau : \mathbf{F} \to \mathbf{F}'$ and $\sigma : \mathbf{F}' \to \mathbf{F}''$ between parallel functors $\mathbf{F}, \mathbf{F}', \mathbf{F}'' : \mathbf{K} \to \mathbf{K}'$

$$\tau;\sigma : \mathbf{F} \to \mathbf{F}''$$

is a natural transformation given by $(\tau;\sigma)_A = \tau_A;\sigma_A$ for all $A \in |\mathbf{K}|$.



Horizontal composition



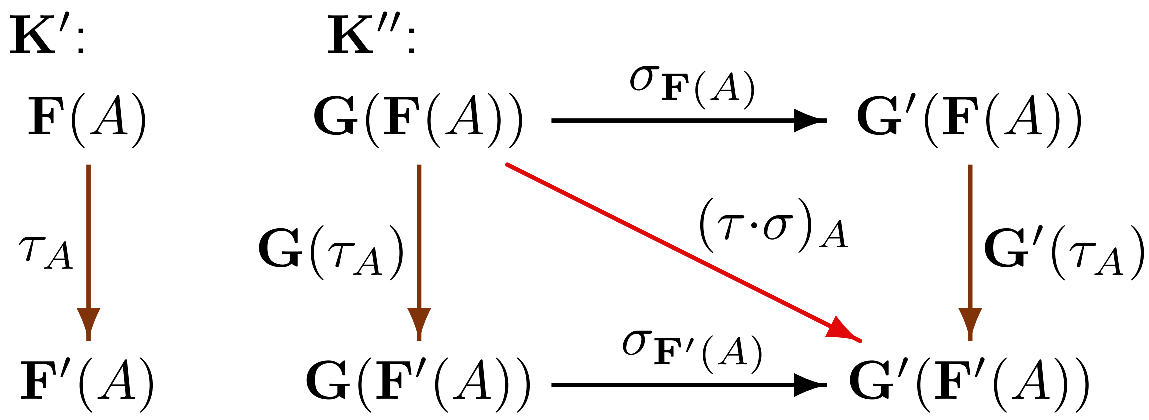
The *horizontal composition* of natural transformations $\tau : \mathbf{F} \to \mathbf{F}'$ and $\sigma : \mathbf{G} \to \mathbf{G}'$ between composable pairs of parallel functors $\mathbf{F}, \mathbf{F}' : \mathbf{K} \to \mathbf{K}'$, $\mathbf{G}, \mathbf{G}' : \mathbf{K}' \to \mathbf{K}''$

$$\tau{\cdot}\sigma : \mathbf{F};\mathbf{G} \to \mathbf{F}';\mathbf{G}'$$

is a natural transformation given by $(\tau{\cdot}\sigma)_A = \mathbf{G}(\tau_A);\sigma_{\mathbf{F}'(A)} = \sigma_{\mathbf{F}(A)};\mathbf{G}'(\tau_A)$ for all $A \in |\mathbf{K}|$.

Multiplication by functor:

- $\tau{\cdot}\mathbf{G} = \tau{\cdot}id_{\mathbf{G}} : \mathbf{F};\mathbf{G} \to \mathbf{F}';\mathbf{G}$, i.e., $(\tau{\cdot}\mathbf{G})_A = \mathbf{G}(\tau_A)$
- $\mathbf{F}{\cdot}\sigma = id_{\mathbf{F}}{\cdot}\sigma : \mathbf{F};\mathbf{G} \to \mathbf{F};\mathbf{G}'$, i.e., $(\mathbf{F}{\cdot}\sigma)_A = \sigma_{\mathbf{F}(A)}$



Show that indeed, $\tau{\cdot}\sigma$ is a natural transformation

Functor categories

Given two categories $\mathbf{K}, \mathbf{K}'$, define the *category of functors from* $\mathbf{K}'$ *to* $\mathbf{K}$, $\mathbf{K}^{\mathbf{K}'}$, as follows:

- objects: functors from $\mathbf{K}'$ to $\mathbf{K}$
- morphisms: natural transformations between them
- composition: vertical composition of the natural transformations

Exercises:

- View the category of S-sorted sets, $\mathbf{Set}^S$, as a functor category
- Show how any functor $\mathbf{F} : \mathbf{K}'' \to \mathbf{K}'$ induces a functor $(\mathbf{F};\_) : \mathbf{K}^{\mathbf{K}'} \to \mathbf{K}^{\mathbf{K}''}$
- Check whether $\mathbf{K}^{\mathbf{K}'}$ is (finitely) (co)complete whenever $\mathbf{K}$ is so.
- Check when $(\mathbf{F};\_) : \mathbf{K}^{\mathbf{K}'} \to \mathbf{K}^{\mathbf{K}''}$ is (finitely) (co)continuous, for a given functor $\mathbf{F} : \mathbf{K}'' \to \mathbf{K}'$

Diagrams as functors

Each diagram D over graph G in category $\mathbf{K}$ yields a functor $\mathbf{F}_D : \mathbf{Path}(G) \to \mathbf{K}$ given by:

- $\mathbf{F}_D(n) = D_n$, for all nodes $n \in |G|_{nodes}$
- $\mathbf{F}_D(n_0 e_1 n_1 \ldots n_{k-1} e_k n_k) = D_{e_1};\ldots;D_{e_k}$, for paths $n_0 e_1 n_1 \ldots n_{k-1} e_k n_k$ in G

Moreover:

- for distinct diagrams D and D' of shape G, $\mathbf{F}_D$ and $\mathbf{F}_{D'}$ are different
- all functors from $\mathbf{Path}(G)$ to $\mathbf{K}$ are given by diagrams over G

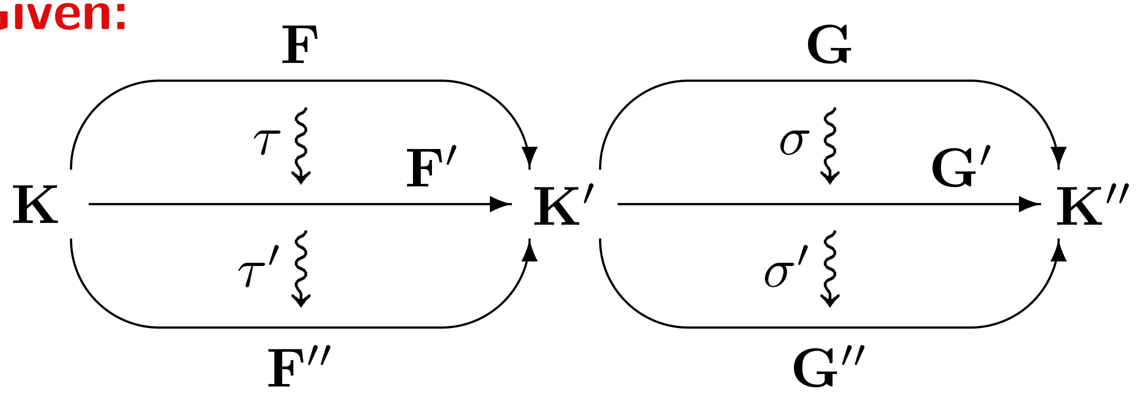
Diagram morphisms $\mu : D \to D'$ between diagrams of the same shape G are exactly natural transformations $\mu : \mathbf{F}_D \to \mathbf{F}_{D'}$.

$$\mathbf{Diag}^G_{\mathbf{K}} \cong \mathbf{K}^{\mathbf{Path}(G)}$$

Diagrams are functors from small (shape) categories

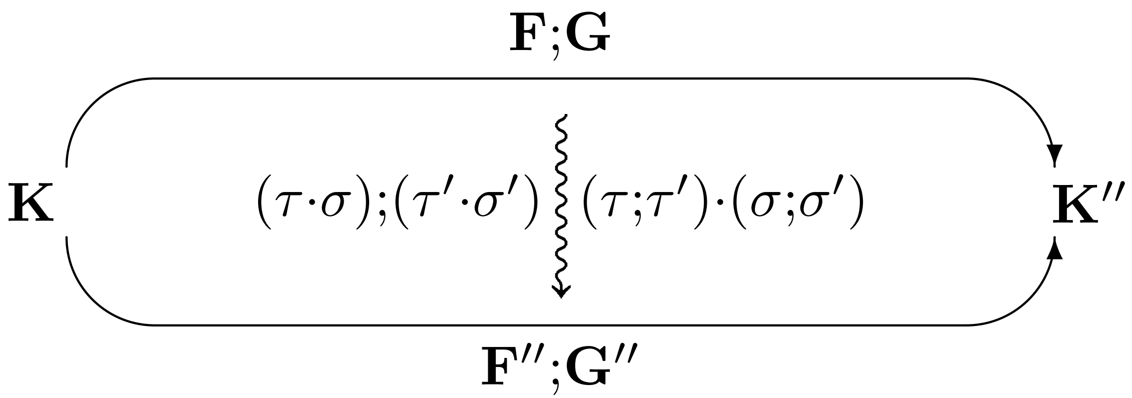
Double law

Given:



then:

$$(\tau \cdot \sigma);(\tau' \cdot \sigma') = (\tau;\tau') \cdot (\sigma;\sigma')$$



This holds in $\mathbf{Cat}$, which is a paradigmatic example of a two-category.

A category $\mathbf{K}$ is a *two-category* when for all objects $A, B \in |\mathbf{K}|$, $\mathbf{K}(A, B)$ is again a category, with *1-morphisms* (the usual $\mathbf{K}$-morphisms) as objects and *2-morphisms* between them. Those 2-morphisms compose vertically (in the categories $\mathbf{K}(A, B)$) and horizontally, subject to the double law as stated here.

In two-category $\mathbf{Cat}$, we have $\mathbf{Cat}(\mathbf{K}', \mathbf{K}) = \mathbf{K}^{\mathbf{K}'}$.

Equivalence of categories

- Two categories $\mathbf{K}$ and $\mathbf{K}'$ are *isomorphic* if there are functors $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ and $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ such that $\mathbf{F};\mathbf{G} = \mathbf{Id}_{\mathbf{K}}$ and $\mathbf{G};\mathbf{F} = \mathbf{Id}_{\mathbf{K}'}$.
- Two categories $\mathbf{K}$ and $\mathbf{K}'$ are *equivalent* if there functors $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ and $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ and natural isomorphisms $\eta : \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$ and $\epsilon : \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$.
- A category is *skeletal* if any two isomorphic objects are identical.
- A *skeleton* of a category is any of its maximal skeletal subcategory.

Fact: *Two categories are equivalent iff they have isomorphic skeletons.*

> *All "categorical" properties are preserved under equivalence of categories*

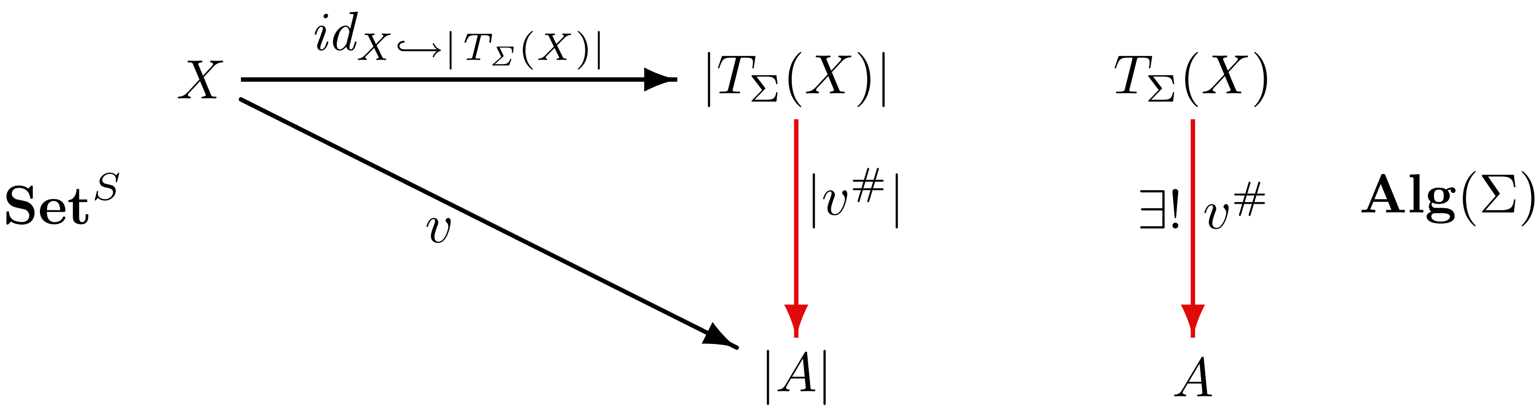
Adjunctions

Recall:

Term algebras

Fact: *For any S-sorted set X of variables, Σ-algebra A and valuation $v : X \to |A|$, there is a unique Σ-homomorphism $v^{\#} : T_\Sigma(X) \to A$ that extends v, so that*

$$id_{X \hookrightarrow |T_\Sigma(X)|};v^{\#} = v$$



Free objects

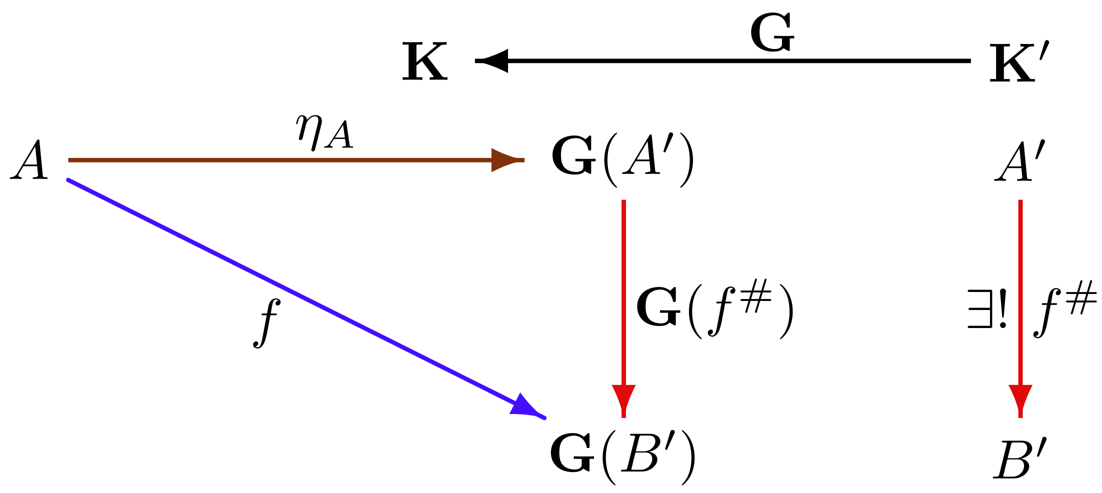
Consider any functor $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$

Definition: *Given an object $A \in |\mathbf{K}|$, a free object over A w.r.t. $\mathbf{G}$ is a $\mathbf{K}'$-object $A' \in |\mathbf{K}'|$ together with a $\mathbf{K}$-morphism $\eta_A : A \to \mathbf{G}(A')$ (called unit morphism) such that given any $\mathbf{K}'$-object $B' \in |\mathbf{K}'|$ with $\mathbf{K}$-morphism $f : A \to \mathbf{G}(B')$, for a unique $\mathbf{K}'$-morphism $f^{\#} : A' \to B'$ we have*

$$\eta_A;\mathbf{G}(f^{\#}) = f$$

Paradigmatic example:

Term algebra $T_\Sigma(X)$ with unit $id_{X \hookrightarrow |T_\Sigma(X)|} : X \to |T_\Sigma(X)|$ is free over $X \in |\mathbf{Set}^S|$ w.r.t. the carrier functor $|\_| : \mathbf{Alg}(\Sigma) \to \mathbf{Set}^S$



Examples

- Consider inclusion $i : \mathbf{Int} \hookrightarrow \mathbf{Real}$, viewing $\mathbf{Int}$ and $\mathbf{Real}$ as (thin) categories, and i as a functor between them. For any real $r \in \mathbf{Real}$, the ceiling of r, $\lceil r \rceil \in \mathbf{Int}$ is free over r w.r.t. i.

What about free objects w.r.t. the inclusion of rationals into reals?

- For any set $X \in |\mathbf{Set}|$, the "free monoid" $\mathbf{List}(X) = \langle X^*, \frown, \epsilon \rangle$ is free over X w.r.t. $|\_| : \mathbf{Monoid} \to \mathbf{Set}$.
- For any graph $G \in |\mathbf{Graph}|$, the category of its paths, $\mathbf{Path}(G) \in |\mathbf{Cat}|$, is free over G w.r.t. the graph functor $G : \mathbf{Cat} \to \mathbf{Graph}$.
- Discrete topologies, completion of metric spaces, free groups, ideal completion of partial orders, ideal completion of free partial algebras, ...

Makes precise these and other similar examples

Indicate unit morphisms!

Free equational models

- Recall: for any algebraic signature $\Sigma = \langle S, \Omega \rangle$, term algebra $T_\Sigma(X)$ is free over $X \in |\mathbf{Set}^S|$ w.r.t. the carrier functor $|\_| : \mathbf{Alg}(\Sigma) \to \mathbf{Set}^S$.
- For any set of Σ-equations Φ, for any set $X \in |\mathbf{Set}^S|$, there exist a model $\mathbf{F}_\Phi(X) \in Mod(\Phi)$ that is free over X w.r.t. the carrier functor $|\_| : \mathbf{Mod}(\langle \Sigma, \Phi \rangle) \to \mathbf{Set}^S$, where $\mathbf{Mod}(\langle \Sigma, \Phi \rangle)$ is the full subcategory of $\mathbf{Alg}(\Sigma)$ given by the models of Φ.
- For any algebraic signature morphism $\sigma : \Sigma \to \Sigma'$, for any Σ-algebra $A \in |\mathbf{Alg}(\Sigma)|$, there exist a Σ'-algebra $\mathbf{F}_\sigma(A) \in |\mathbf{Alg}(\Sigma')|$ that is free over A w.r.t. the reduct functor $\_|_\sigma : \mathbf{Alg}(\Sigma') \to \mathbf{Alg}(\Sigma')$.
- For any equational specification morphism $\sigma : \langle \Sigma, \Phi \rangle \to \langle \Sigma', \Phi' \rangle$, for any model $A \in Mod(\Phi)$, there exist a model $\mathbf{F}_\sigma(A) \in Mod(\Phi')$ that is free over A w.r.t. the reduct functor $\_|_\sigma : \mathbf{Mod}(\langle \Sigma, \Phi \rangle) \to \mathbf{Mod}(\langle \Sigma', \Phi' \rangle)$.

Prove the above.

Facts

Consider a functor $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$, and object $A \in |\mathbf{K}|$, and an object $A' \in |\mathbf{K}'|$ free over A w.r.t. $\mathbf{G}$ with unit $\eta_A : A \to \mathbf{G}(A')$.

- A free objects over A w.r.t. $\mathbf{G}$ the initial objects in the comma category $(\mathbf{C}_A, \mathbf{G})$, where $\mathbf{C}_A : \mathbf{1} \to \mathbf{K}$ is the constant functor.
- A free object over A w.r.t. $\mathbf{G}$, if exists, is unique up to isomorphism.
- The function $(\_)^{\#} : \mathbf{K}(A, \mathbf{G}(B')) \to \mathbf{K}'(A', B')$ is bijective for each $B' \in |\mathbf{K}'|$.
- For any morphisms $g_1, g_2 : A' \to B'$ in $\mathbf{K}'$, $g_1 = g_2$ iff $\eta_A;\mathbf{G}(g_1) = \eta_A;\mathbf{G}(g_2)$.

Colimits as free objects

Fact: *In a category $\mathbf{K}$, given a diagram D of shape $G(D)$, the colimit of D in $\mathbf{K}$ is a free object over D w.r.t. the diagonal functor $\Delta_{\mathbf{K}}^{G(D)} : \mathbf{K} \to \mathbf{Diag}_{\mathbf{K}}^{G(D)}$.*

Spell this out for initial objects, coproducts, coequalisers, and pushouts

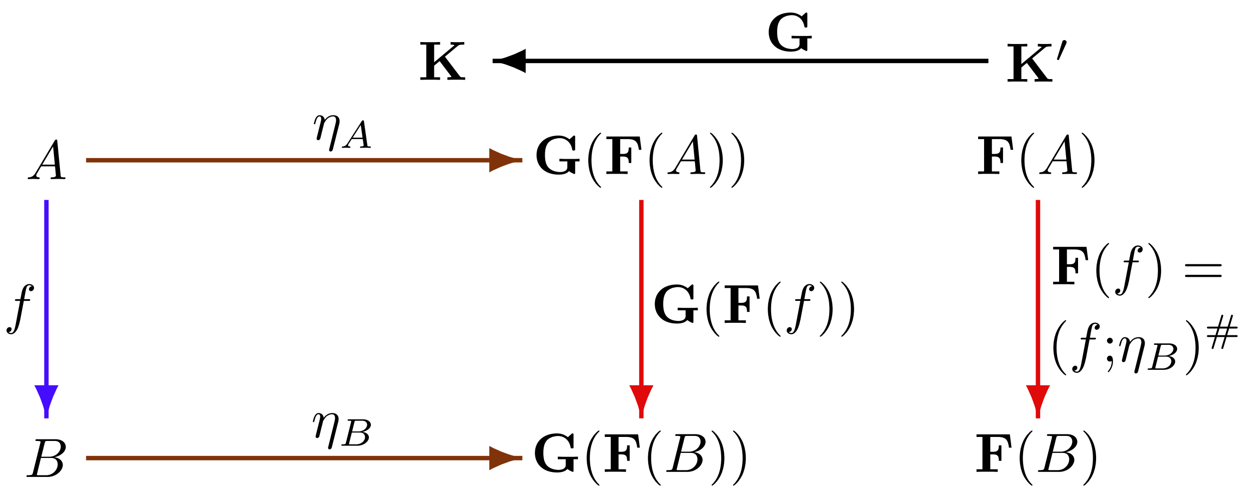
Left adjoints

Consider a functor $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$.

Fact: *Assume that for each object $A \in |\mathbf{K}|$ there is a free object over A w.r.t. $\mathbf{G}$, say $\mathbf{F}(A) \in |\mathbf{K}'|$ is free over A with unit $\eta_A : A \to \mathbf{G}(\mathbf{F}(A))$. Then the mapping:*

- $(A \in |\mathbf{K}|) \mapsto (\mathbf{F}(A) \in |\mathbf{K}'|)$
- $(f : A \to B) \mapsto ((f;\eta_B)^{\#} : \mathbf{F}(A) \to \mathbf{F}(B))$

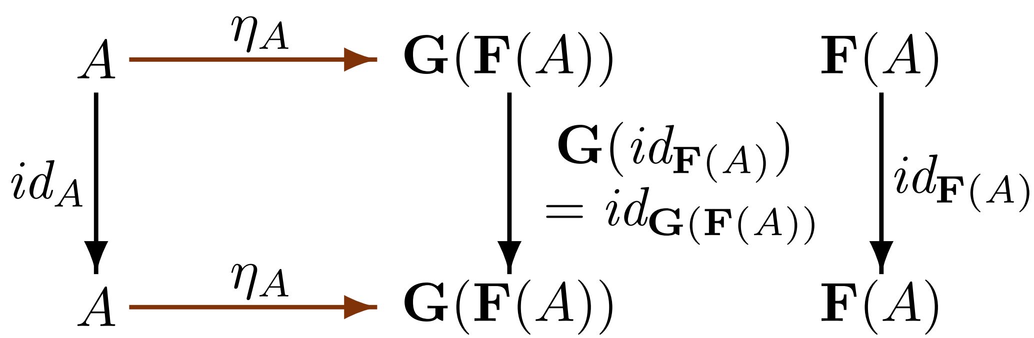
form a functor $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$. Moreover, $\eta : \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$ is a natural transformation.



Proof

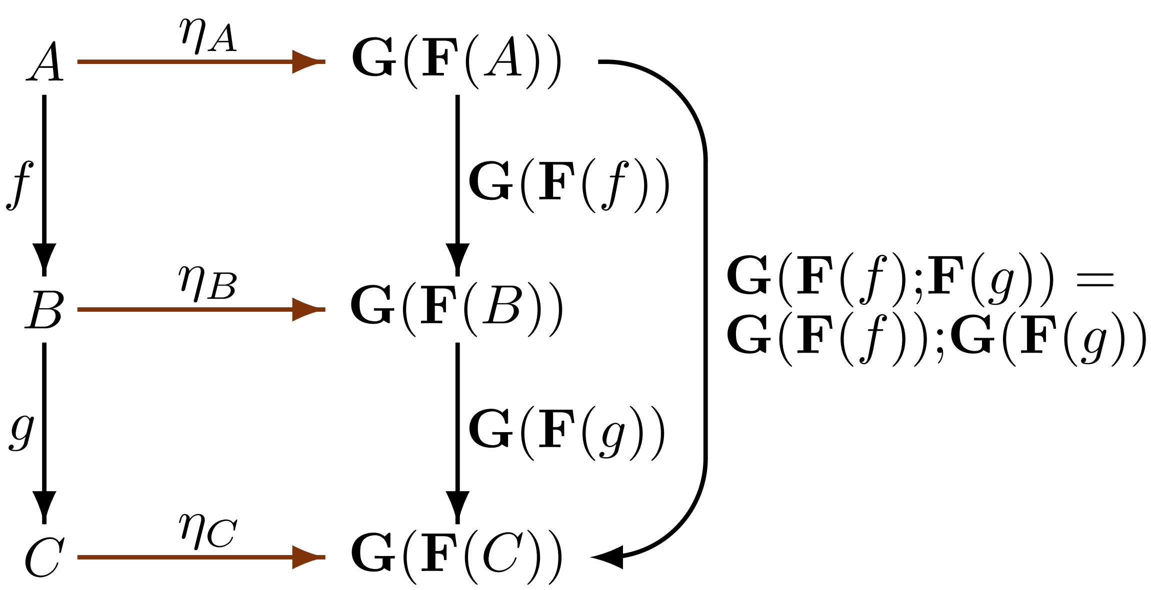
$\mathbf{F}$ preserves identities:

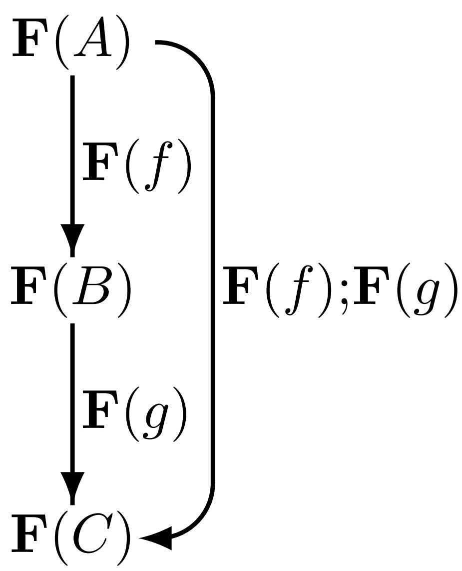
$$\mathbf{F}(id_A) = (id_A;\eta_A)^\# = id_{\mathbf{F}(A)}$$



$\mathbf{F}$ preserves composition:

$$\mathbf{F}(f;g) = (f;g;\eta_C)^\# = \mathbf{F}(f);\mathbf{F}(g)$$





Left adjoints

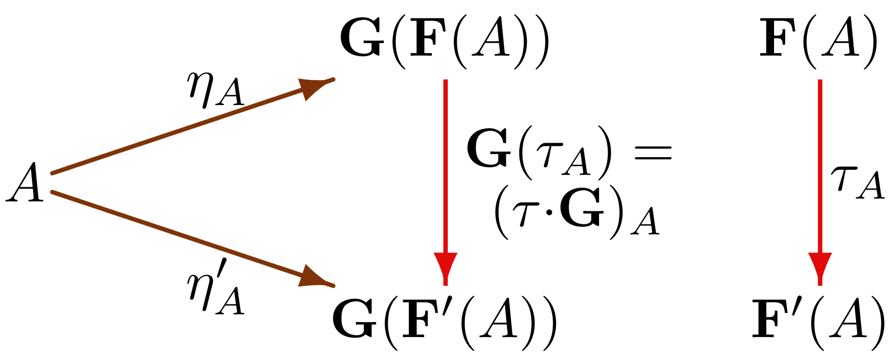
Definition: *A functor* $\mathbf{F}:\mathbf{K}\to\mathbf{K}'$ *is left adjoint to (a functor)* $\mathbf{G}:\mathbf{K}'\to\mathbf{K}$ *with unit (natural transformation)* $\eta:\mathbf{Id}_{\mathbf{K}}\to\mathbf{F};\mathbf{G}$ *if for all objects* $A\in|\mathbf{K}|$, $\mathbf{F}(A)\in|\mathbf{K}'|$ *is free over* A *with unit morphism* $\eta_A:A\to\mathbf{G}(\mathbf{F}(A))$.

Examples

- The term-algebra functor $T_\Sigma:\mathbf{Set}^S\to\mathbf{Alg}(\Sigma)$ is left adjoint to the carrier functor $|\_|:\mathbf{Alg}(\Sigma)\to\mathbf{Set}^S$, for any algebraic signature $\Sigma=\langle S,\Omega\rangle$.
- The ceiling $\lceil\_\rceil:\mathbf{Real}\to\mathbf{Int}$ is left adjoint to the inclusion $i:\mathbf{Int}\hookrightarrow\mathbf{Real}$ of integers into reals.
- The path-category functor $\mathbf{Path}:\mathbf{Graph}\to\mathbf{Cat}$ is left adjoint to the graph functor $G:\mathbf{Cat}\to\mathbf{Graph}$.
- . . . other examples given by the examples of free objects above . . .

Uniqueness of left adjoints

Fact: *A left adjoint to any functor $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$, if exists, is determined uniquely up to a natural isomorphism: if $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ and $\mathbf{F}' : \mathbf{K} \to \mathbf{K}'$ are left adjoint to $\mathbf{G}$ with units $\eta : \mathbf{Id_K} \to \mathbf{F};\mathbf{G}$ and $\eta' : \mathbf{Id_K} \to \mathbf{F}';\mathbf{G}$, respectively, then there exists a natural isomorphism $\tau : \mathbf{F} \to \mathbf{F}'$ such that $\eta;(\tau{\cdot}\mathbf{G}) = \eta'$.*



Proof: For each $A \in |\mathbf{K}|$, $\tau_A = (\eta'_A)^{\#}$.

Put also $\tau_A^{-1} = (\eta_A)^{\#'}$.

Then show:

– $\tau_A;\tau_A^{-1} = id_{\mathbf{F}(A)}$ and $\tau_A^{-1};\tau_A = id_{\mathbf{F}'(A)}$
– $\tau : \mathbf{F} \to \mathbf{F}'$ is indeed a natural transformation

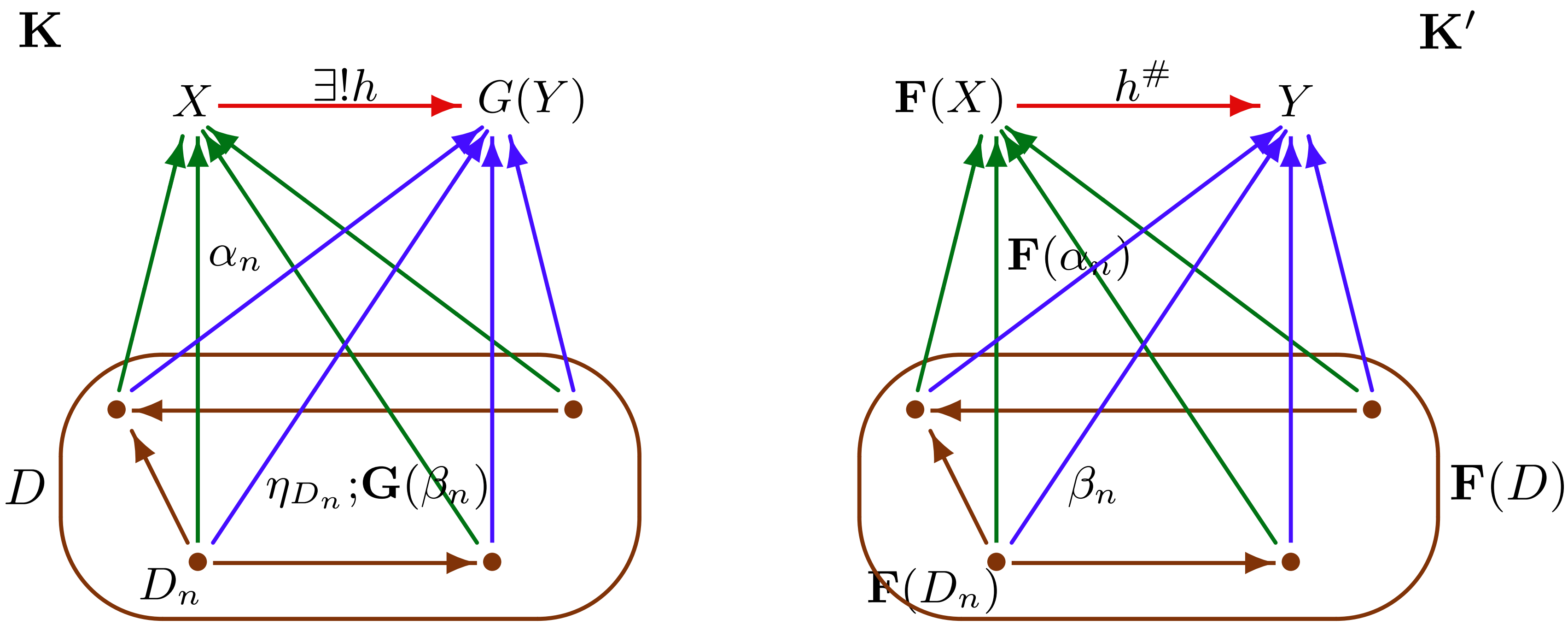
– For $f : A \to B$, $\mathbf{F}(f) = (f;\eta_B)^{\#}$.
– For $g_1, g_2 : \mathbf{F}(A) \to \bullet$, if $\eta_A;\mathbf{G}(g_1) = \eta_A;\mathbf{G}(g_2)$ then $g_1 = g_2$.

Left adjoints and colimits

Let $\mathbf{F} : \mathbf{K} \rightarrow \mathbf{K}'$ be left adjoint to $\mathbf{G} : \mathbf{K}' \rightarrow \mathbf{K}$ with unit $\eta : \mathbf{Id}_{\mathbf{K}} \rightarrow \mathbf{F};\mathbf{G}$.

Fact: $\mathbf{F}$ *is cocontinuous (preserves colimits).*

Proof:

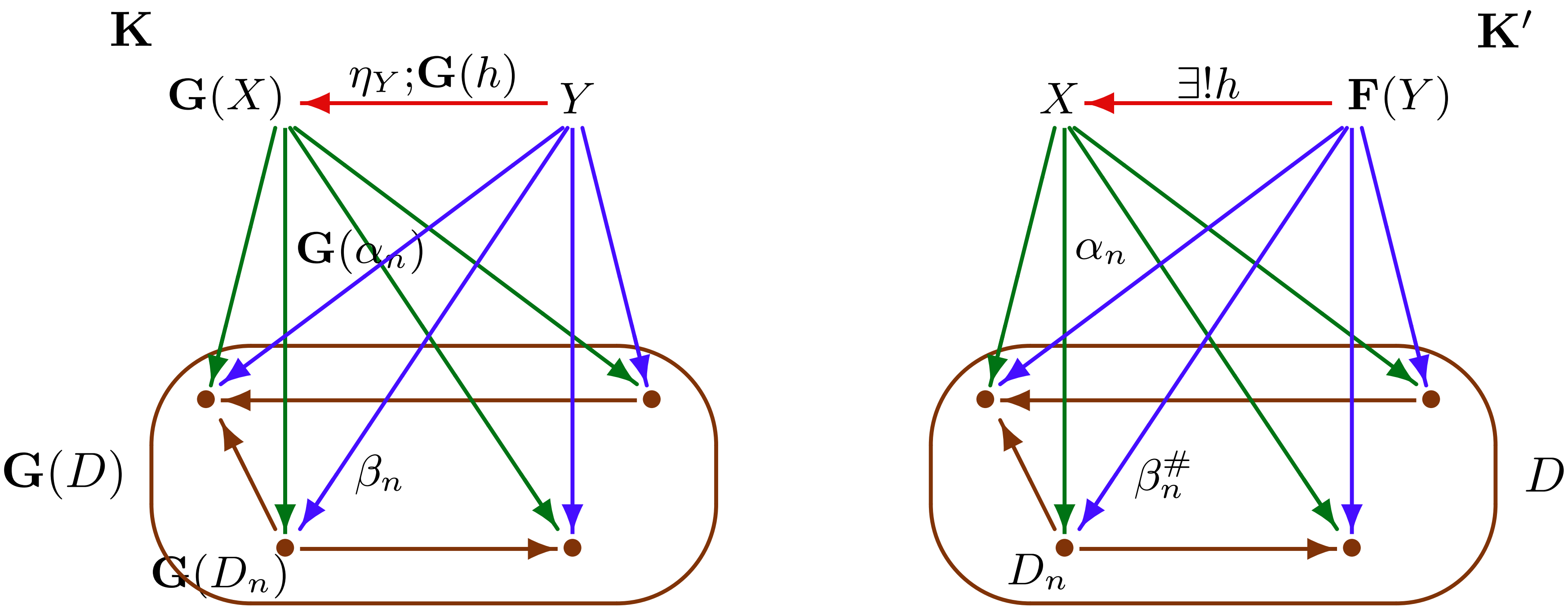


Left adjoints and limits

Let $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ be left adjoint to $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ with unit $\eta : \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$.

Fact: $\mathbf{G}$ *is continuous (preserves limits).*

Proof:



Existence of left adjoints

Fact: *Let $\mathbf{K}'$ be a locally small complete category. Then a functor $\mathbf{G}: \mathbf{K} \to \mathbf{K}'$ has a left adjoint iff*

1. *$\mathbf{G}$ is continuous, and*
2. *for each $A \in |\mathbf{K}|$ there exists a set $\{f_i : A \to \mathbf{G}(X_i) \mid i \in \mathcal{I}\}$ (of objects $X_i \in |\mathbf{K}'|$ with morphisms $f_i : A \to \mathbf{G}(X_i)$, $\imath \in \mathcal{I}$) such that for each $B \in |\mathbf{K}'|$ and $h : A \to \mathbf{G}(B)$, for some $f : X_i \to B$, $i \in \mathcal{I}$, we have $h = f_i;f$.*

Proof:

"$\Rightarrow$": Let $\mathbf{F}: \mathbf{K} \to \mathbf{K}'$ be left adjoint to $\mathbf{G}$ with unit $\eta : \mathbf{Id_K} \to \mathbf{F};\mathbf{G}$. Then 1 follows by the previous fact, and for 2 just put $\mathcal{I} = \{*\}$, $X_* = \mathbf{F}(A)$, and $f_* = \eta_A : A \to \mathbf{G}(\mathbf{F}(A))$

"$\Leftarrow$": It is enough to show that for each $A \in |\mathbf{K}|$ the comma category $(\mathbf{C}_A, \mathbf{G})$ has an initial object. Under our assumptions, $(\mathbf{C}_A, \mathbf{G})$ is complete. The rest follows by the next fact.

On the existence of initial objects

Fact: *A locally small complete category $\mathbf{K}$ has an initial object if there exists a set of objects $\mathcal{I} \subseteq |\mathbf{K}|$ such that for all $B \in |\mathbf{K}|$, for some $X \in \mathcal{I}$ there is $f : X \to B$.*

Proof: Let $P \in |\mathbf{K}|$ be a products of $\mathcal{I}$, with projections $p_X : P \to X$ for $X \in \mathcal{I}$. Let $e : E \to P$ be an "equaliser" (limit) of all morphisms in $\mathbf{K}(P, P)$. Then E is initial in $\mathbf{K}$, since for any $B \in |\mathbf{K}|$:

- $e;p_X;f : E \to B$, where $f : X \to B$ for some $X \in \mathcal{I}$.
- Given $g_1, g_2 : E \to B$, take their equaliser $e' : E' \to E$. As in the previous item, we have $h : P \to E'$. Then $h;e;e' : P \to P$, and by the construction of $e : E \to P$, $e;h;e';e = e;id_P = id_E;e$. Now, since e is mono, $e;h;e' = id_E$, and so e' is a mono retraction, hence an isomorphism, which proves $g_1 = g_2$.

Cofree objects

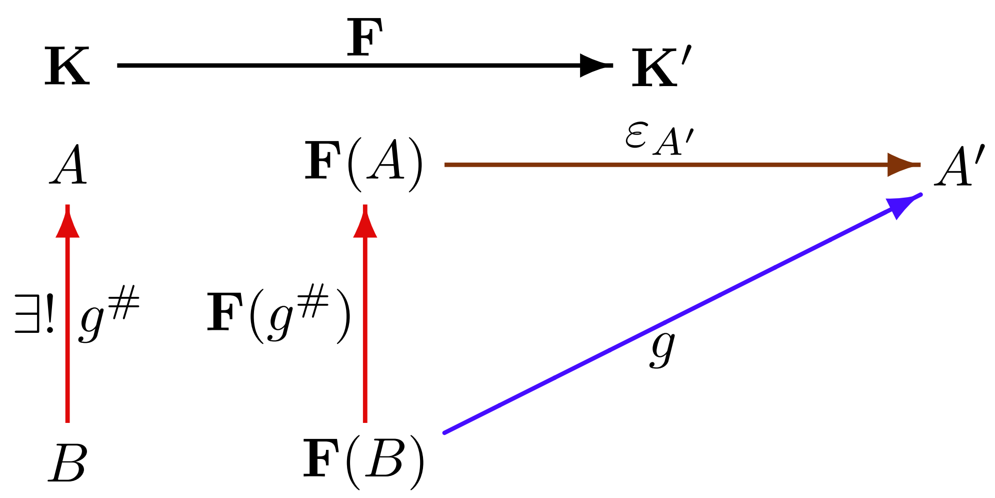
Consider any functor $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$

Definition: *Given an object $A' \in |\mathbf{K}'|$, a cofree object under A' w.r.t. $\mathbf{F}$ is a $\mathbf{K}$-object $A \in |\mathbf{K}|$ together with a $\mathbf{K}$-morphism $\varepsilon_{A'} : \mathbf{F}(A) \to A'$ (called counit morphism) such that given any $\mathbf{K}$-object $B \in |\mathbf{K}|$ with $\mathbf{K}'$-morphism $g : \mathbf{F}(B) \to A'$, for a unique $\mathbf{K}$-morphism $g^{\#} : B \to A$ we have*

$$\mathbf{G}(g^{\#});\varepsilon_{A'} = g$$

Paradigmatic example:

Function spaces, coming soon



Examples

- Consider inclusion $i : \mathbf{Int} \hookrightarrow \mathbf{Real}$, viewing $\mathbf{Int}$ and $\mathbf{Real}$ as (thin) categories, and i as a functor between them. For any real $r \in \mathbf{Real}$, the floor of r, $\lfloor r \rfloor \in \mathbf{Int}$ is cofree under r w.r.t. i.

  What about cofree objects w.r.t. the inclusion of rationals into reals?

- Fix a set $X \in |\mathbf{Set}|$. Consider functor $\mathbf{F}_X : \mathbf{Set} \to \mathbf{Set}$ defined by:
  - for any set $A \in |\mathbf{Set}|$, $\mathbf{F}_X(A) = A \times X$
  - for any function $f : A \to B$, $\mathbf{F}_X(f) : A \times X \to B \times X$ is a function given by $\mathbf{F}_X(f)(\langle a, x \rangle) = \langle f(a), x \rangle$.

  Then for any set $A \in |\mathbf{Set}|$, the powerset $A^X \in |\mathbf{Set}|$ (i.e., the set of all functions from X to A) is a cofree objects under A w.r.t. $\mathbf{F}_X$. The counit morphism $\varepsilon_A : \mathbf{F}_X(A^X) = A^X \times X \to A$ is the evaluation function: $\varepsilon_A(\langle f, x \rangle) = f(x)$.

  A generalisation to deal with exponential objects will (not) be discussed later

Facts

Dual to those for free objects: Consider a functor $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$, object $A' \in |\mathbf{K}'|$, and an object $A \in |\mathbf{K}|$ cofree under A' w.r.t. $\mathbf{F}$ with counit $\varepsilon_{A'} : \mathbf{F}(A) \to A'$.

- Cofree objects under A' w.r.t. $\mathbf{F}$ are the terminal objects in the comma category $(\mathbf{F}, \mathbf{C}_{A'})$, where $\mathbf{C}_{A'} : \mathbf{1} \to \mathbf{K}'$ is the constant functor.
- A cofree object under A' w.r.t. $\mathbf{F}$, if exists, is unique up to isomorphism.
- The function $(\_)^{\#} : \mathbf{K}'(\mathbf{F}(B), A') \to \mathbf{K}(B, A)$ is bijective for each $B \in |\mathbf{K}|$.
- For any morphisms $g_1, g_2 : B \to A$ in $\mathbf{K}$, $g_1 = g_2$ iff $g_1;\varepsilon_{A'} = g_2;\varepsilon_{A'}$.

Limits as cofree objects

Fact: *In a category $\mathbf{K}$, given a diagram D of shape $G(D)$, the limit of D in $\mathbf{K}$ is a cofree object under D w.r.t. the diagonal functor $\Delta^{G(D)}_{\mathbf{K}} : \mathbf{K} \to \mathbf{Diag}^{G(D)}_{\mathbf{K}}$.*

Spell this out for terminal objects, products, equalisers, and pullbacks

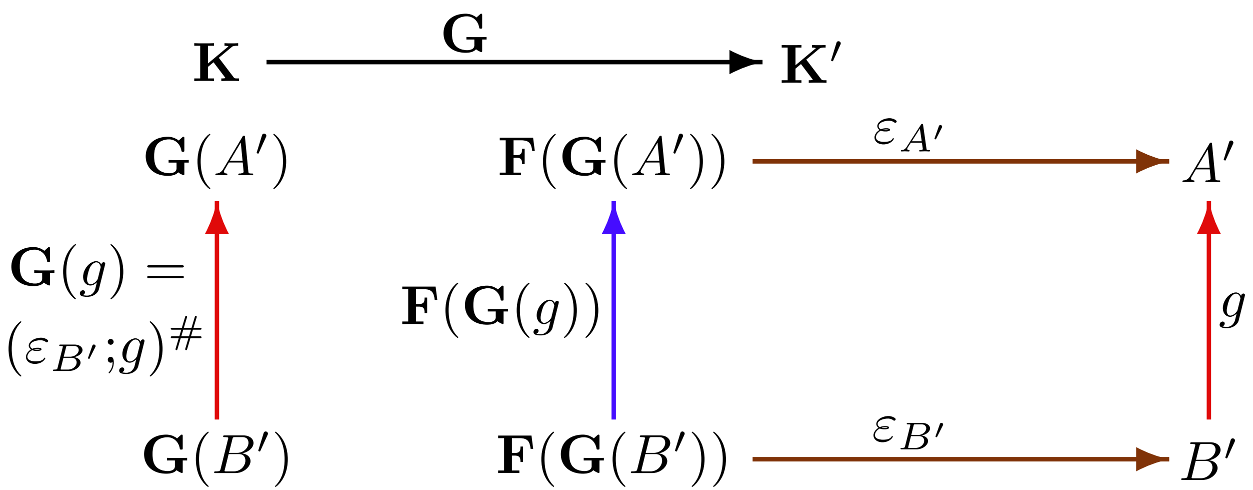
Right adjoints

Consider a functor $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$.

Fact: *Assume that for each object $A' \in |\mathbf{K}'|$ there is a cofree object under A' w.r.t. $\mathbf{F}$, say $\mathbf{G}(A') \in |\mathbf{K}'|$ is cofree under A' with counit $\varepsilon_{A'} : \mathbf{F}(\mathbf{G}(A')) \to A'$. Then the mapping:*

– $(A' \in |\mathbf{K}'|) \mapsto (\mathbf{G}(A') \in |\mathbf{K}|)$
– $(g : B' \to A') \mapsto ((\varepsilon_{B'};g)^{\#} : \mathbf{G}(B') \to \mathbf{G}(A'))$

form a functor $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$. Moreover, $\varepsilon : \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$ is a natural transformation.



Right adjoints

Definition: *A functor* $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ *is right adjoint to (a functor)* $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ *with counit (natural transformation)* $\varepsilon : \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$ *if for all objects* $A' \in |\mathbf{K}'|$, $\mathbf{G}(A') \in |\mathbf{K}|$ *is cofree under* A' *with counit morphism* $\varepsilon_{A'} : \mathbf{F}(\mathbf{G}(A')) \to A'$.

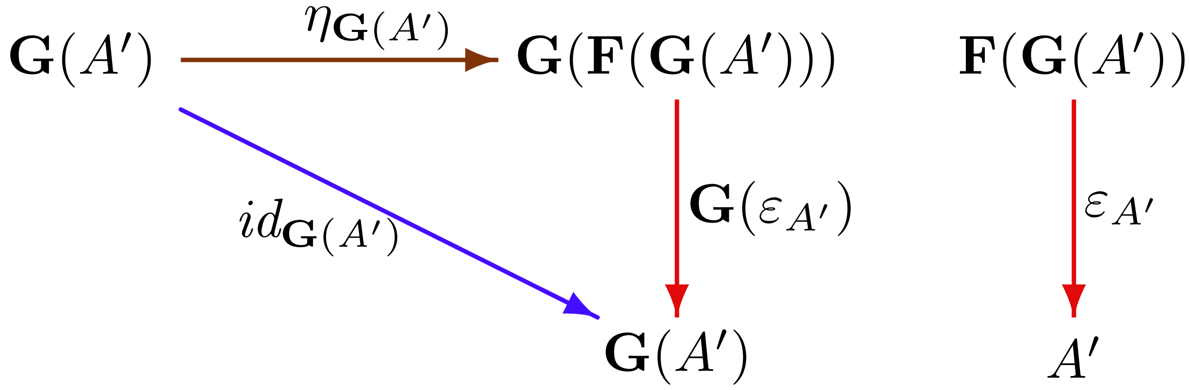
Fact: *A right adjoint to any functor* $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$*, if exists, is determined uniquely up to a natural isomorphism: if* $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ *and* $\mathbf{G}' : \mathbf{K}' \to \mathbf{K}$ *are right adjoint to* $\mathbf{F}$ *with counits* $\varepsilon : \mathbf{G};\mathbf{F}$ *and* $\varepsilon' : \mathbf{G}';\mathbf{F}$*, respectively, then there exists a natural isomorphism* $\tau : \mathbf{G} \to \mathbf{G}'$ *such that* $(\tau \cdot \mathbf{F});\varepsilon' = \varepsilon$.

Fact: *Let* $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ *be right adjoint to* $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ *with counit* $\varepsilon : \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$*. Then* $\mathbf{G}$ *is continuous (preserves limits) and* $\mathbf{F}$ *is cocontinuous (preserves colimits).*

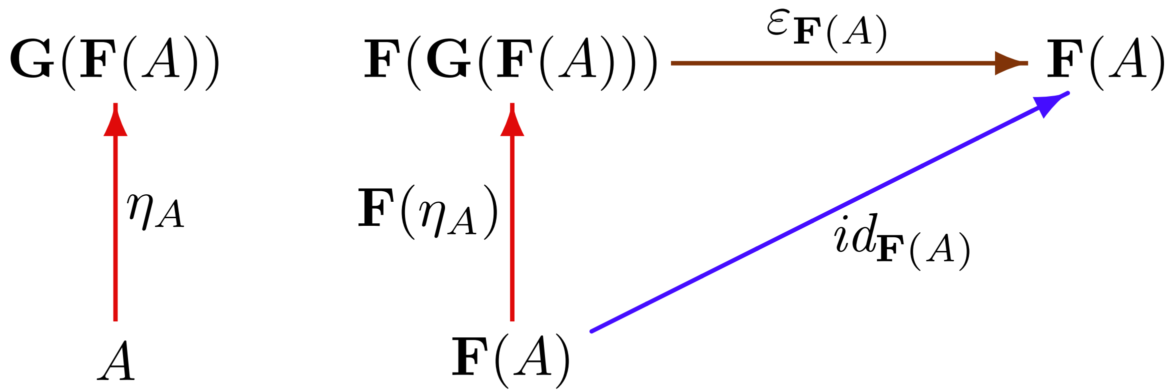
From left adjoints to adjunctions

Fact: *Let* $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ *be left adjoint to* $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ *with unit* $\eta : \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$*.*
Then there is a natural transformation $\varepsilon : \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$ *such that:*

- $(\mathbf{G}\cdot\eta);(\varepsilon\cdot\mathbf{G}) = id_{\mathbf{G}}$



- $(\eta\cdot\mathbf{F});(\mathbf{F}\cdot\varepsilon) = id_{\mathbf{F}}$

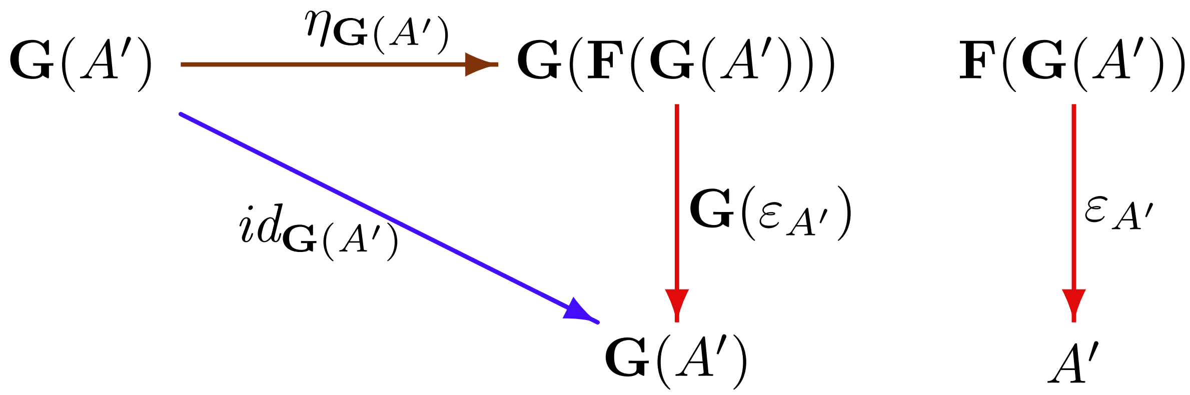


Proof (idea):
Put $\varepsilon_{A'} = (id_{\mathbf{G}(A')})^{\#}$.

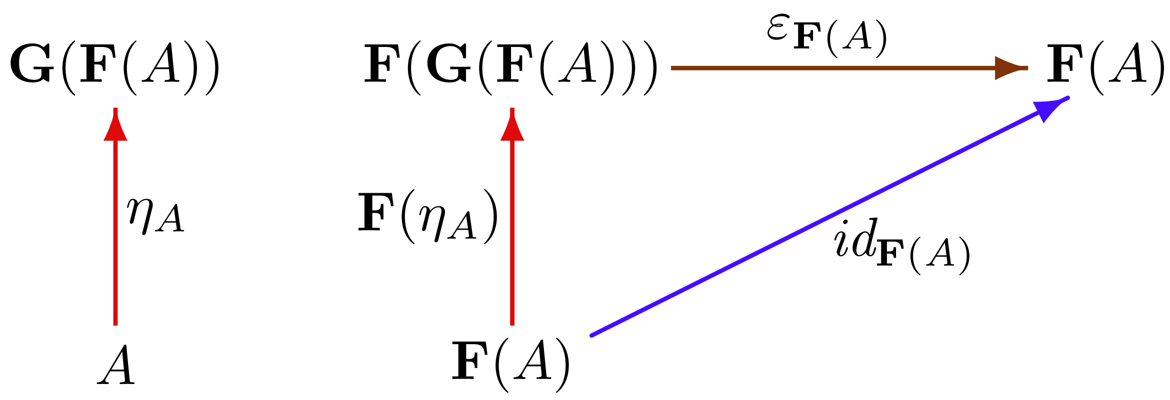
From right adjoints to adjunctions

Fact: *Let* $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ *be right adjoint to* $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ *with counit* $\varepsilon : \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$. *Then there is a natural transformation* $\eta : \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$ *such that:*

- $(\mathbf{G}\cdot\eta);(\varepsilon\cdot\mathbf{G}) = id_{\mathbf{G}}$



- $(\eta\cdot\mathbf{F});(\mathbf{F}\cdot\varepsilon) = id_{\mathbf{F}}$



Proof (idea):

Put $\eta_A = (id_{\mathbf{F}(A)})^{\#}$.

From adjunctions to left and right adjoints

Fact: *Consider two functors* $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ *and* $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ *with natural transformations* $\eta : \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$ *and* $\varepsilon : \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$ *such that:*

- $(\mathbf{G}{\cdot}\eta);(\varepsilon{\cdot}\mathbf{G}) = id_{\mathbf{G}}$
- $(\eta{\cdot}\mathbf{F});(\mathbf{F}{\cdot}\varepsilon) = id_{\mathbf{F}}$

Then:

- $\mathbf{F}$ *is left adjoint to* $\mathbf{G}$ *with unit* η*.*
- $\mathbf{G}$ *is right adjoint to* $\mathbf{F}$ *with counit* ε*.*

Proof: For $A \in |\mathbf{K}|$, $B' \in |\mathbf{K}'|$ and $f : A \to \mathbf{G}(B')$, define $f^{\#} = \mathbf{F}(f);\varepsilon_{B'}$. Then $f^{\#} : \mathbf{F}(A) \to B'$ satisfies $\eta_A;\mathbf{G}(f^{\#}) = f$ and is the only such morphism in $\mathbf{K}'(\mathbf{F}(A), B')$. This proves that $\mathbf{F}(A)$ is free over A with unit η_A, and so indeed, $\mathbf{F}$ is left adjoint to $\mathbf{G}$ with unit η.

The proof that $\mathbf{G}$ is right adjoint to $\mathbf{F}$ with counit ε is similar.

Adjunctions

Definition: *An adjunction between categories* $\mathbf{K}$ *and* $\mathbf{K}'$ *is*

$$\langle \mathbf{F}, \mathbf{G}, \eta, \varepsilon \rangle$$

where $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ *and* $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ *are functors, and* $\eta : \mathbf{Id_K} \to \mathbf{F};\mathbf{G}$ *and* $\varepsilon : \mathbf{G};\mathbf{F} \to \mathbf{Id_{K'}}$ *natural transformations such that:*

- $(\mathbf{G}{\cdot}\eta);(\varepsilon{\cdot}\mathbf{G}) = id_{\mathbf{G}}$
- $(\eta{\cdot}\mathbf{F});(\mathbf{F}{\cdot}\varepsilon) = id_{\mathbf{F}}$

Equivalently, such an adjunction may be given by:

- Functor $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ and all $A \in |\mathbf{K}|$, a free object over A w.r.t. $\mathbf{G}$.
- Functor $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ and its left adjoint.
- Functor $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ and all $A' \in |\mathbf{K}'|$, a cofree object under A' w.r.t. $\mathbf{F}$.
- Functor $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ and its right adjoint.