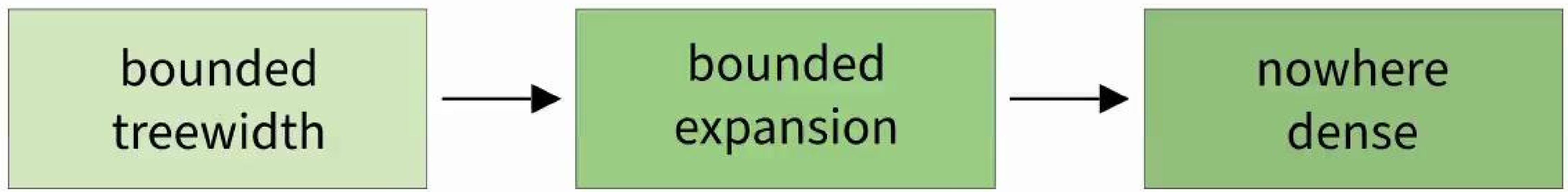
Merge-Width and First-Order Model Checking

Szymon Toruńczyk, joint work with Jan Dreier

LOGALG’25

Overview



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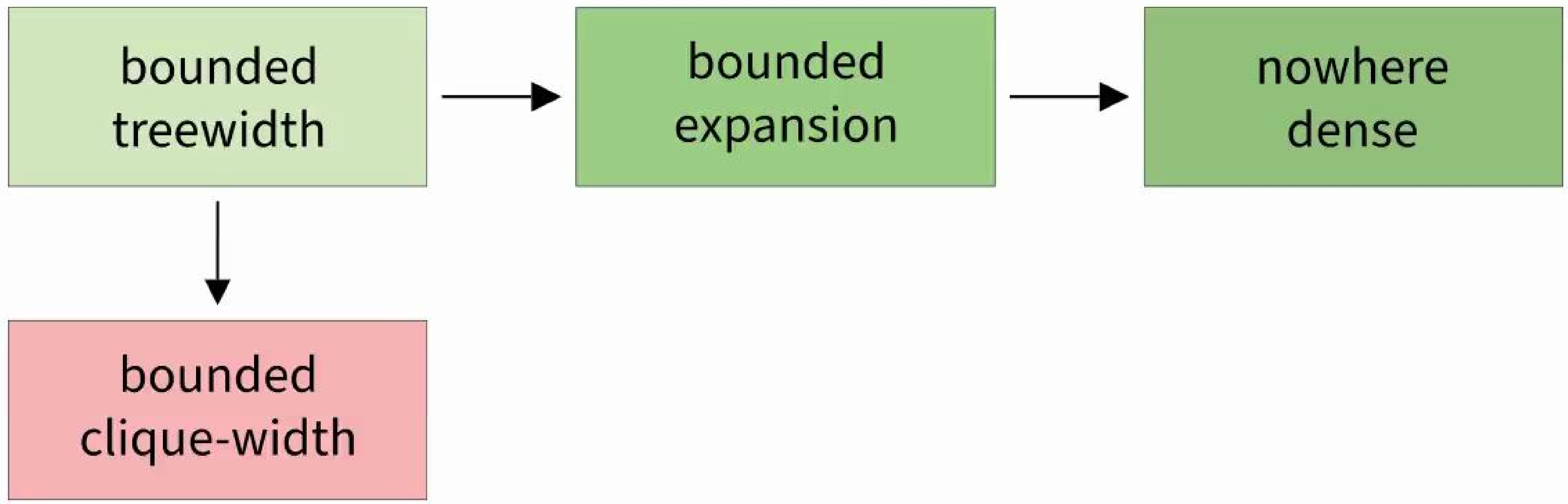
bounded treewidth
bounded expansion
nowhere dense
bounded clique-width

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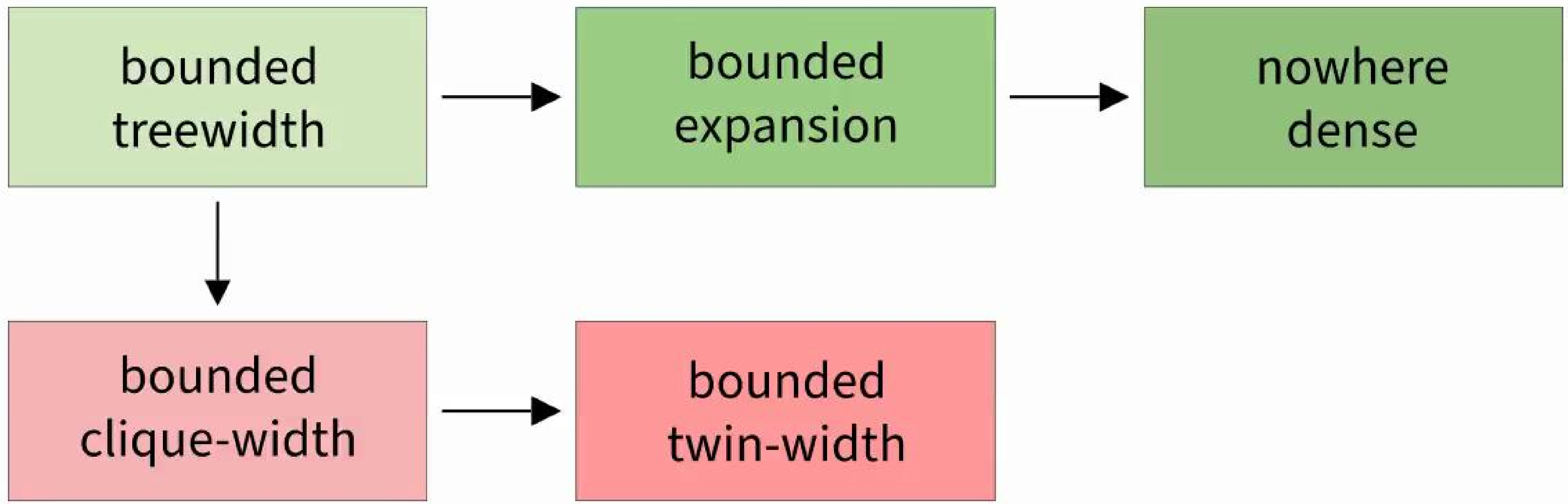

bounded treewidth
bounded expansion
nowhere dense
bounded clique-width
bounded twin-width

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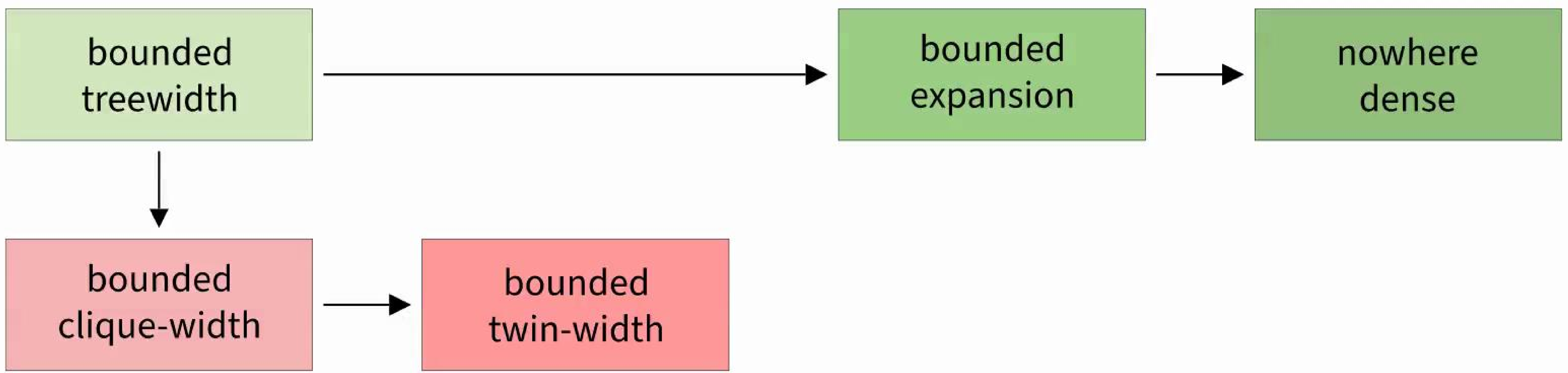
bounded treewidth
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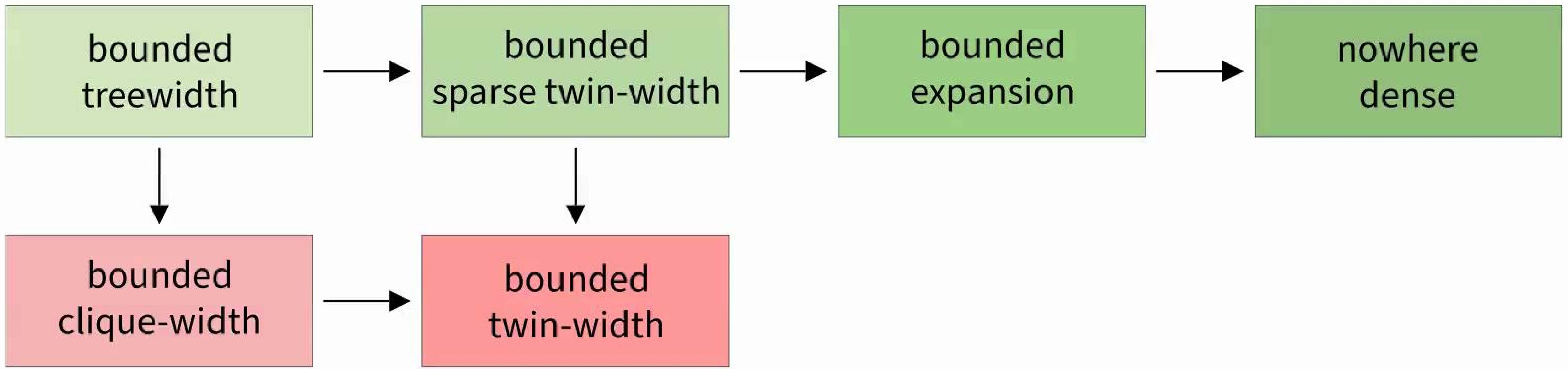
bounded treewidth
bounded sparse twin-width
bounded expansion
nowhere dense
bounded clique-width
bounded twin-width

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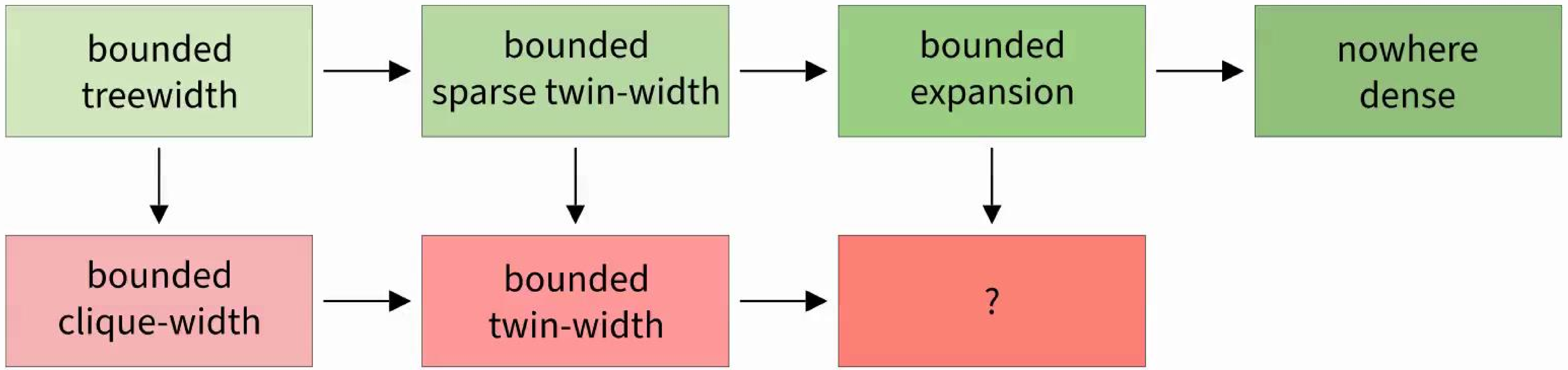
bounded treewidth
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bounded expansion
nowhere dense
bounded clique-width
bounded twin-width
?

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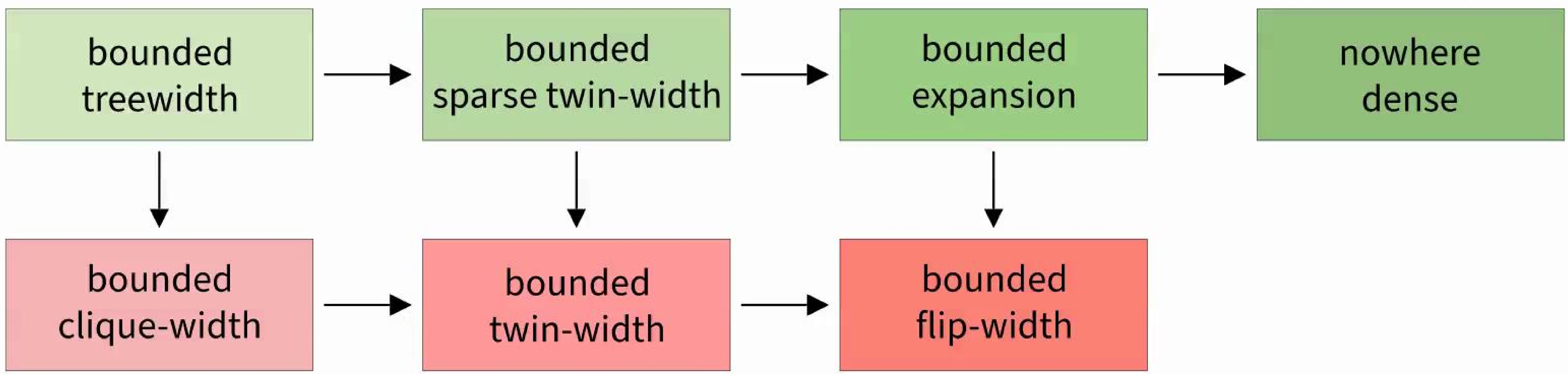

bounded treewidth
bounded sparse twin-width
bounded expansion
nowhere dense
bounded clique-width
bounded twin-width
bounded flip-width

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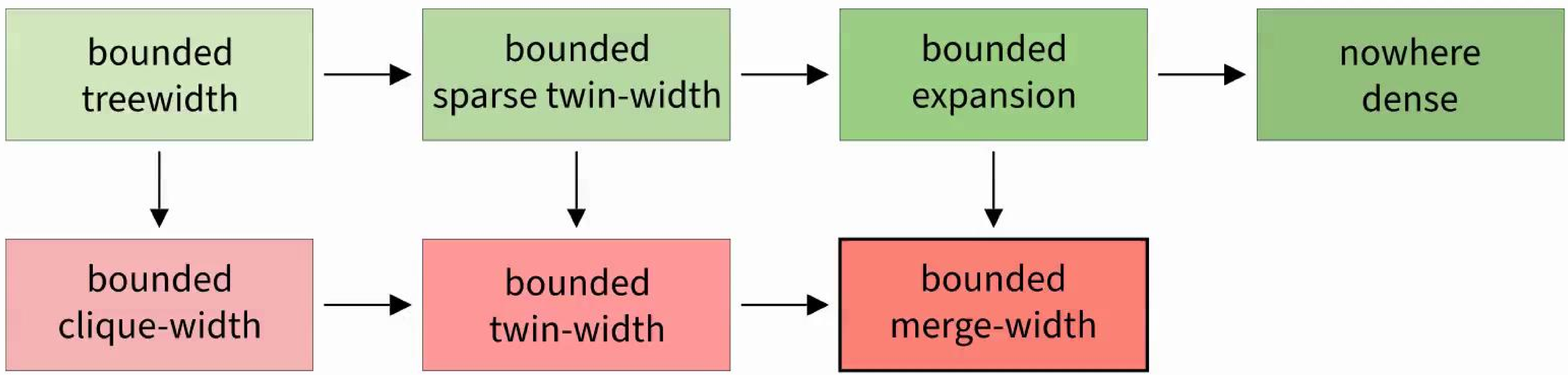
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bounded sparse twin-width
bounded expansion
nowhere dense
bounded clique-width
bounded twin-width
bounded merge-width

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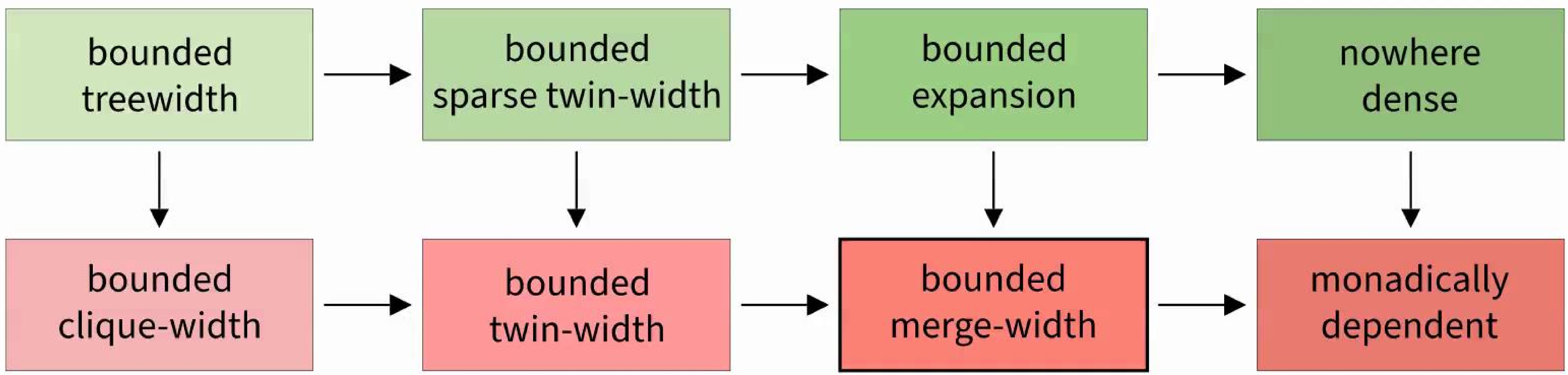
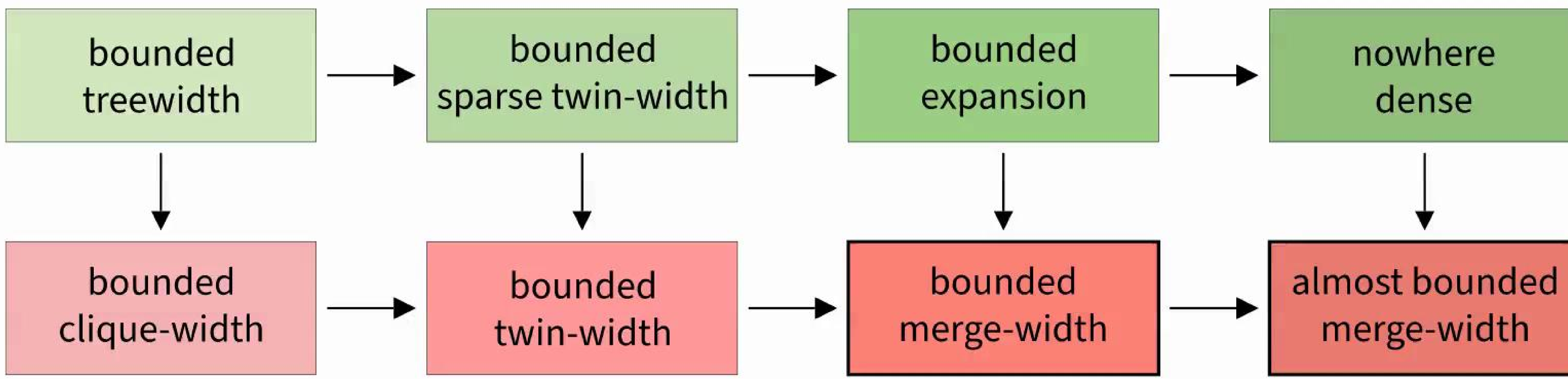
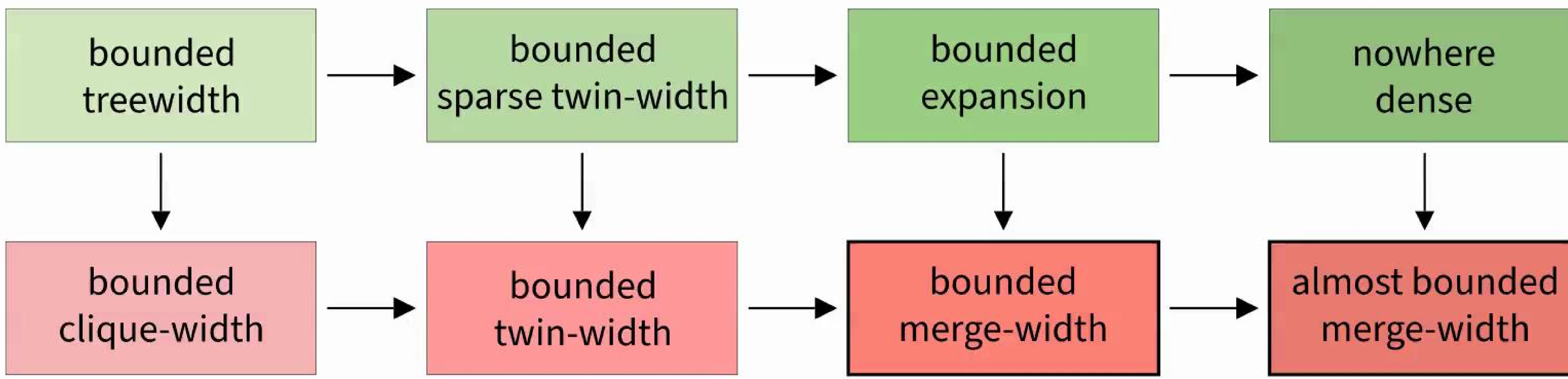

bounded treewidth
bounded sparse twin-width
bounded expansion
nowhere dense
bounded clique-width
bounded twin-width
bounded merge-width
monadically dependent

Overview


bounded treewidth
bounded sparse twin-width
bounded expansion
nowhere dense
bounded clique-width
bounded twin-width
bounded merge-width
almost bounded merge-width

Overview



Courcelle's theorem. Graph problems definable in monadic-second order logic can be verified in linear time on graphs of bounded tree-width.

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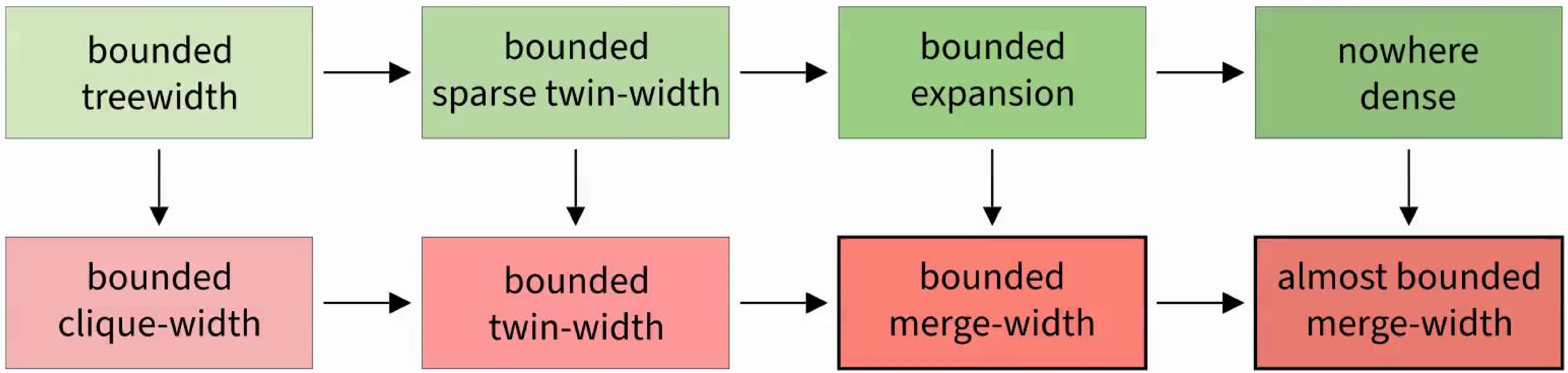

Courcelle's theorem. Graph problems definable in monadic-second order logic can be verified in linear time on graphs of bounded tree-width.

Main result. Graph problems definable in first-order logic can be verified in cubic time on graphs of bounded merge-width, given a decomposition.

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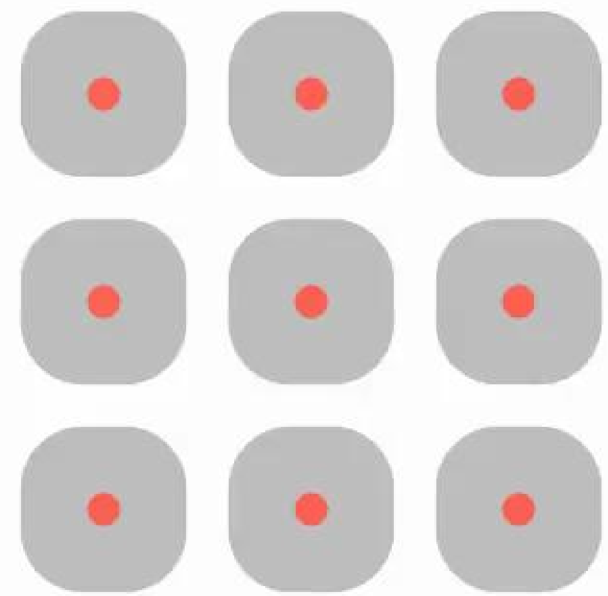

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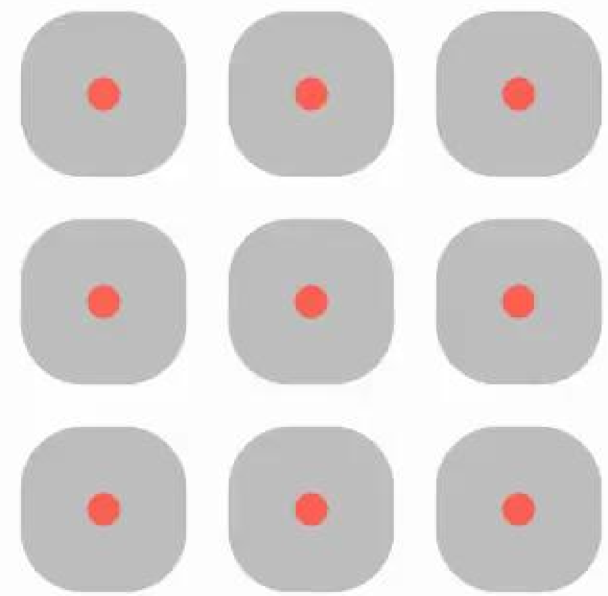

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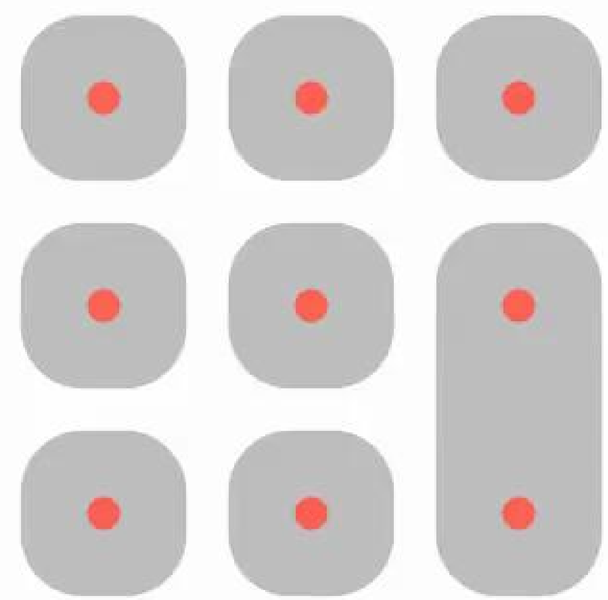

merge

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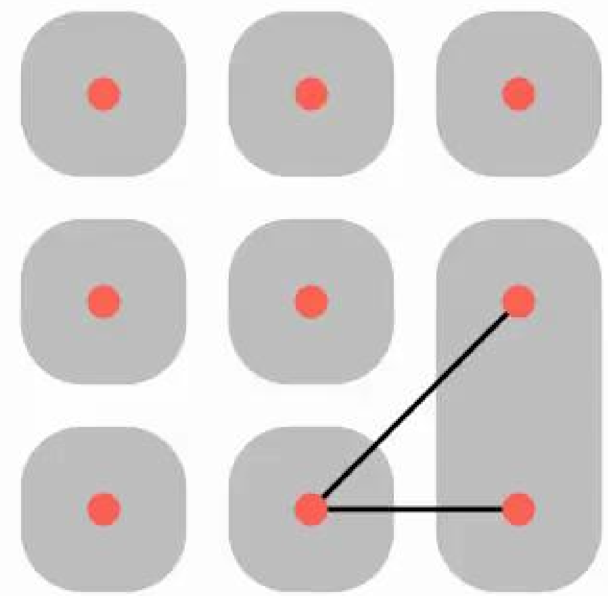

resolve+

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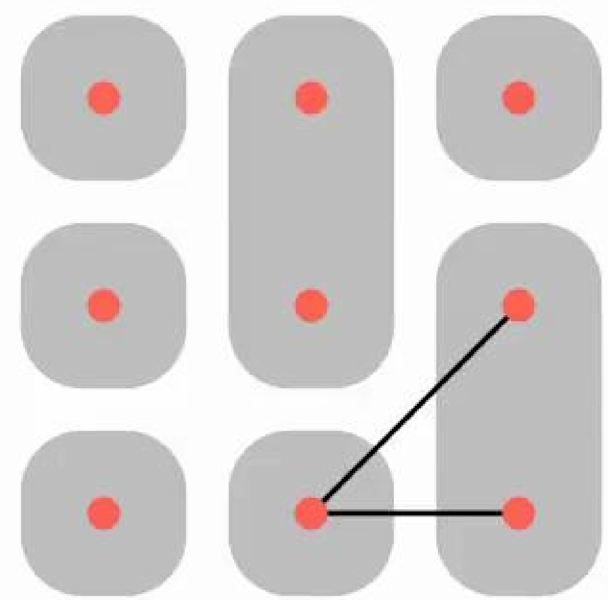

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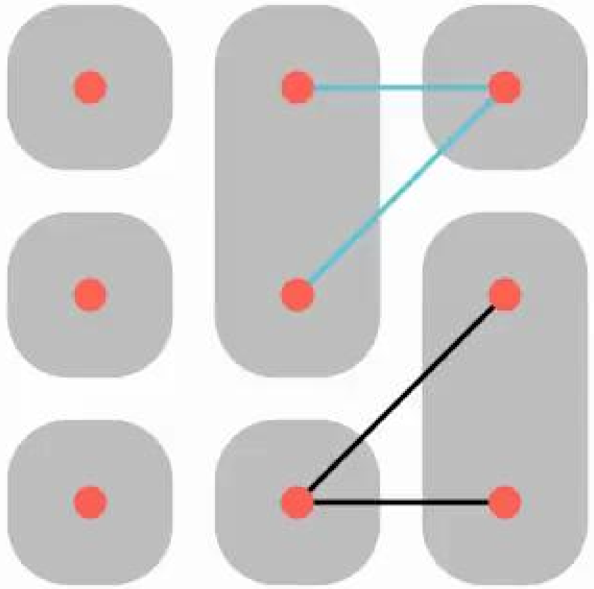

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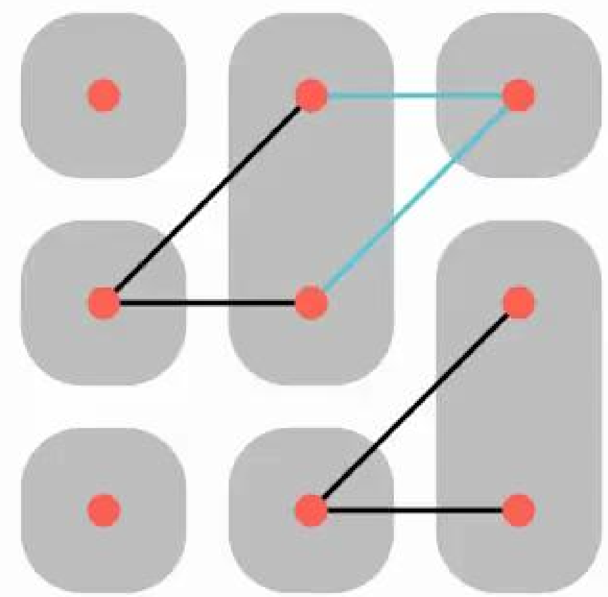

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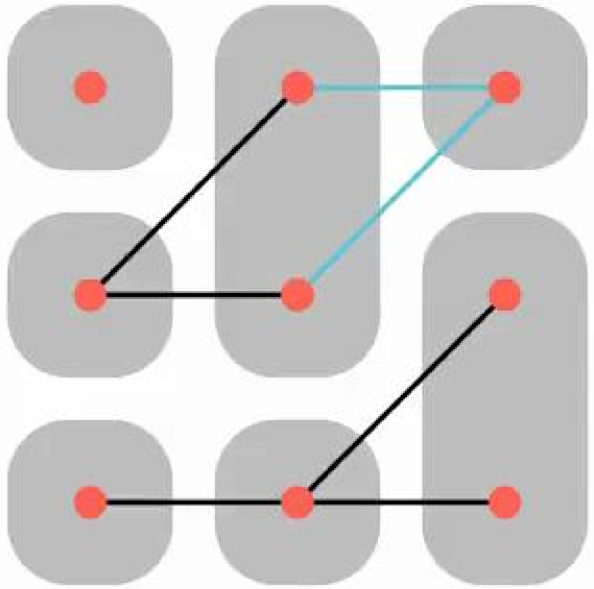

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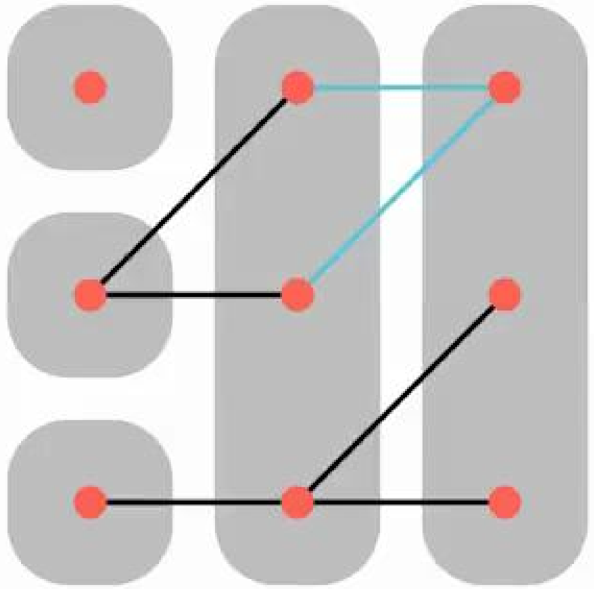

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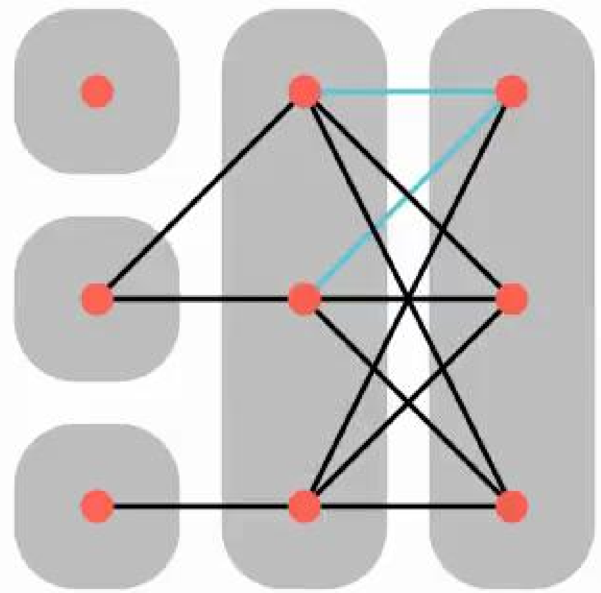

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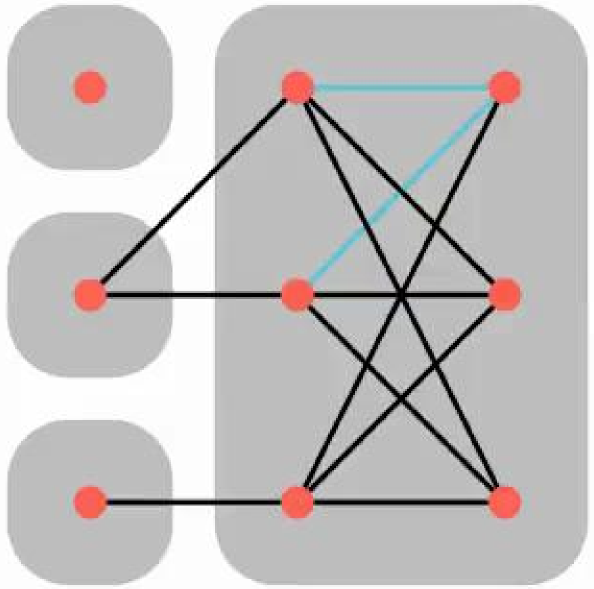

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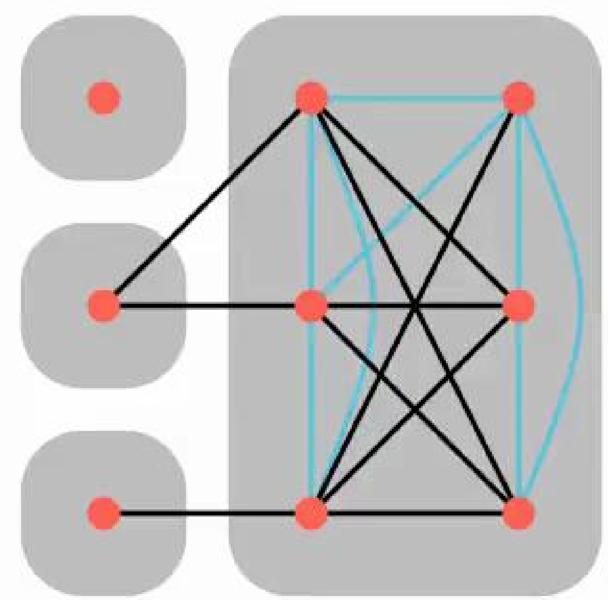

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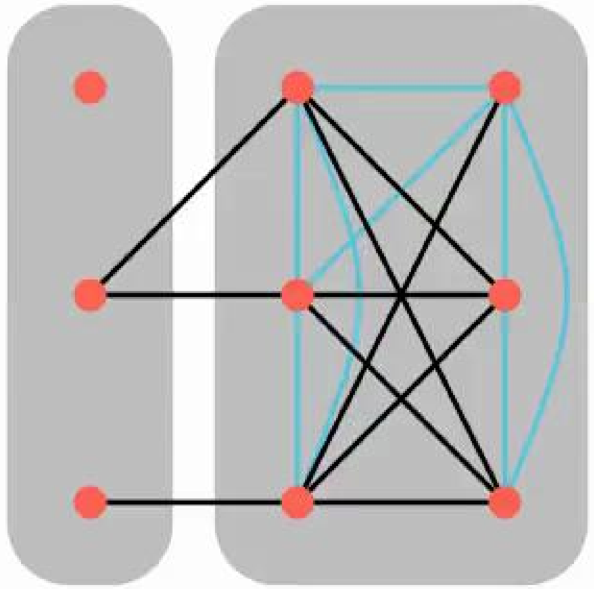

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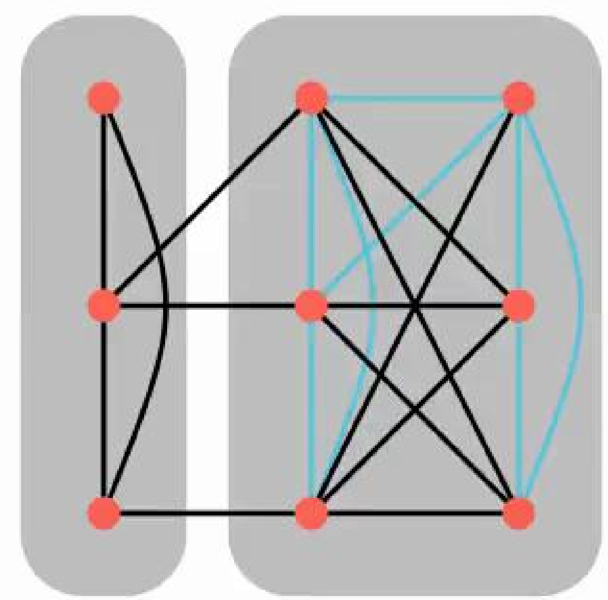

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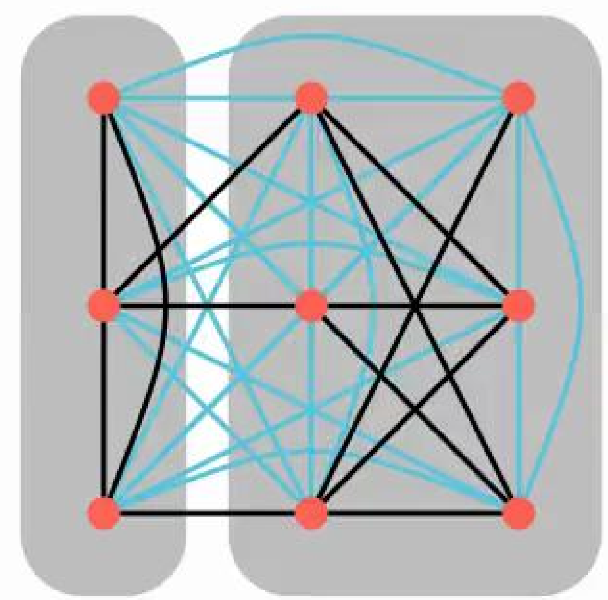

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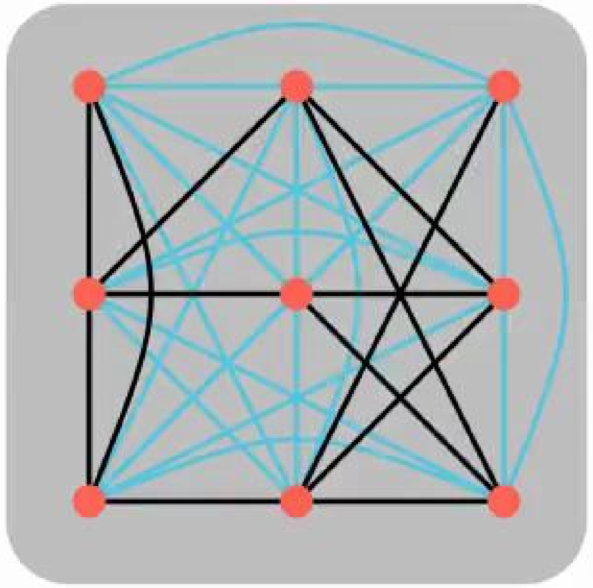

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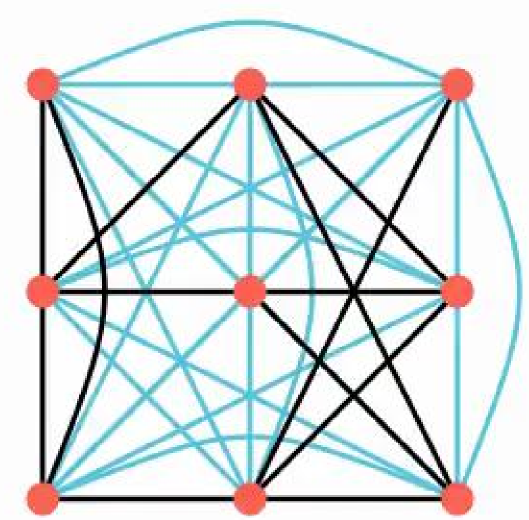
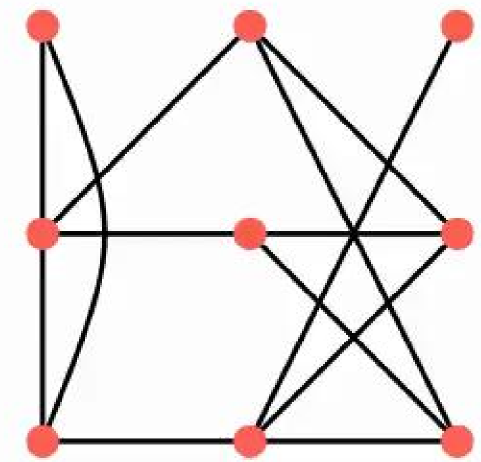
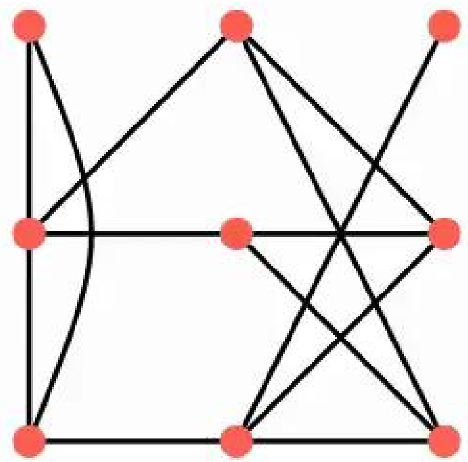


done!



Construction sequence: such a sequence $(\mathcal{P}_1, E_1, N_1), \ldots, (\mathcal{P}_n, E_n, N_n)$.



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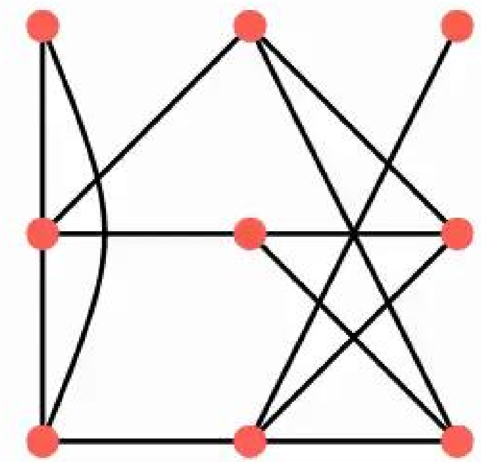

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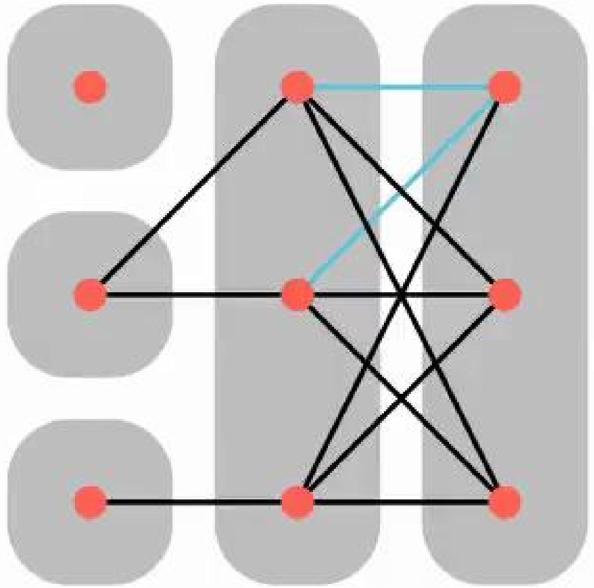

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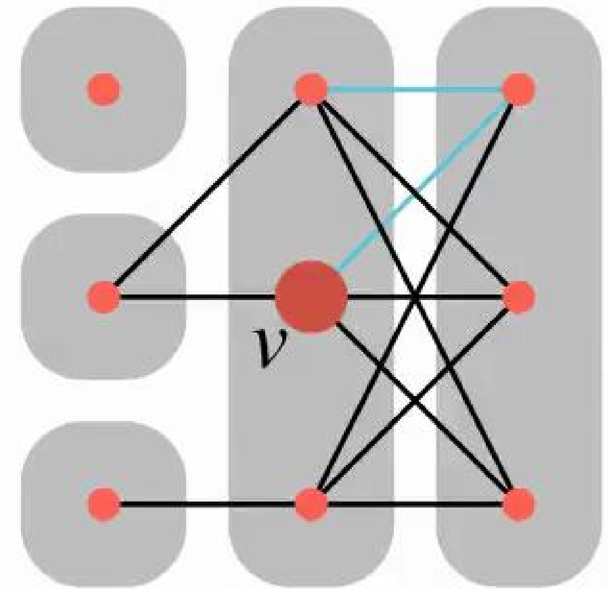

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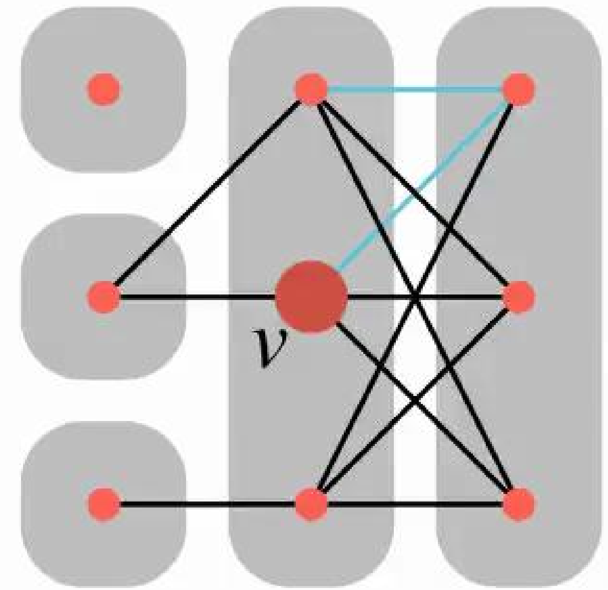

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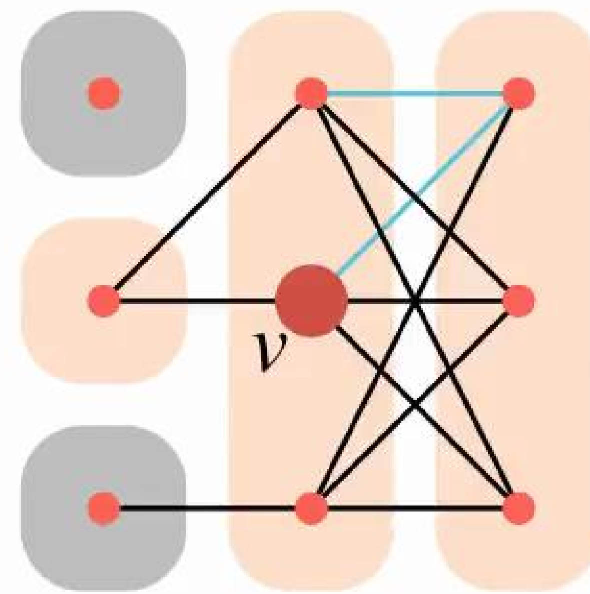

$(r = 2)$

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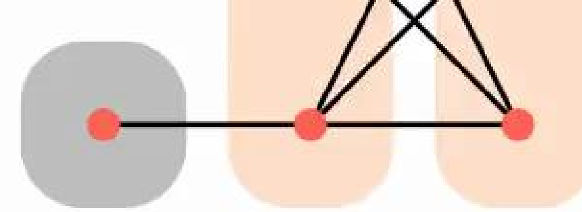

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$\underbrace{\text{radius-}r \text{ merge-width of } G}_{\mathrm{mw}_r(G)}$:= min. radius-r width of a constr. sequence of G

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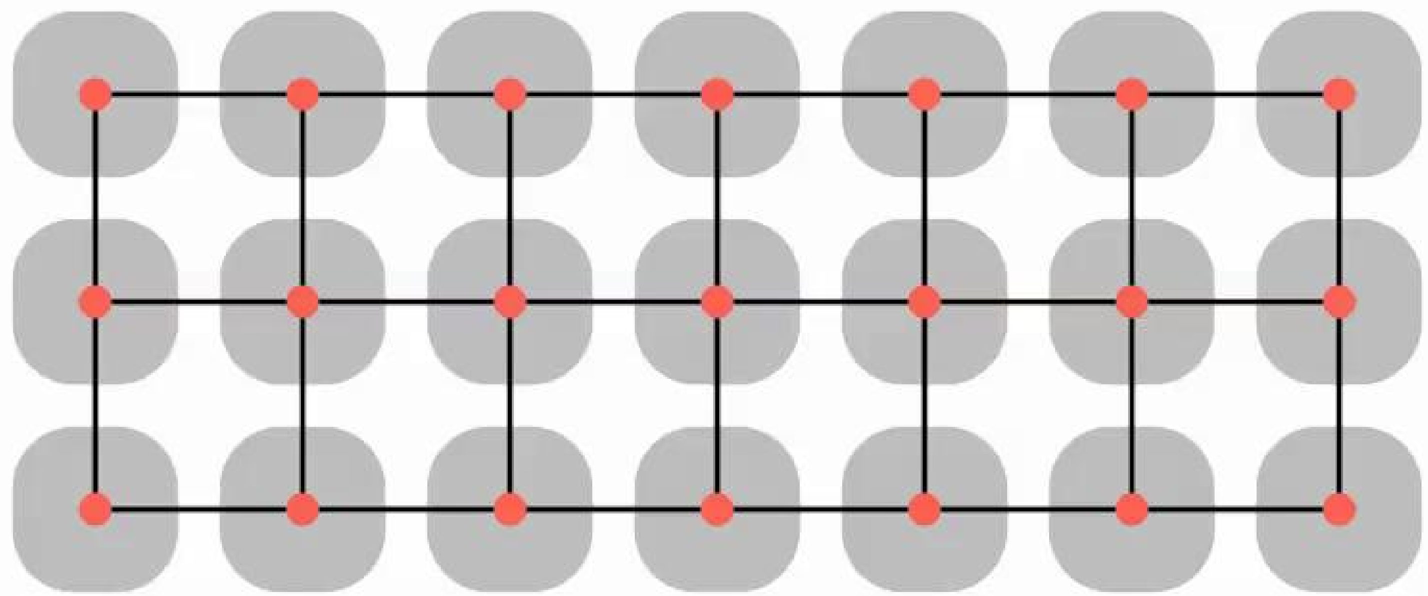
Fix $r \in \mathbb{N}$.

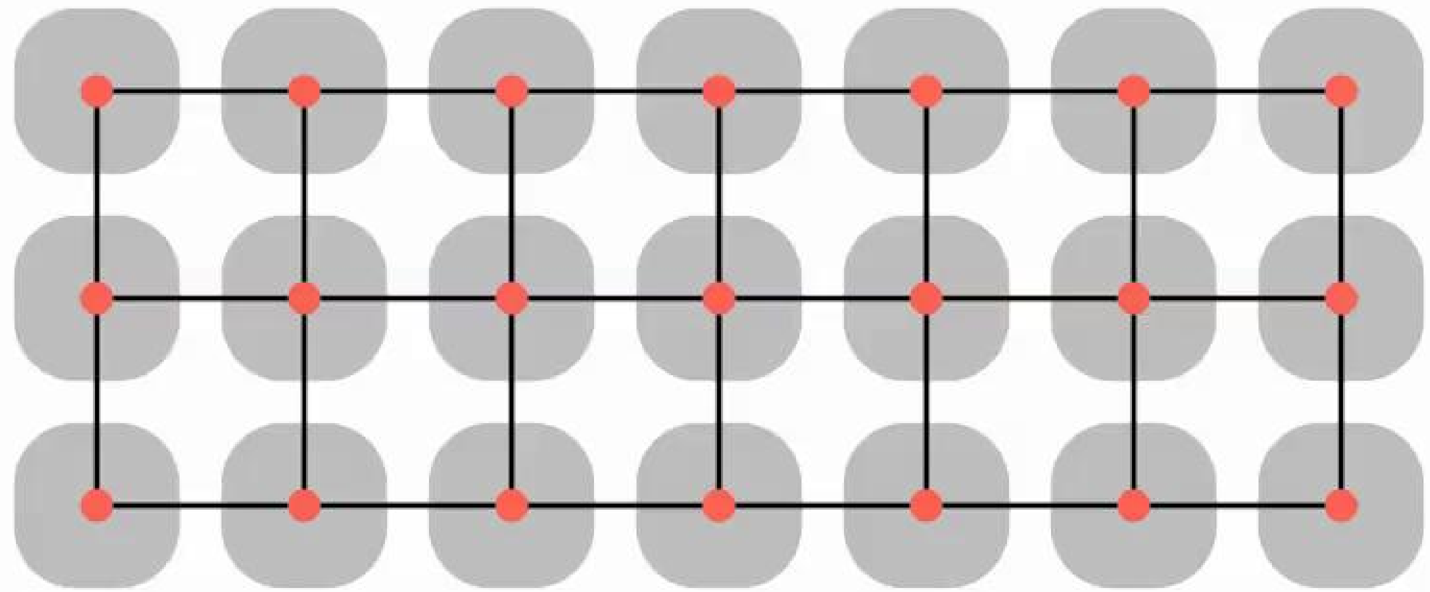
radius-r width of a construction sequence :=

$$\max_{t=1..n} \quad \max_{v \in V} \quad \#\text{parts of } \mathcal{P}_t \text{ at distance } \leqslant r \text{ from } v \text{ in } (V, E_t \cup N_t)$$

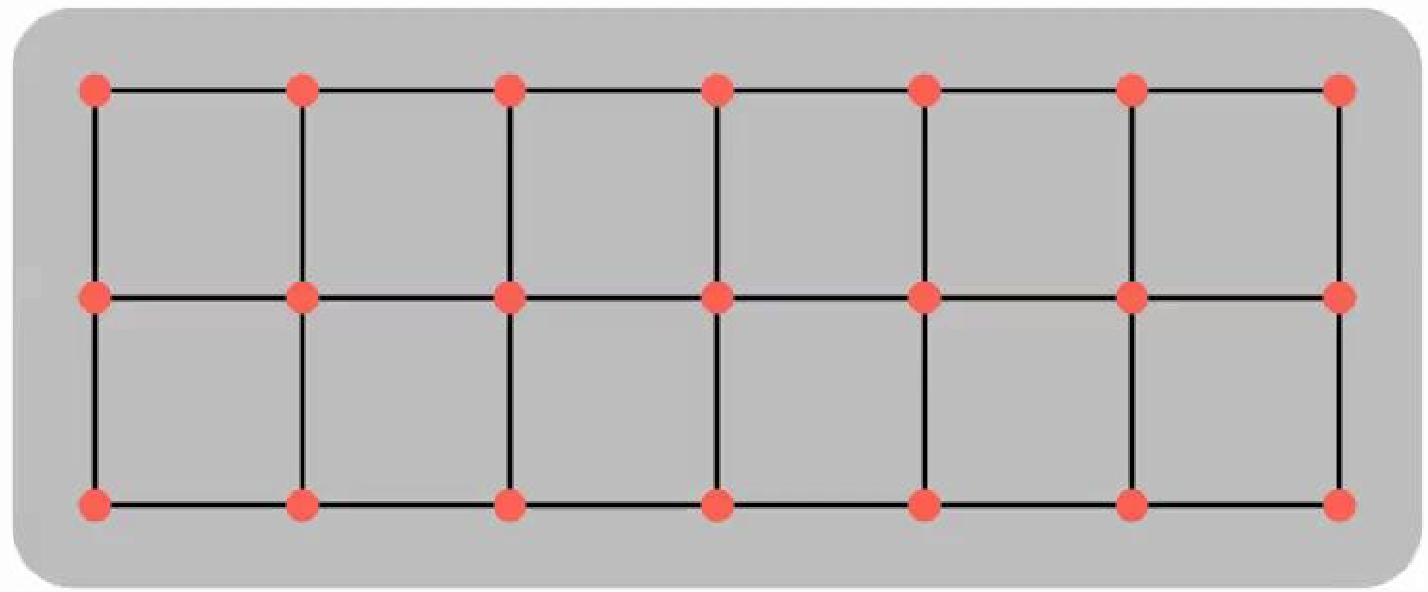
$$\underbrace{\text{radius-}r \text{ merge-width of } G}_{\mathrm{mw}_r(G)} := \text{min. radius-}r\text{ width of a constr. sequence of } G$$

A graph class has *bounded merge-width* if $\mathrm{mw}_r(C) < \infty$ for all $r \in \mathbb{N}$.





$$\mathrm{mw}_r(G) \leqslant O(\text{maximum-degree}(G))^r$$



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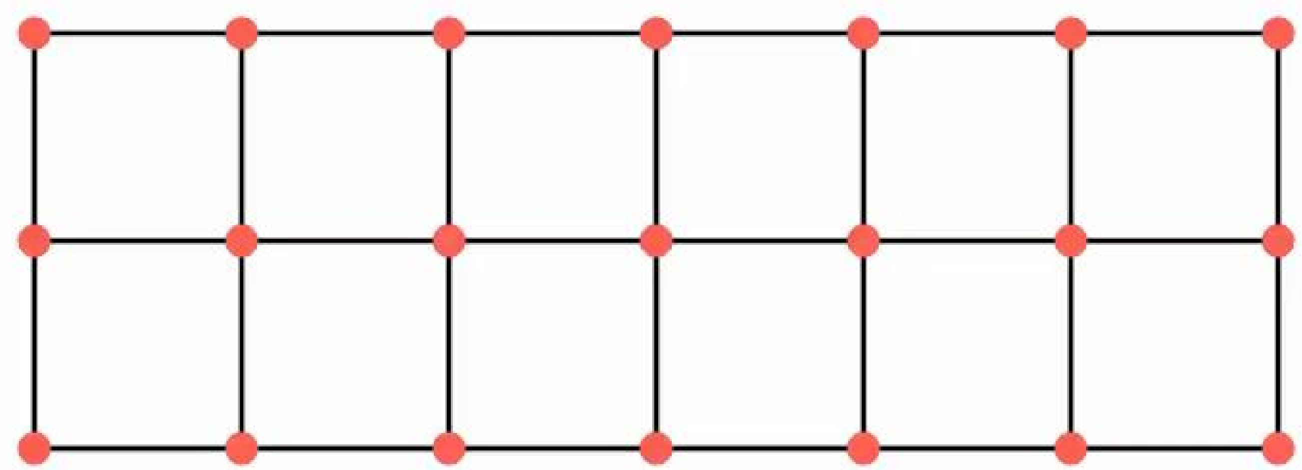

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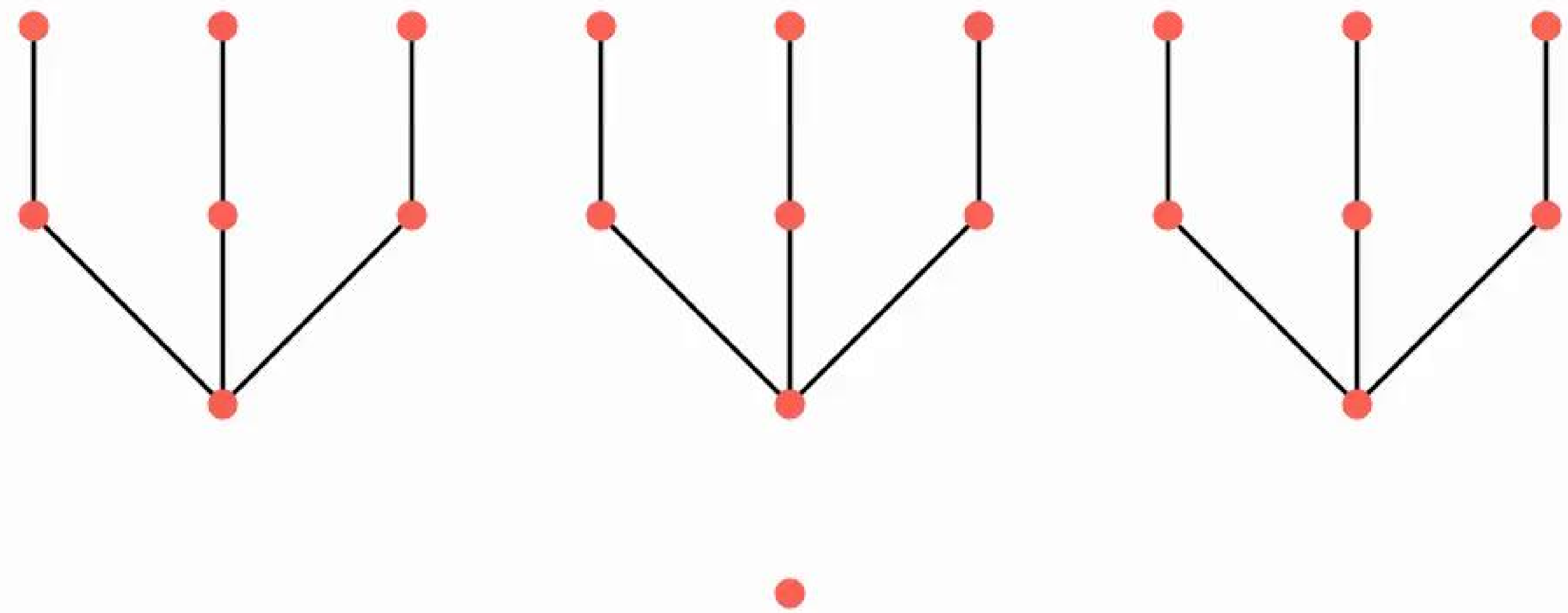
Corollary. Every class of bounded maximum degree has bounded merge-width.

Trees



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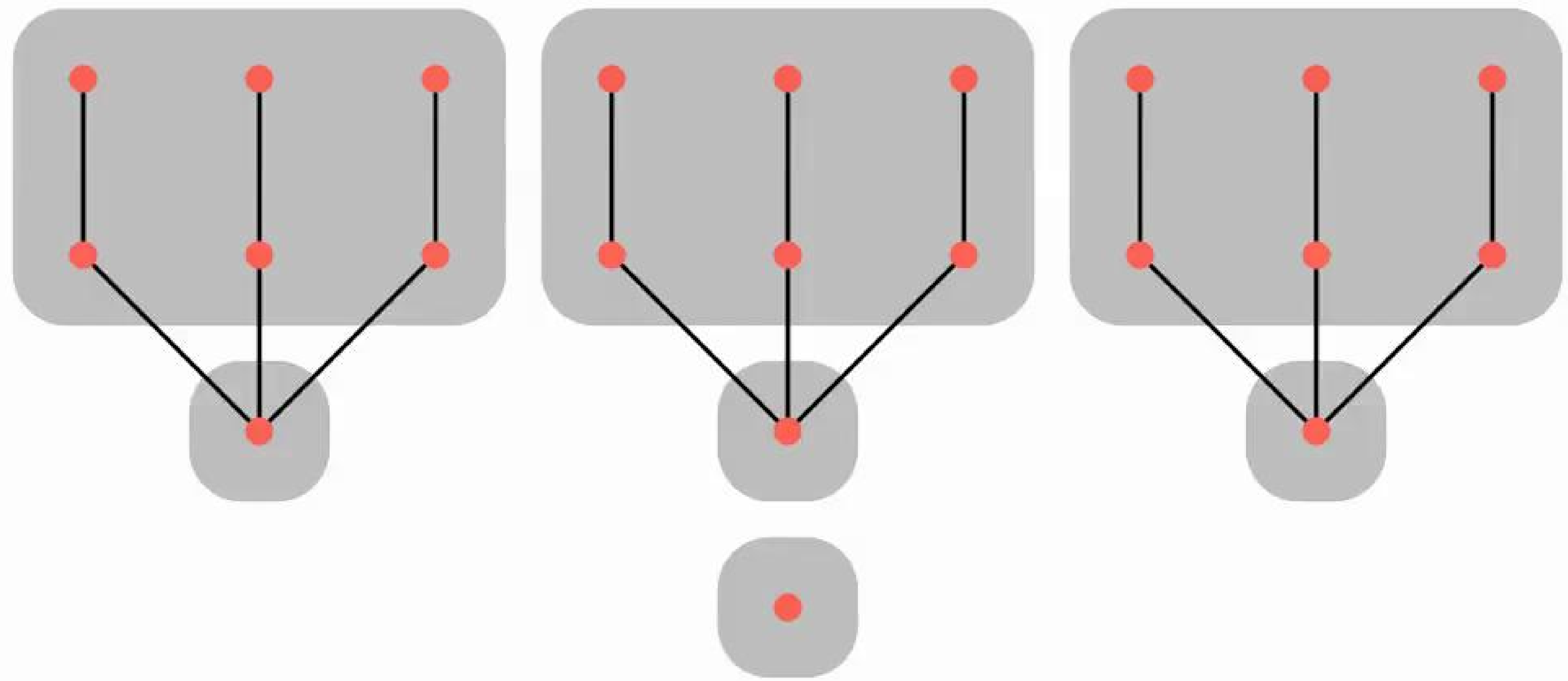

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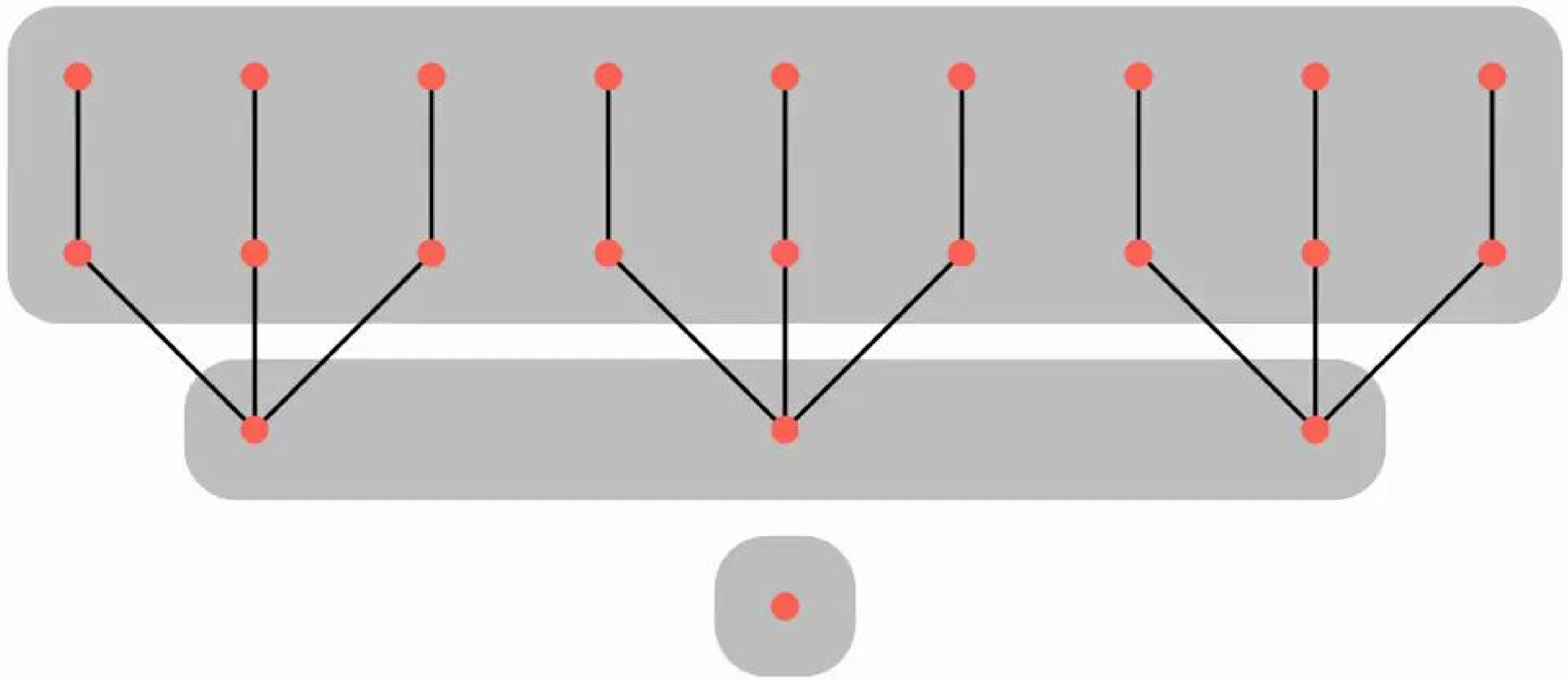

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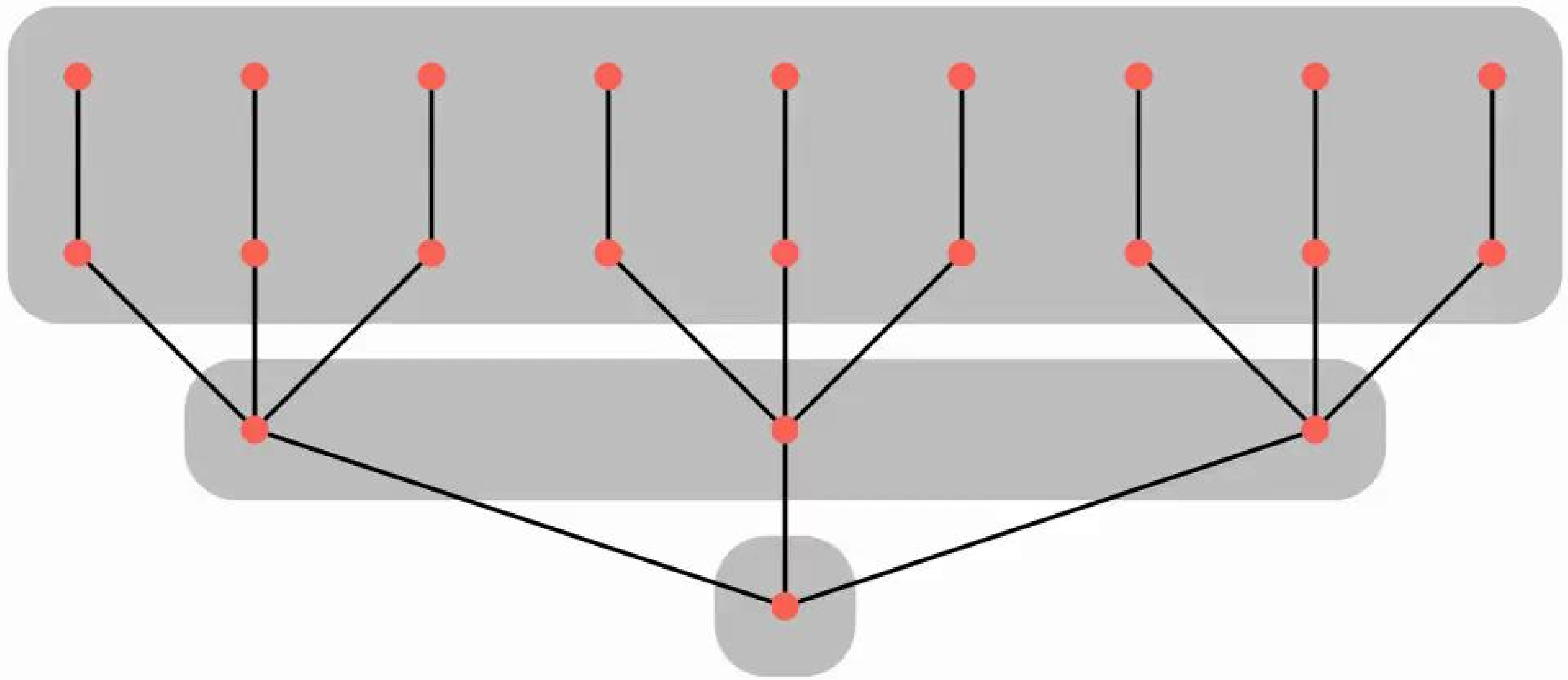

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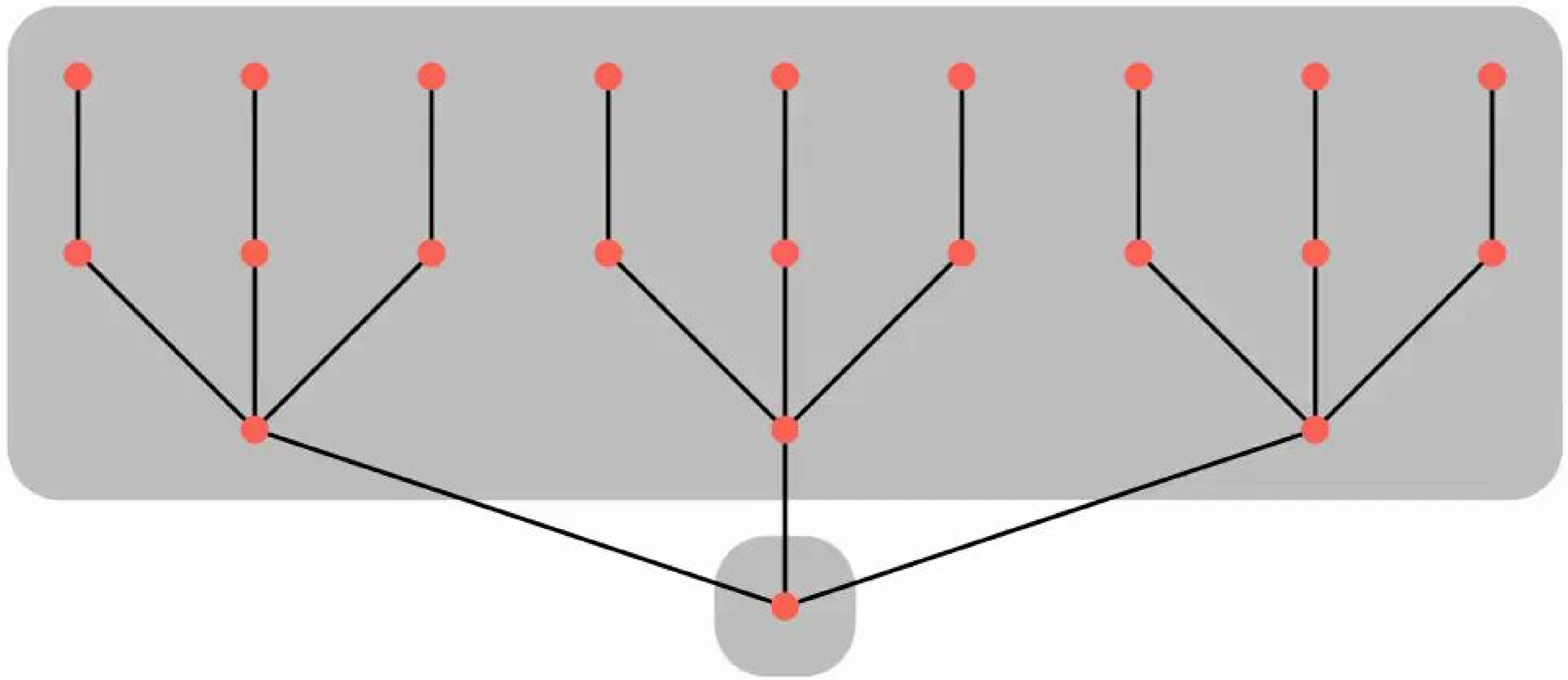

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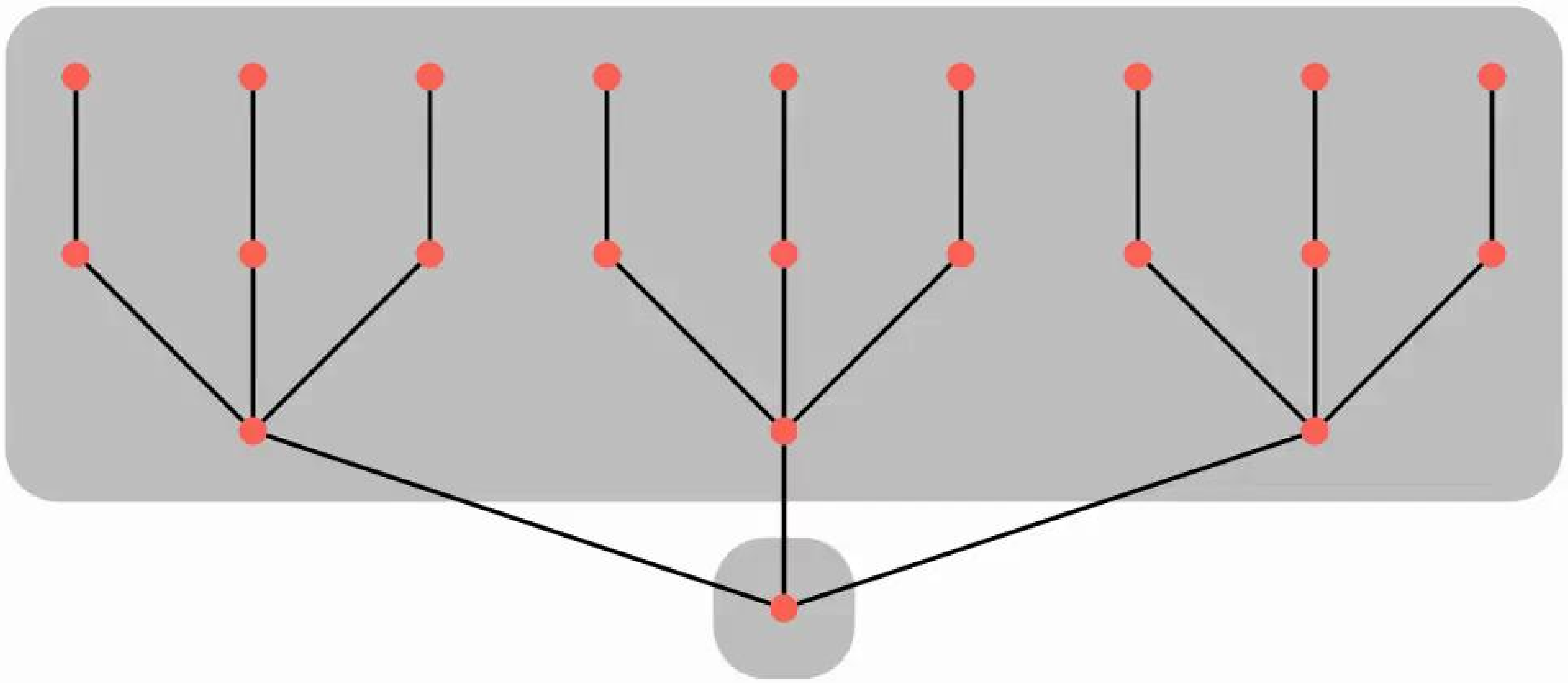
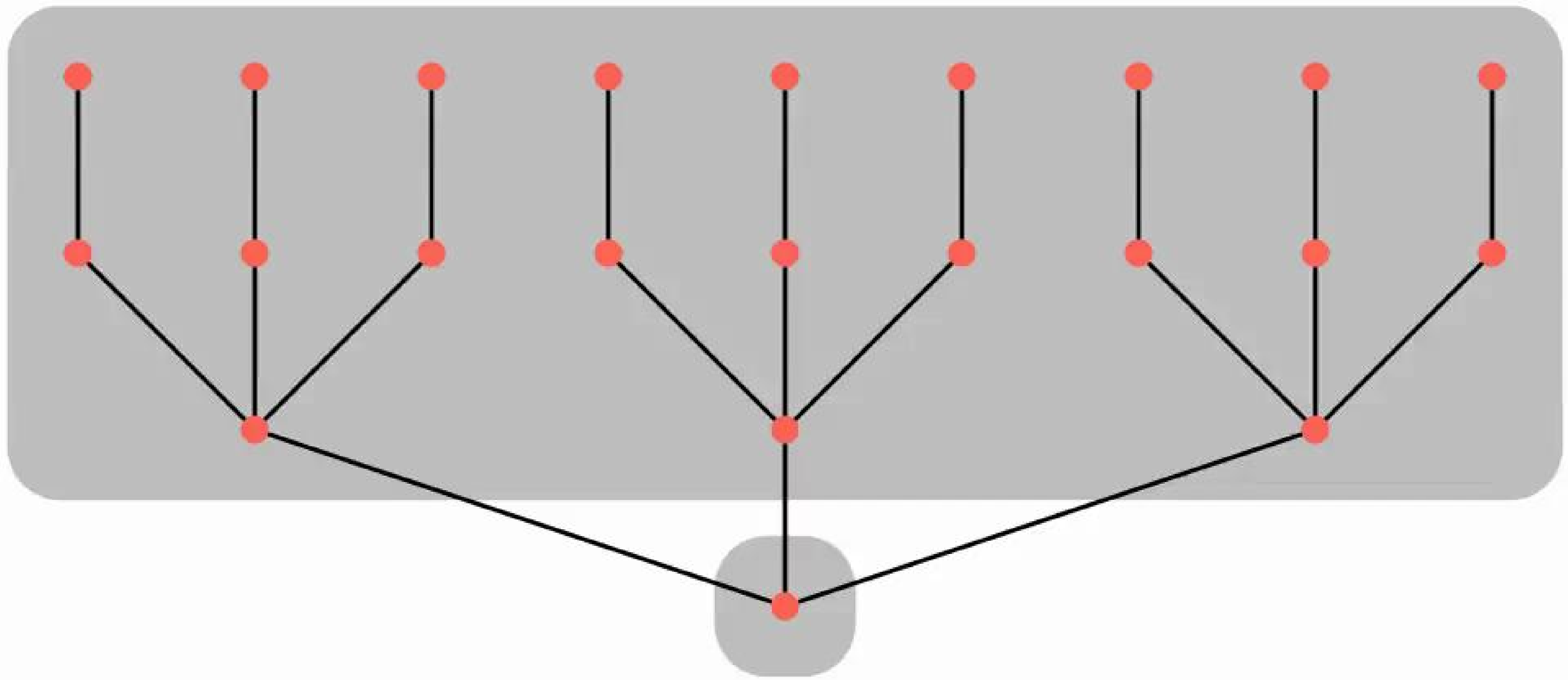


$$\mathrm{mw}_\infty(\mathit{Trees}) \leqslant 3$$

Trees



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Theorem. $\mathrm{mw}_\infty \approx$ clique-width

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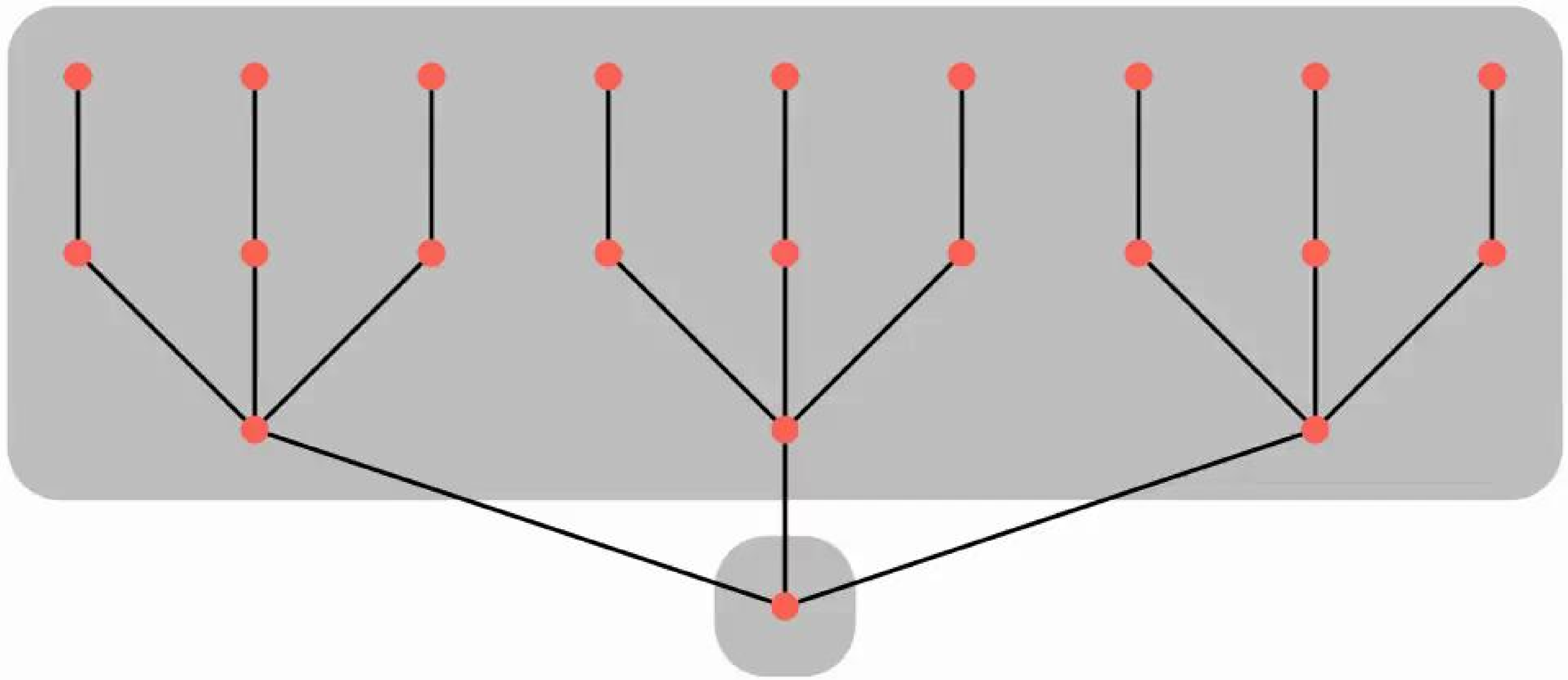

$$\mathrm{mw}_\infty(\mathit{Trees}) \leqslant 3$$

Theorem. $\mathrm{mw}_\infty \approx \text{clique-width}$

Corollary. Every class of bounded clique-width has bounded merge-width.

Twin-width

[Bonnet, Kim, Thomassé, Watrigant, 2021]

$$\text{twin-width}(G) \leqslant k \quad ^*$$

$$\Updownarrow$$

$\exists$ construction sequence such that each *part* neighbors $\leqslant k$ other parts

*differs by ± 1 from original definition

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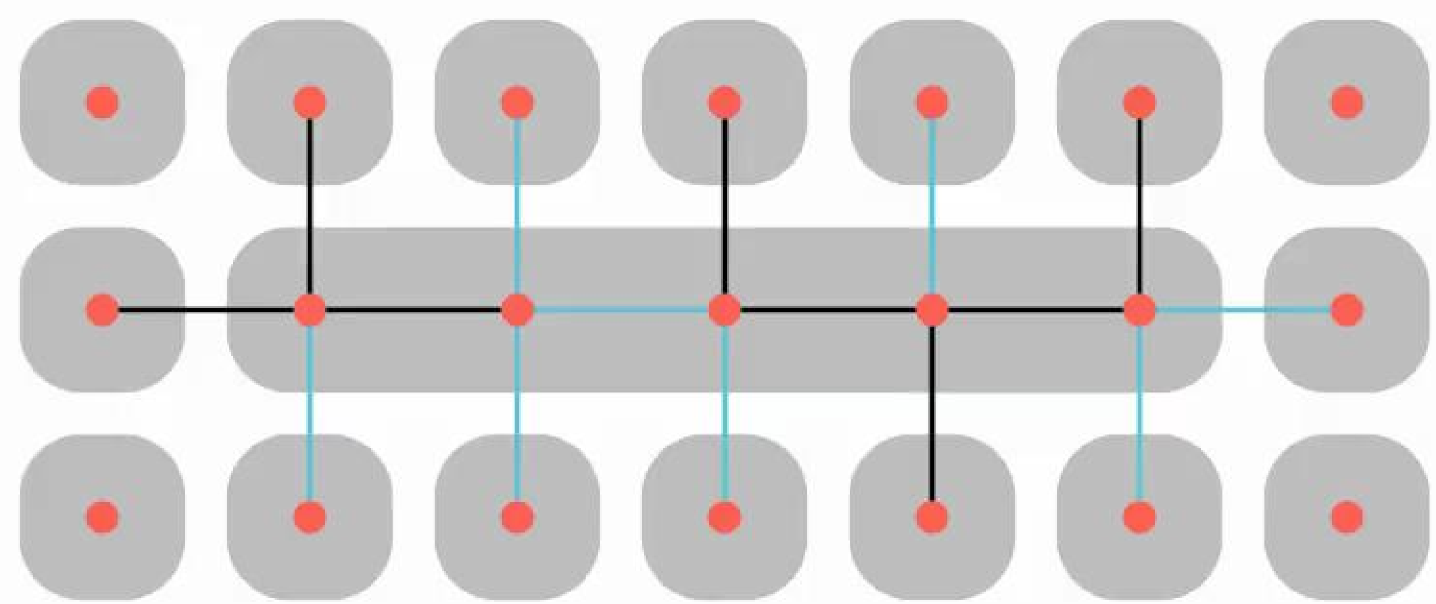

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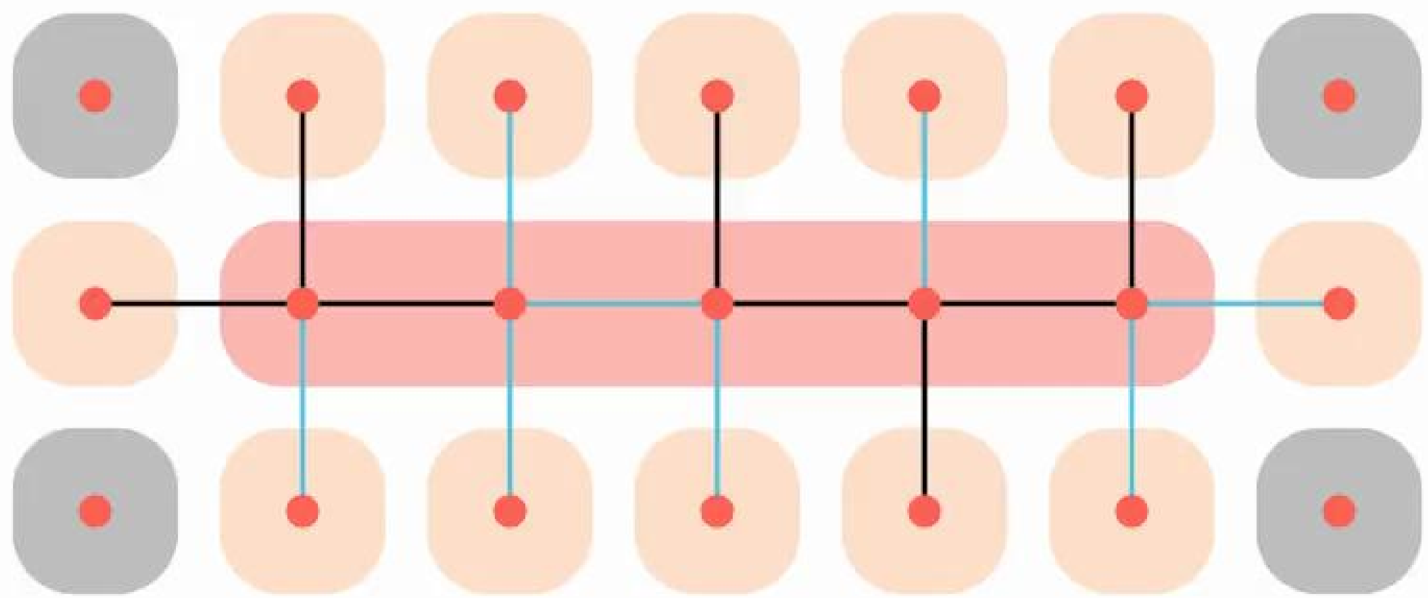

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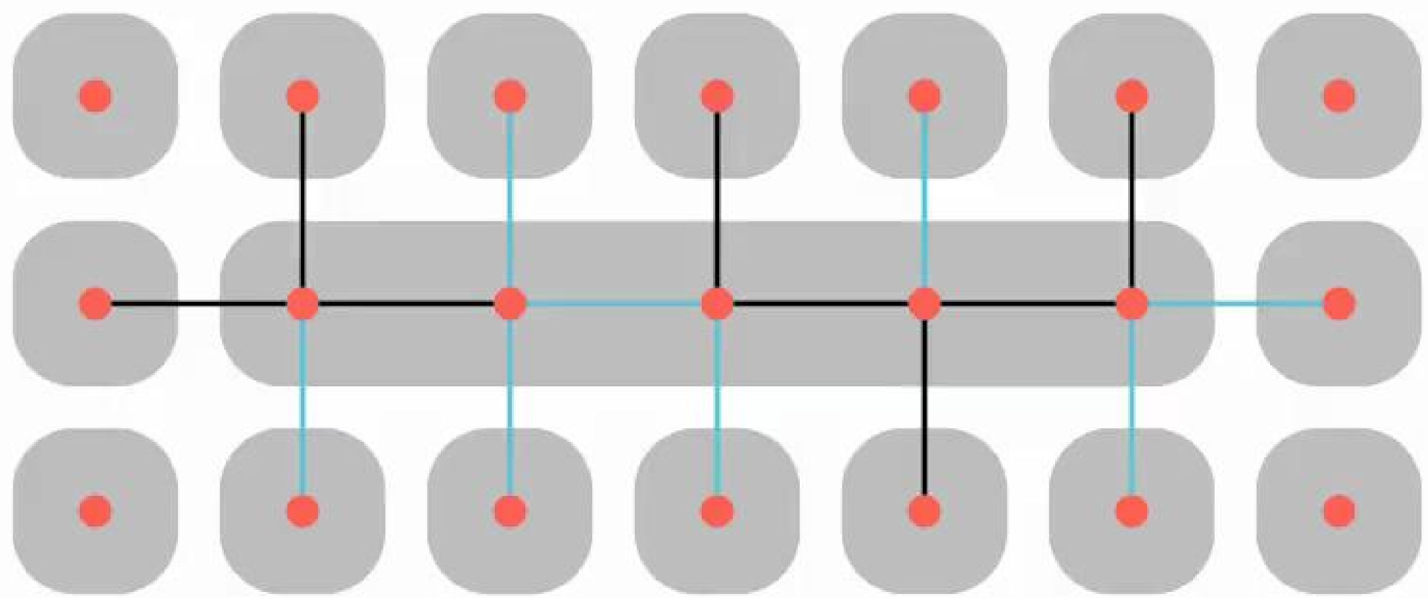

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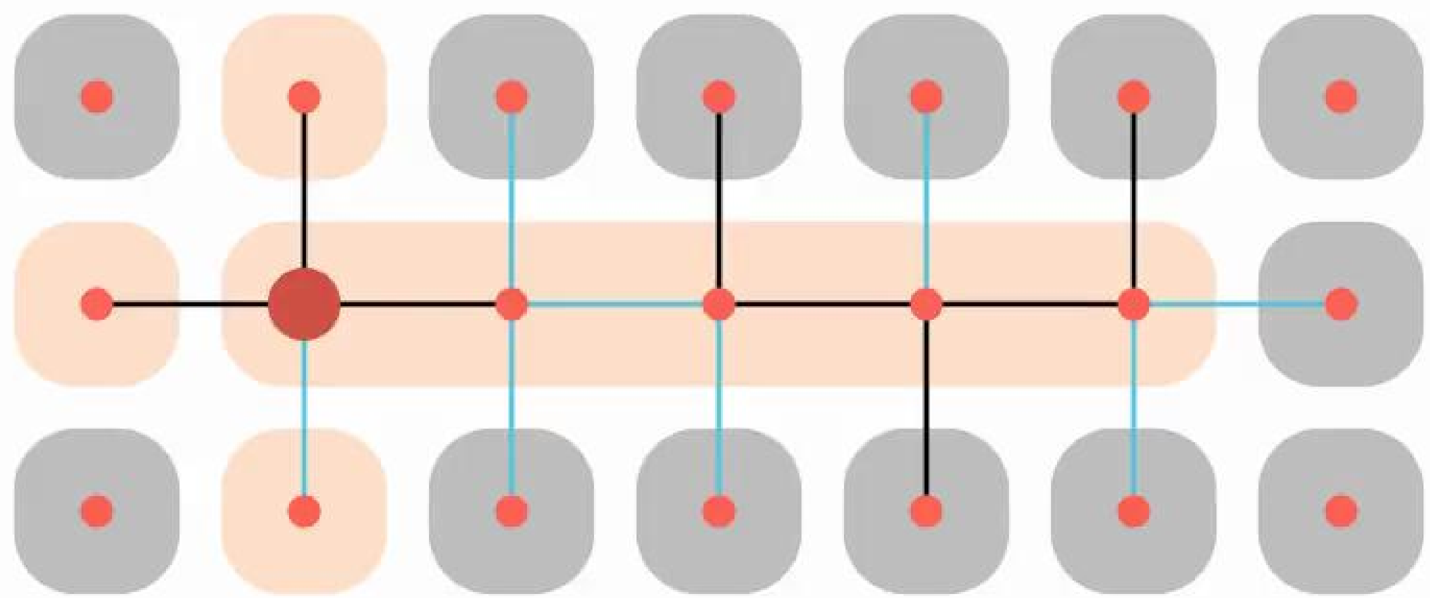

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Degeneracy

$$\mathrm{degeneracy}(G) \leqslant d$$

$$\Updownarrow$$

$\exists$ total order on $V(G)$ such that each vertex has $\leqslant d$ neighbors before it

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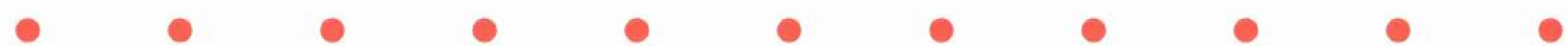

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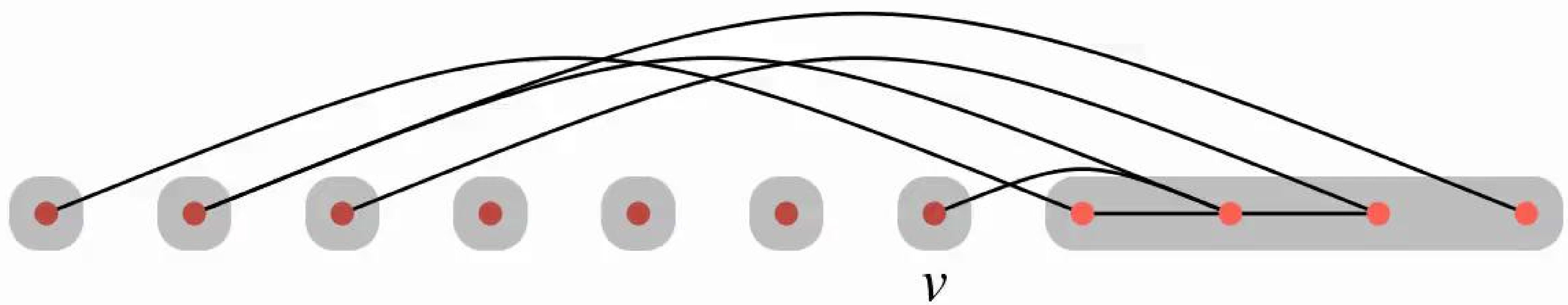

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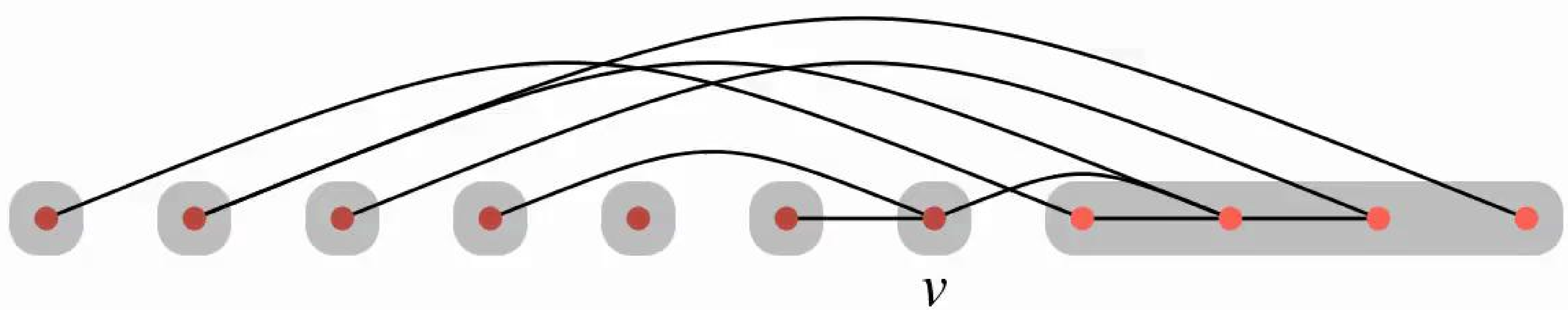

resolve $\{v\}$ with $\leqslant d$ parts to the left

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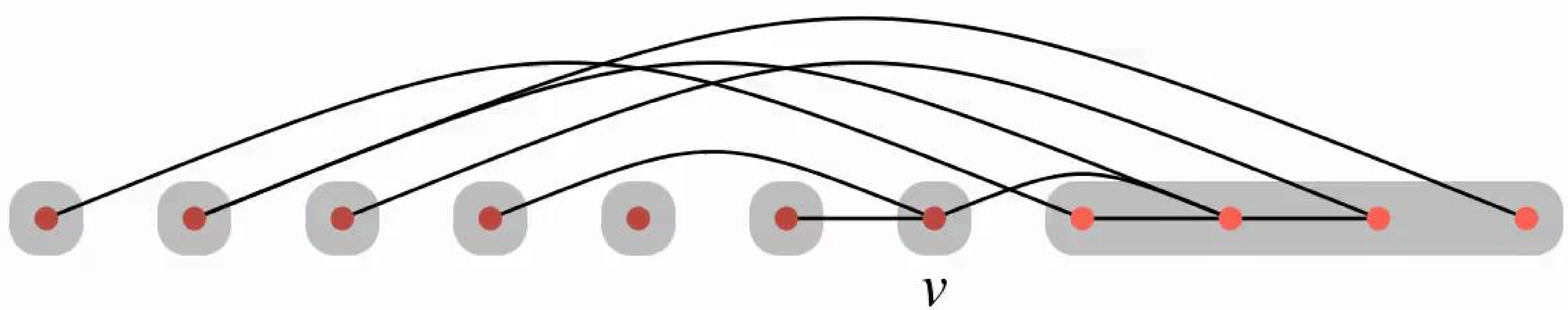

$$\text{mw}_1(G) \leqslant 2 + \text{degeneracy}(G)$$

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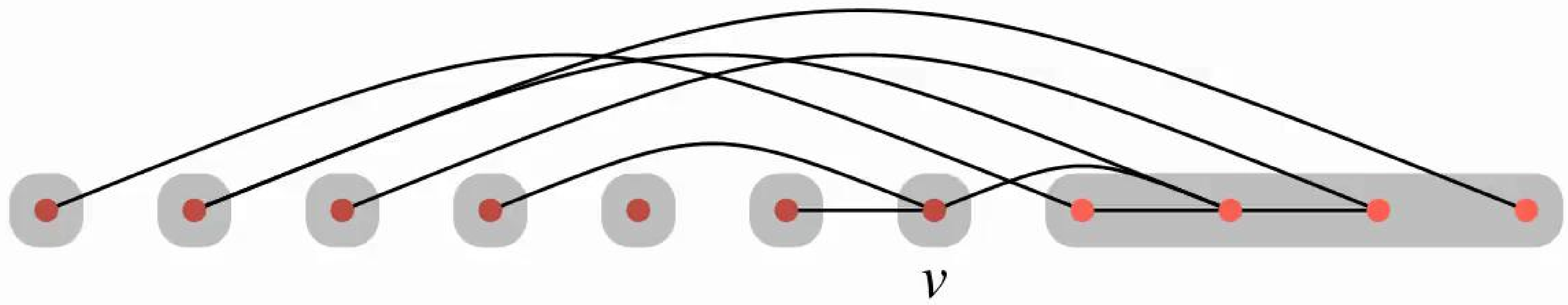

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Theorem. Every class of *bounded expansion* has bounded merge-width.

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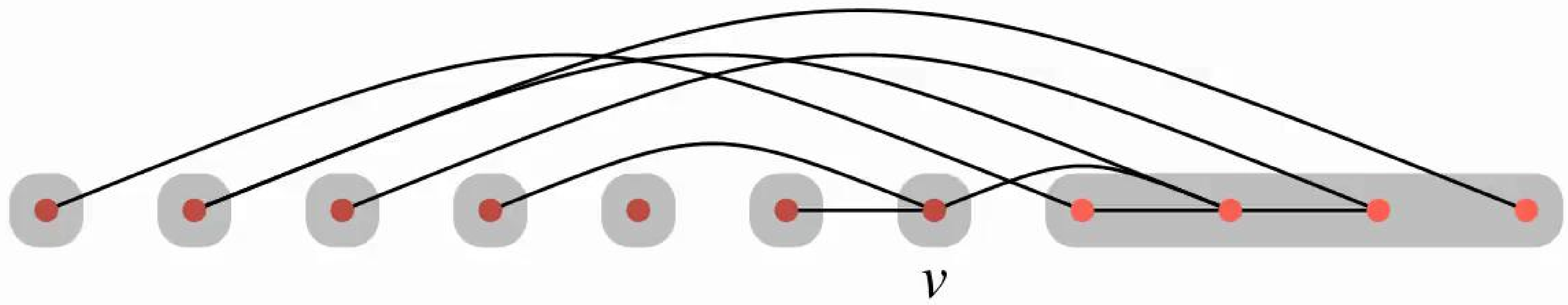

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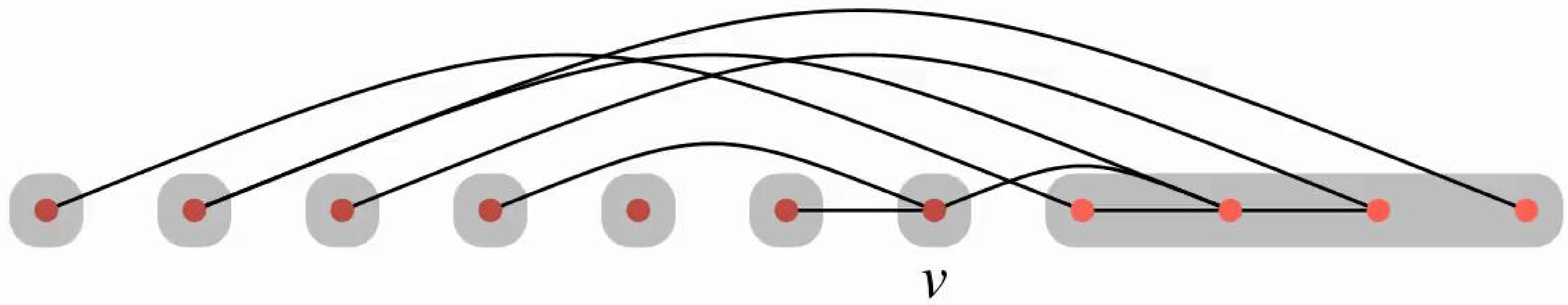

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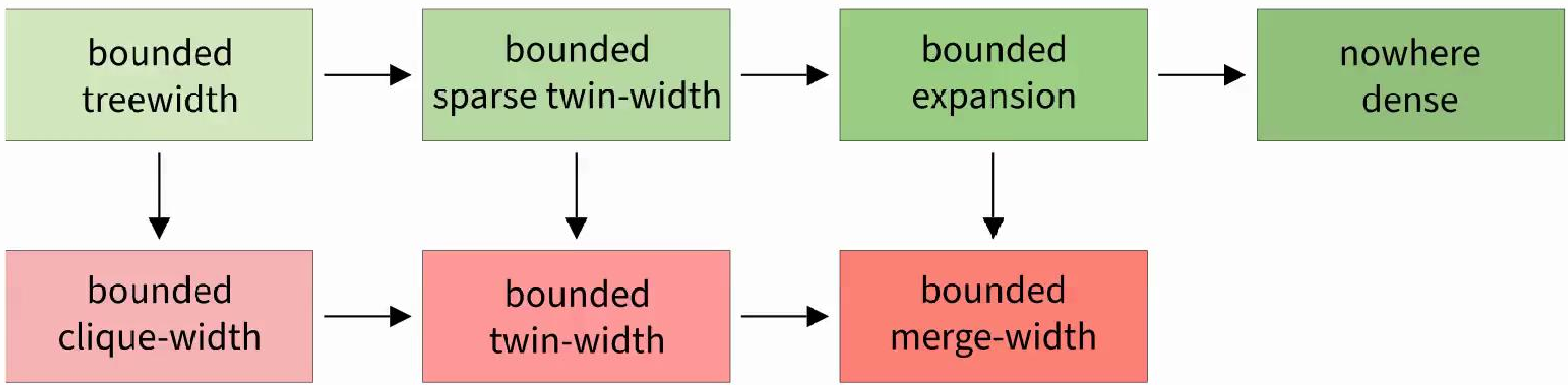
- or as a topological minor.


bounded treewidth
bounded sparse twin-width
bounded expansion
nowhere dense
bounded clique-width
bounded twin-width
bounded merge-width

First-Order Model Checking

fpt algorithm for FO-definable properties

Input: a graph G and first-order sentence φ

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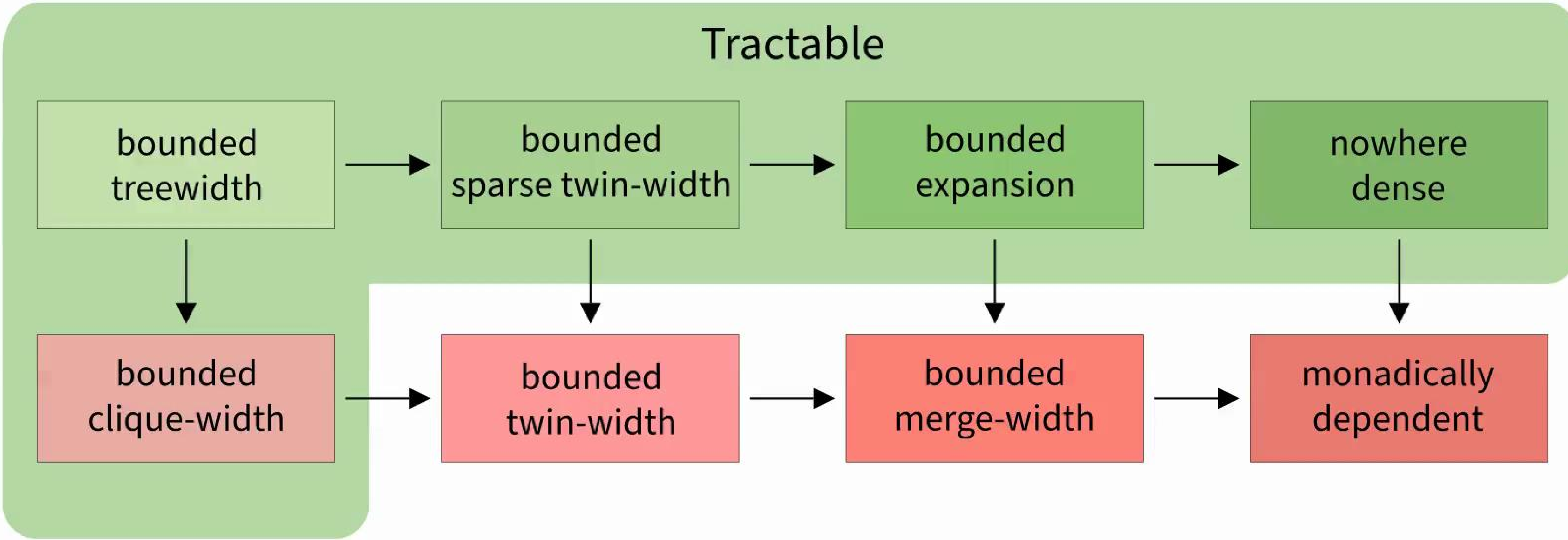

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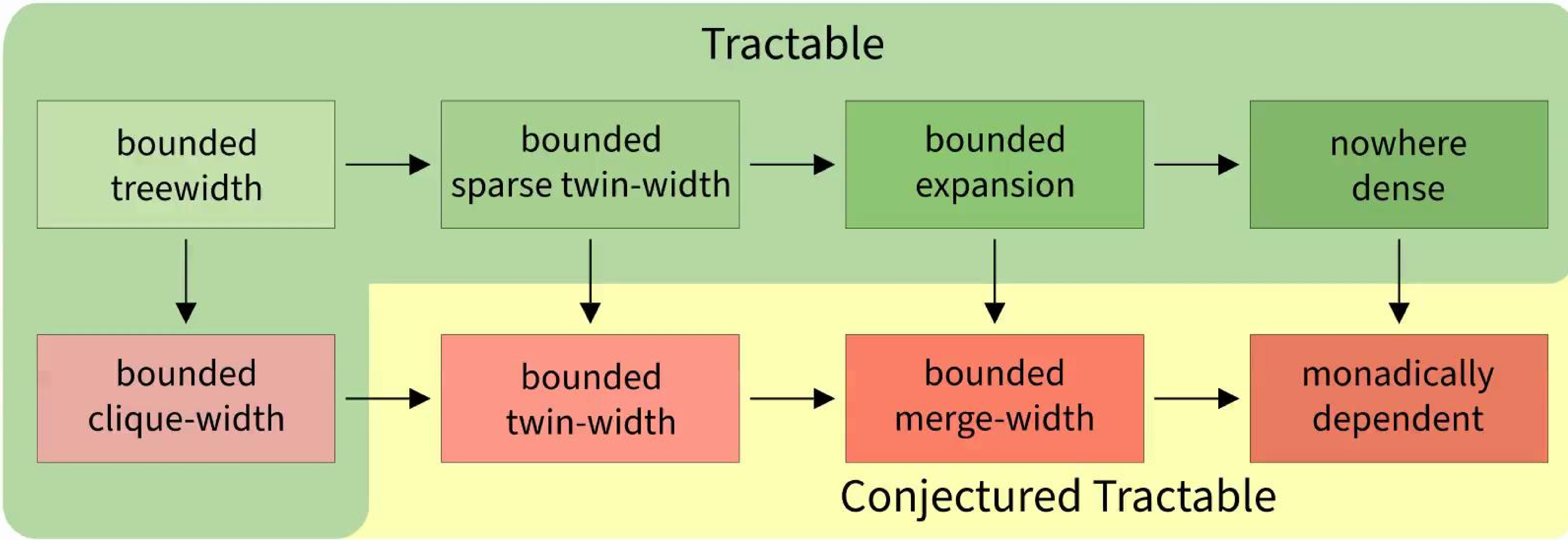

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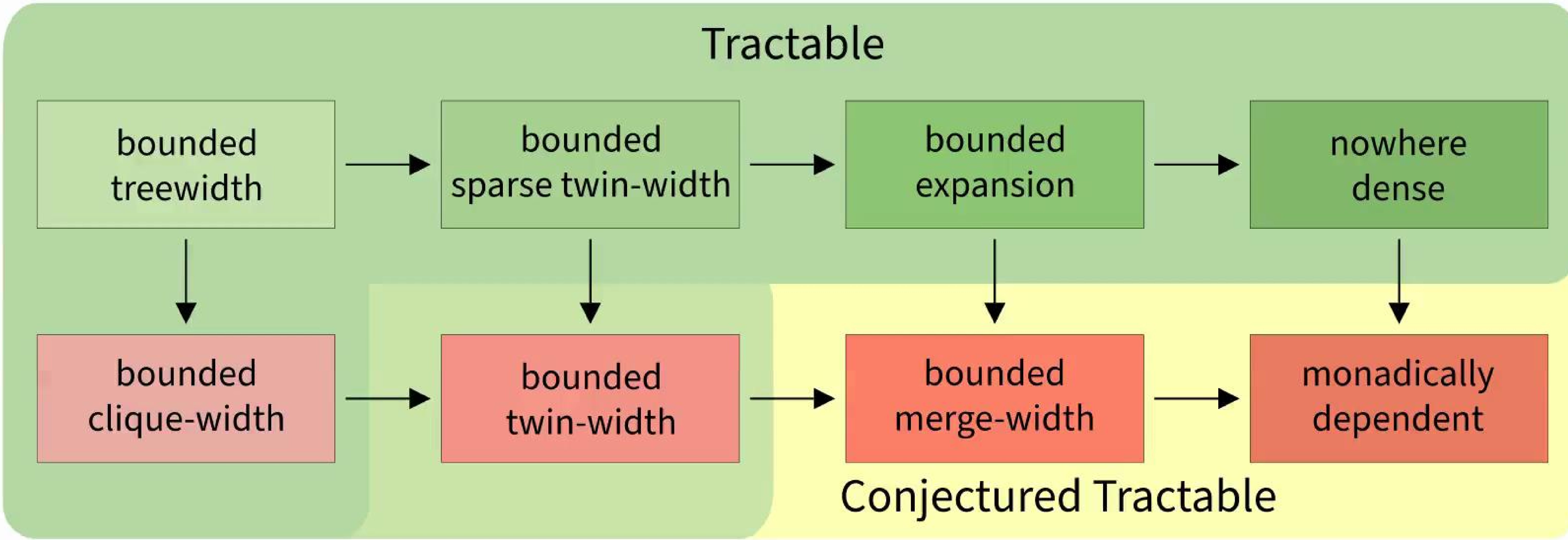

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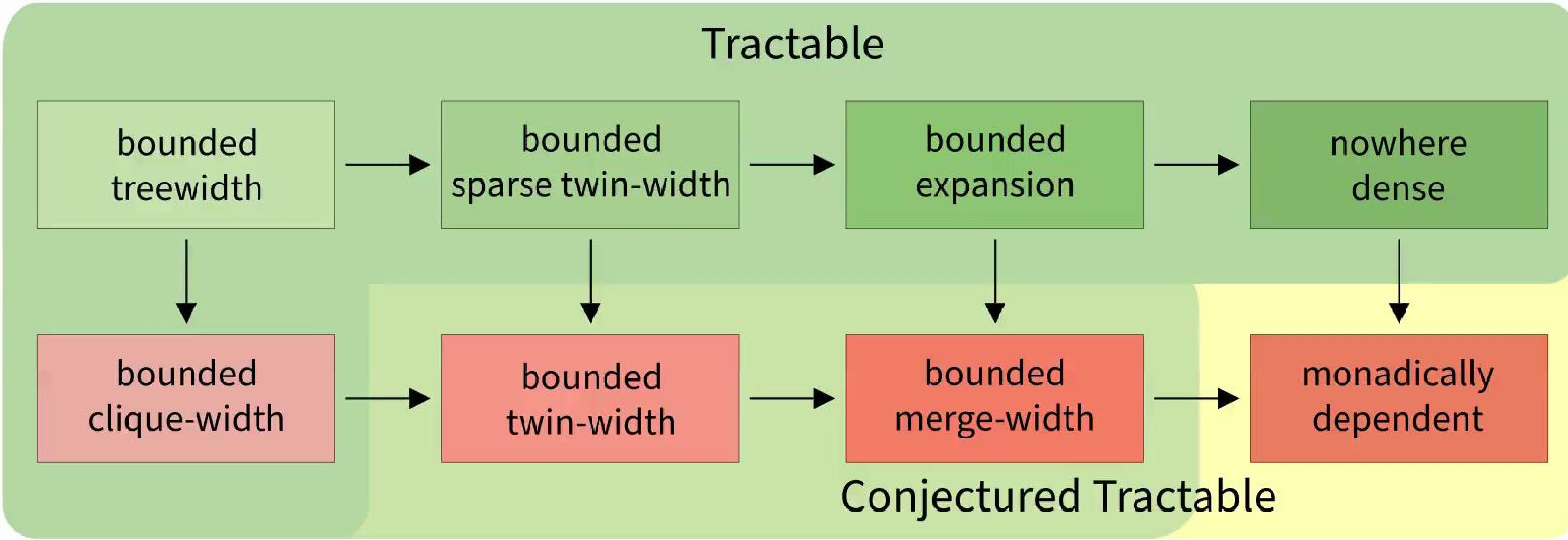

Conjectured Tractable

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Theorem. There is an algorithm which given:

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decides whether G satisfies φ in time

$$f(q,w) \cdot |V(G)|^3,$$

for w=radius-r width of C and $r = 2^{O(q^2)}$

Proof sketch

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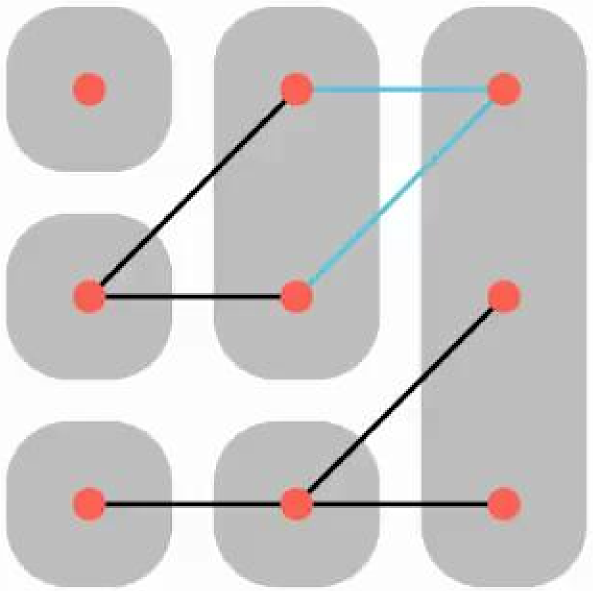

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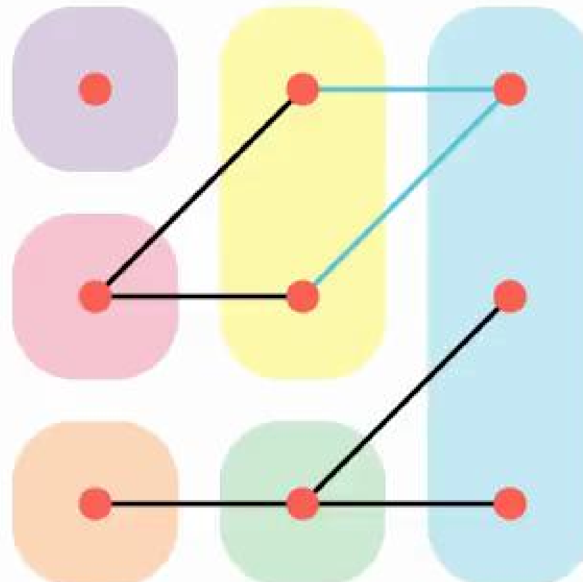

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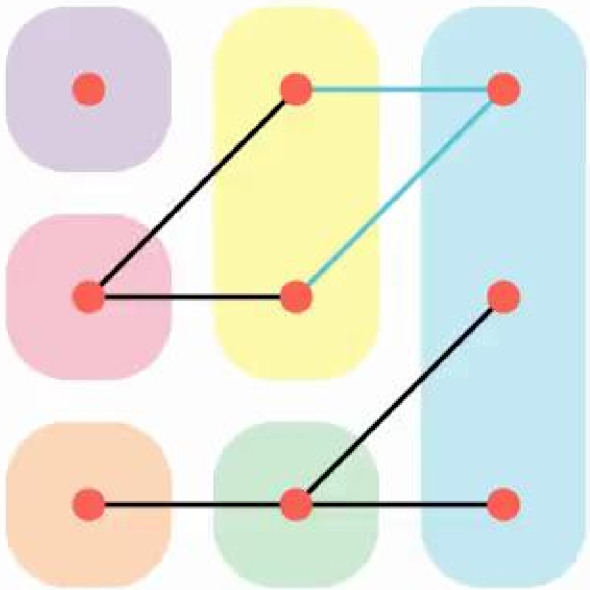

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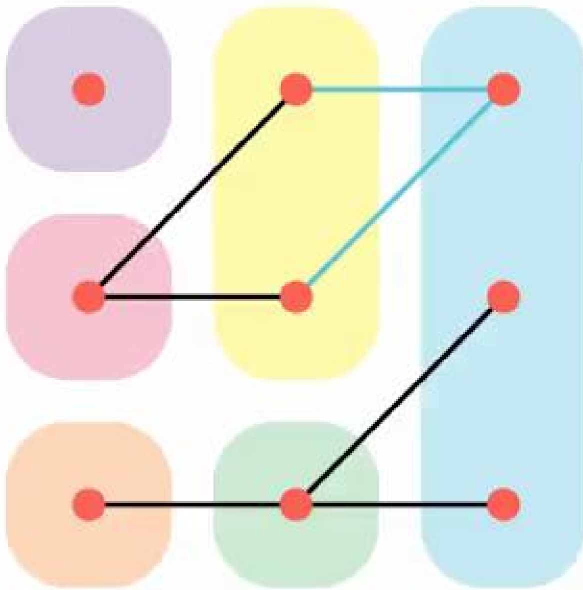

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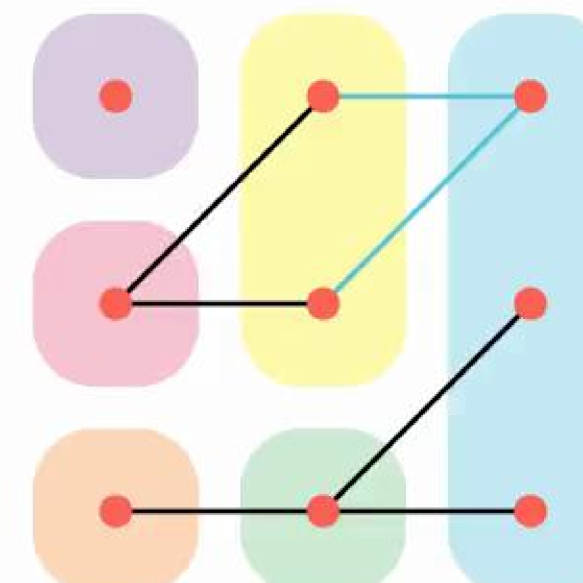

Locality Theorem

Fix a structure A, vertex $v \in V(A)$, $q \in \mathbb{N}$.

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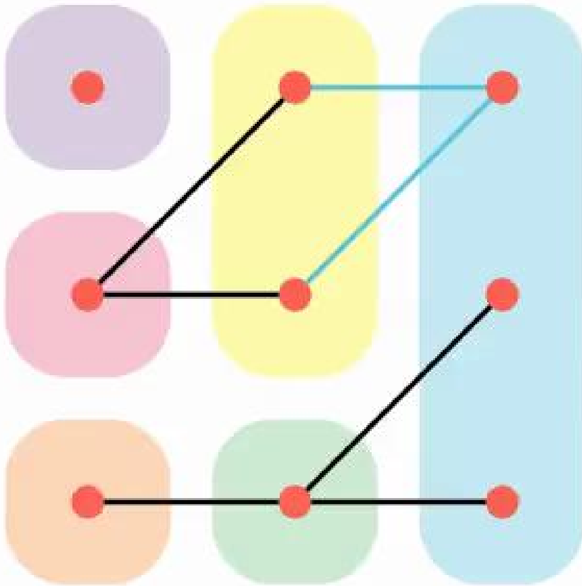

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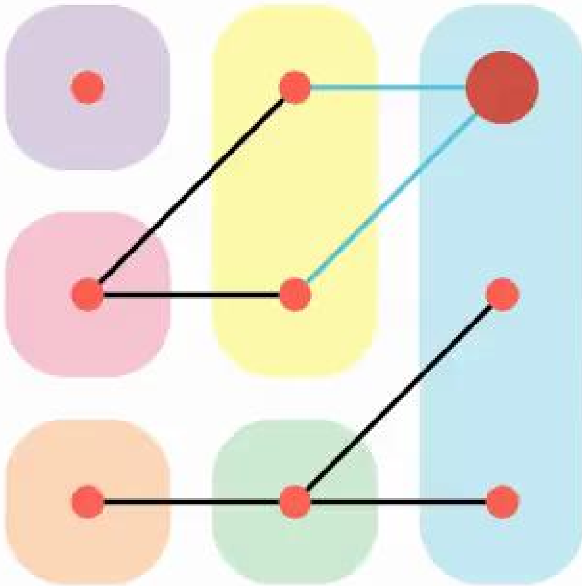

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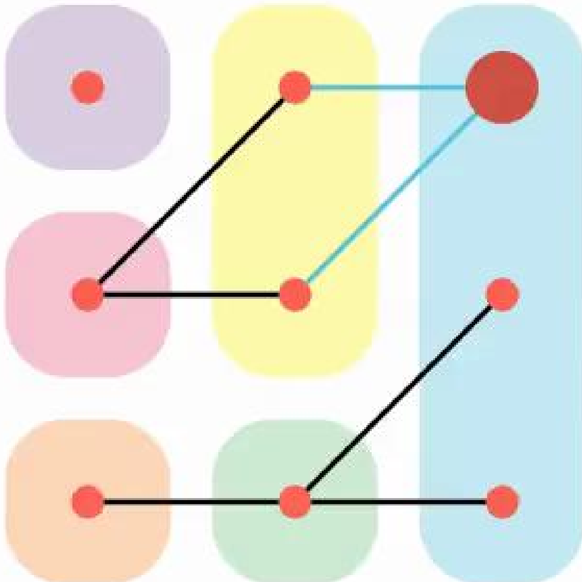

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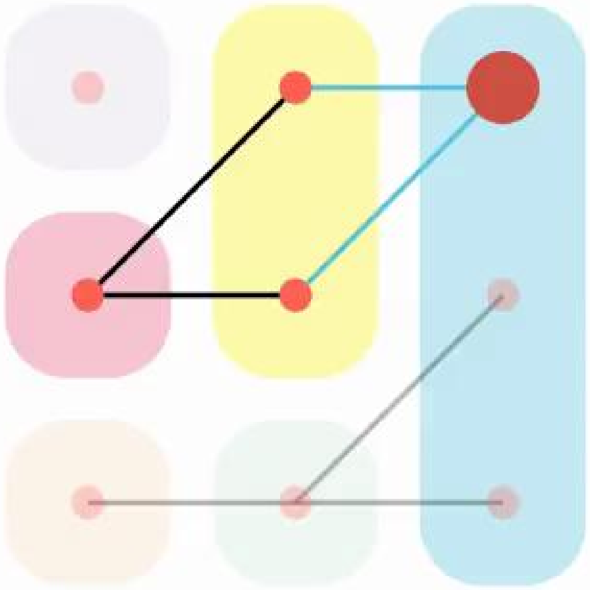

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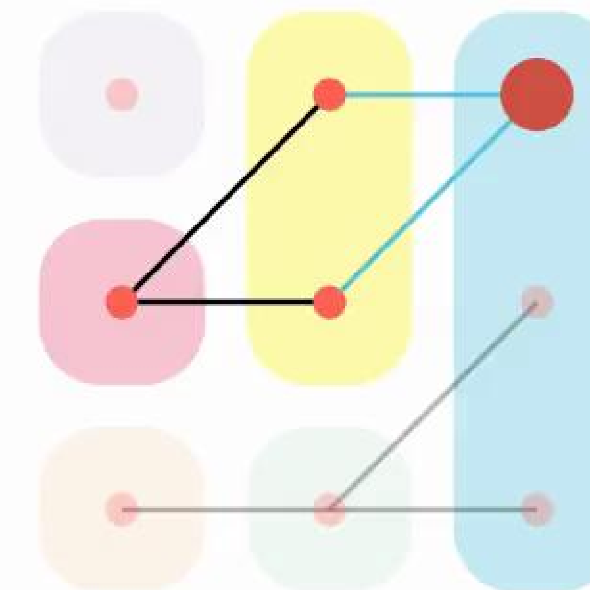

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Locality Theorem. $\mathrm{tp}_q(A, v)$ *is determined by* $\mathrm{ltp}_q(A, v)$ *(and global sentences).*

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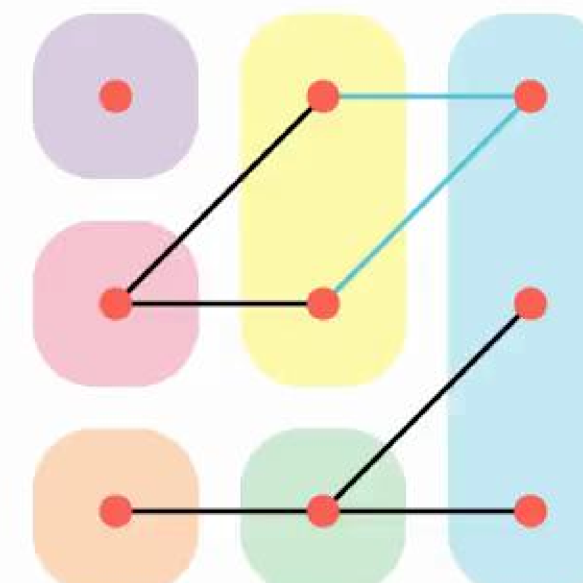


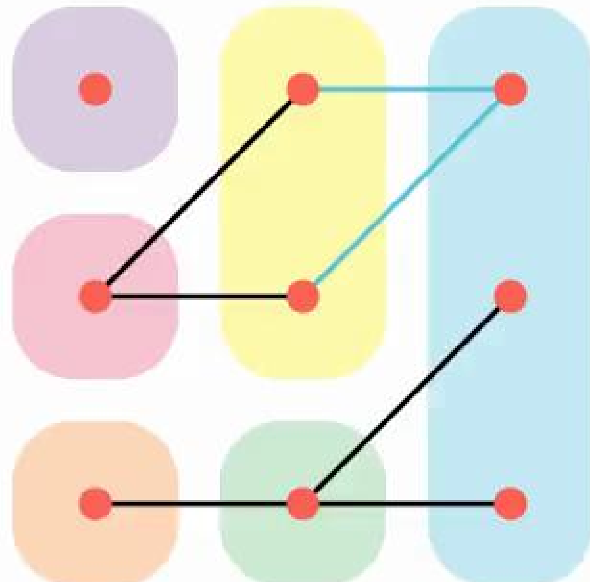
Proof sketch

View a construction sequence as a sequence of structures $G_1, \ldots, G_m$

G_t: binary relations E_t, N_t,
unary relation $P(\cdot)$ for each $P \in \mathcal{P}_t$.

- $\mathrm{Gaifman}(G_t) = (V, E_t \cup N_t)$
- $\#\{\text{unary predicates in an } r\text{-ball}\} \leq \text{radius-}r \text{ width}$

$\rightarrow$ each $\mathrm{ltp}_q(G_t, v)$ has size bounded in terms of mw_r



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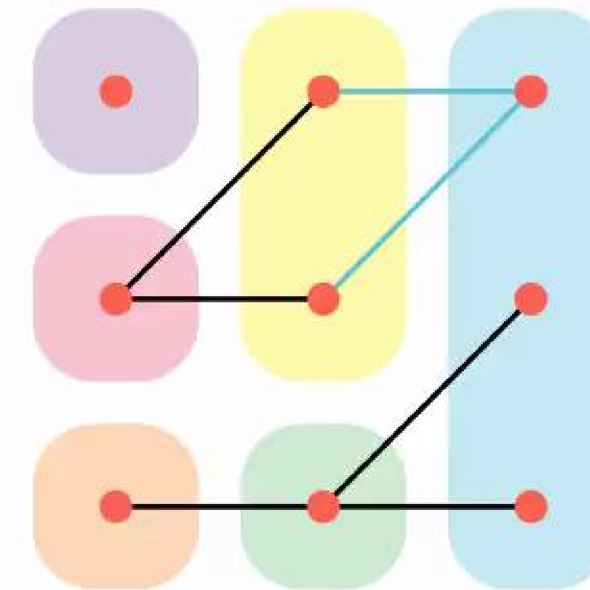

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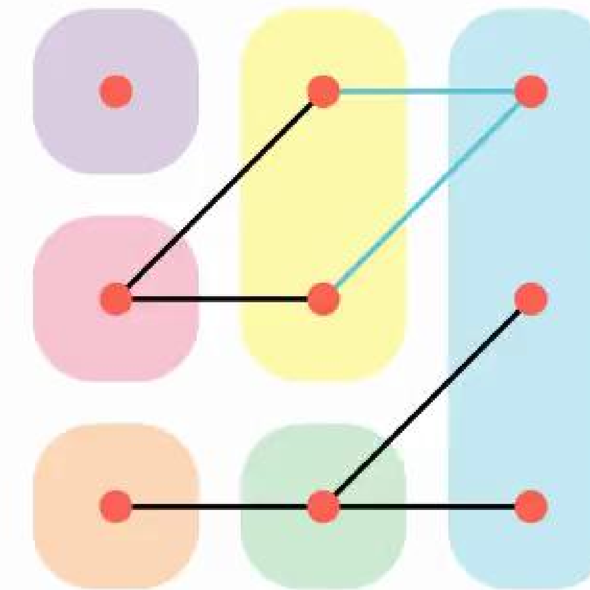

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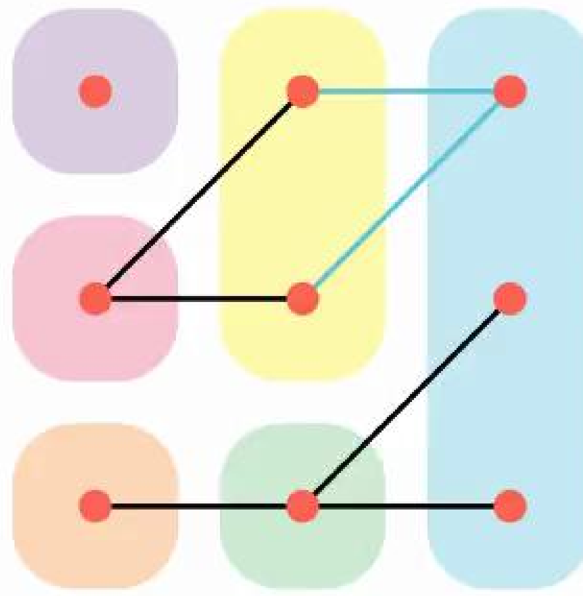

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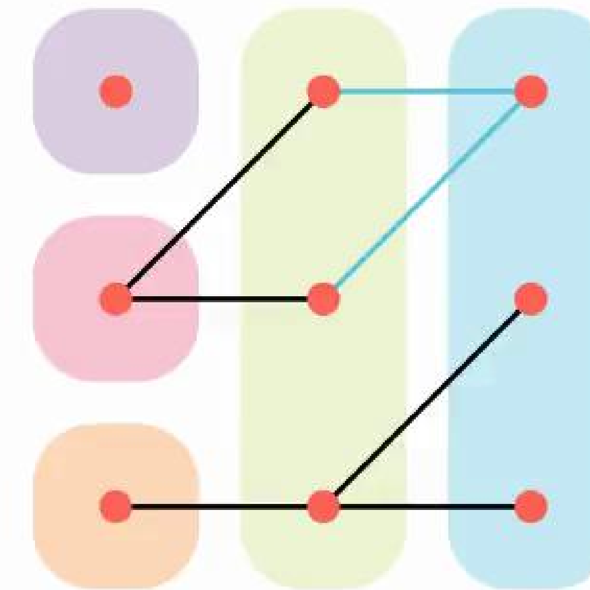

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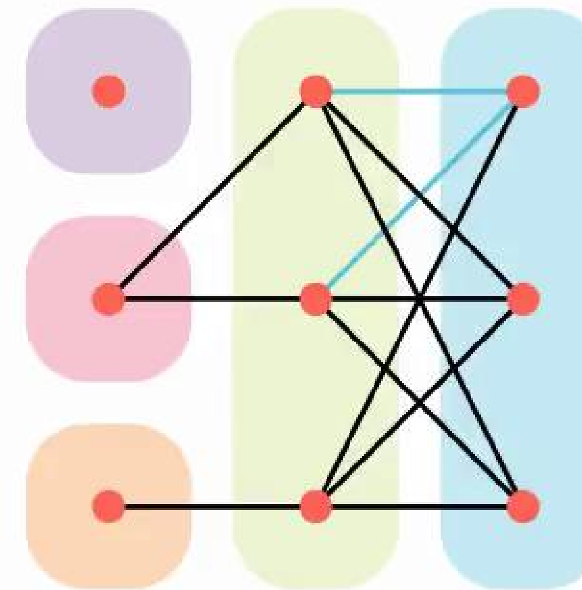

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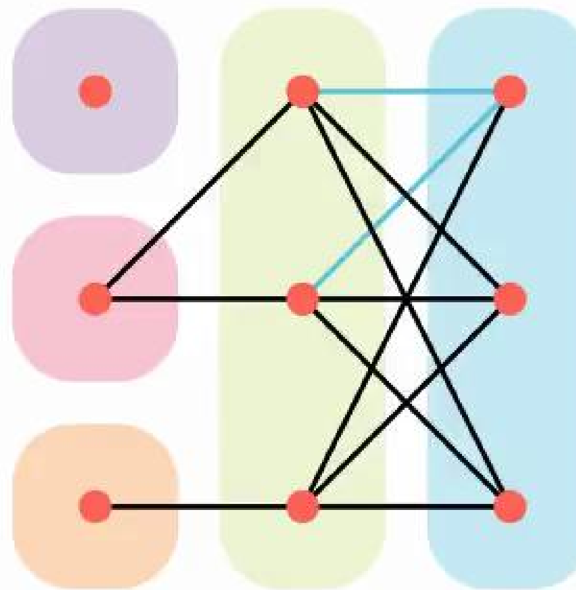

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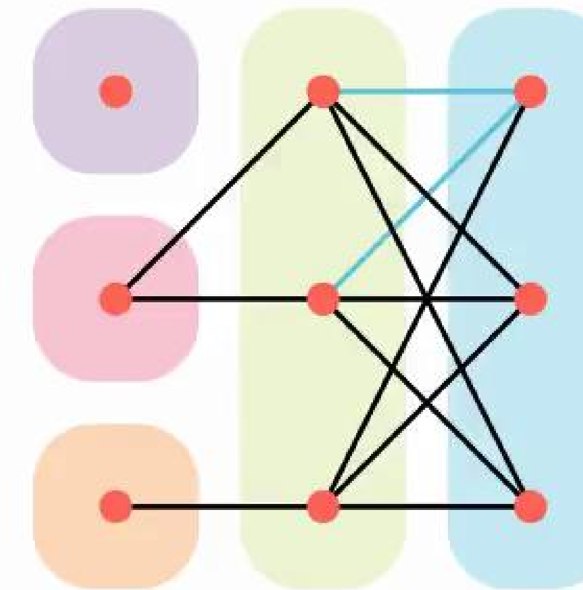

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Remark. *This can be computed in fpt time.*

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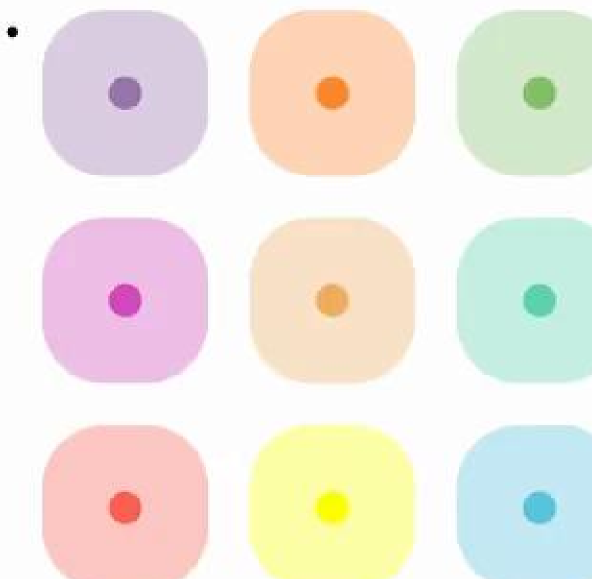

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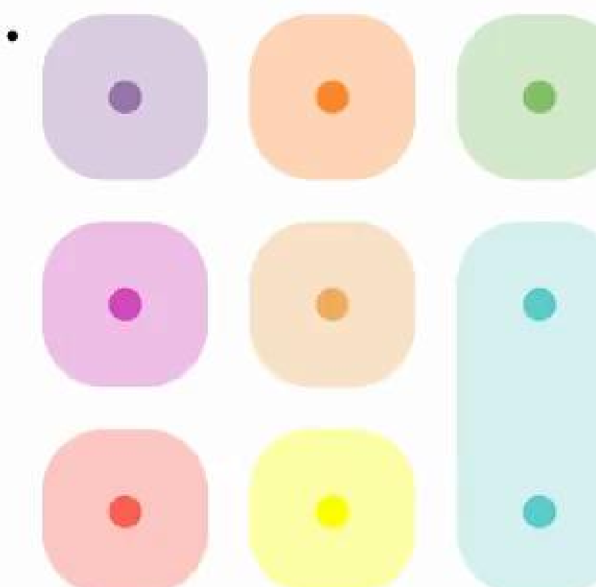

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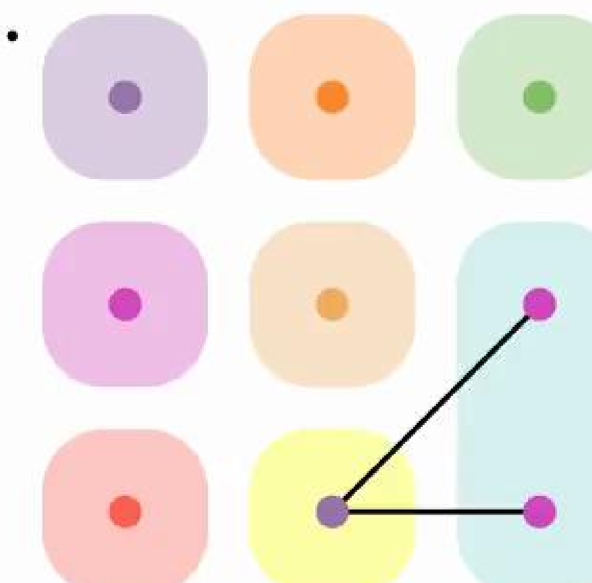

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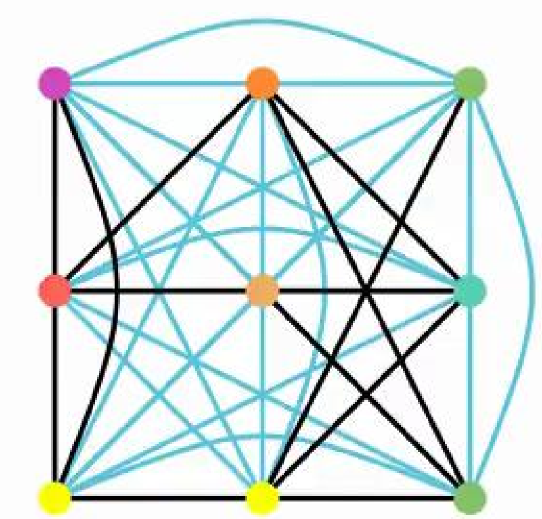

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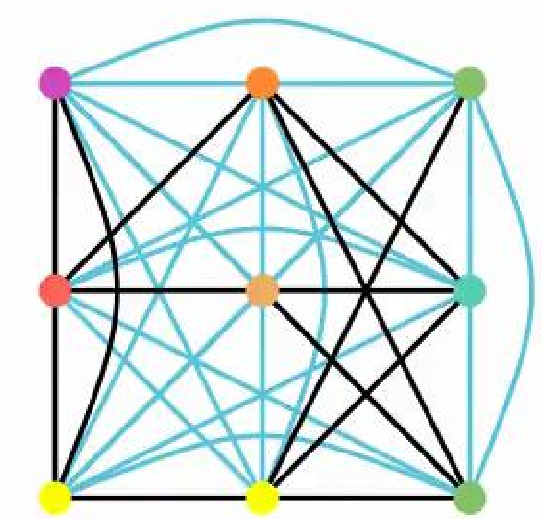

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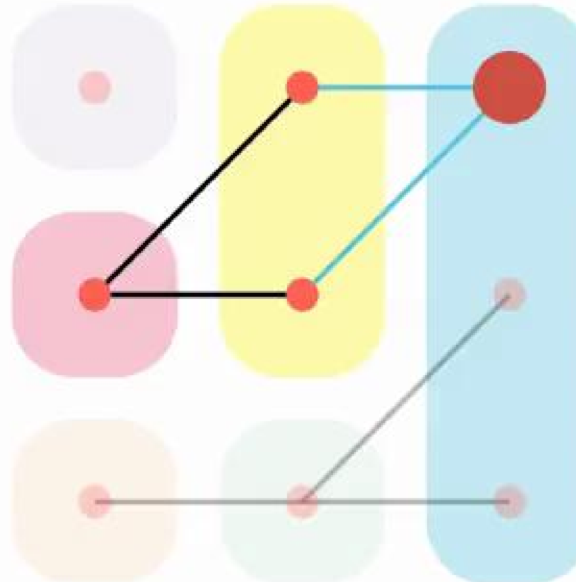
Output: 'yes' iff $\varphi(x) \in \mathrm{ltp}_q(G_m, v_1)$.

Locality Theorem



Fix a structure A, vertex $v \in V(A)$, $q \in \mathbb{N}$.

$\mathrm{tp}_q(A, v) := \{$formulas $\varphi(x)$ of depth q with $A \models \varphi(v)\}$

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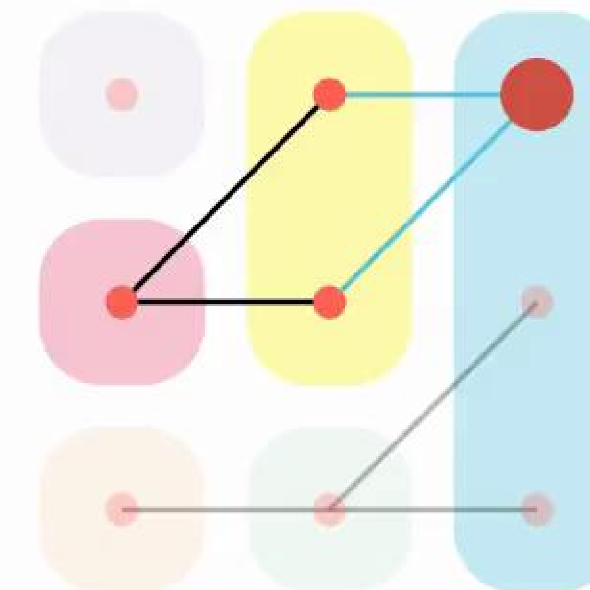

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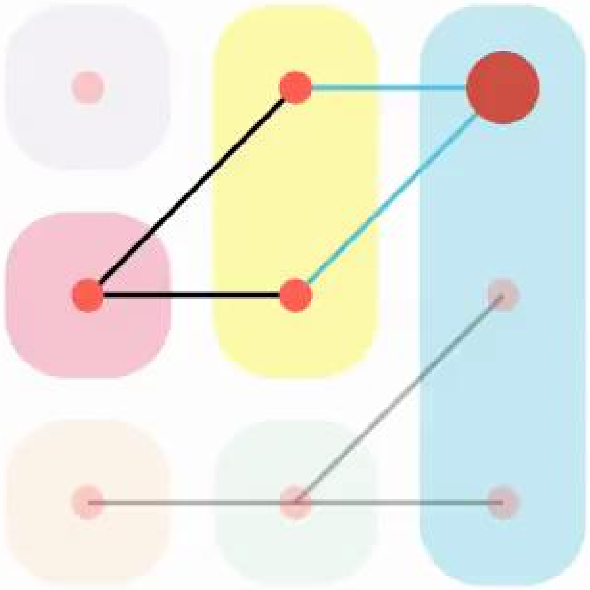

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Rank-Preserving Locality Theorem. (Grohe, Kreutzer, Siebertz 2013)
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- at time t resolve pairs v, w with $\mathrm{dist}_{G_t}(v, w) \leq r$. ■

Example. If $\varphi(x, y) = \exists z. E(x, z) \wedge E(y, z)$ then $\varphi(G) = G^2$.

Theorem. C has bounded merge-width $\Rightarrow \varphi(C) := \{\varphi(G) \mid G \in C\}$ has bounded merge-width.

Corollary. C has bounded merge-width $\Rightarrow C$ is *monadically dependent*: $\varphi(C) \neq \{\textit{Graphs}\}$, for every formula $\varphi(x, y)$.

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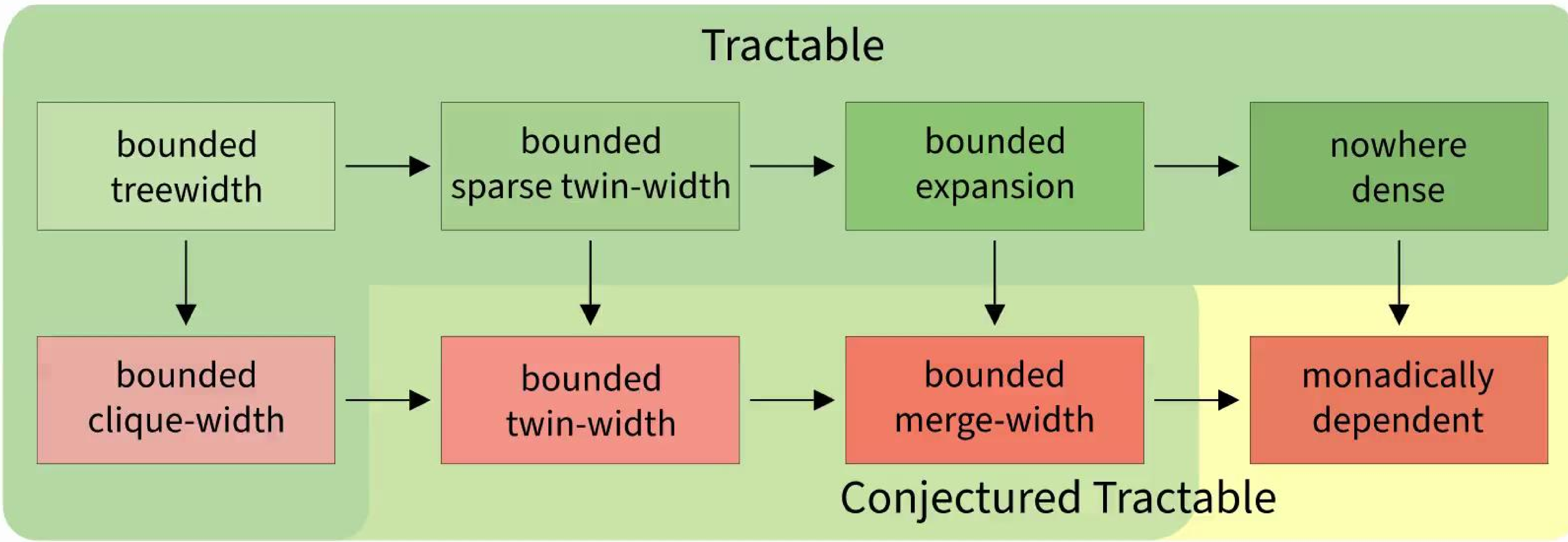

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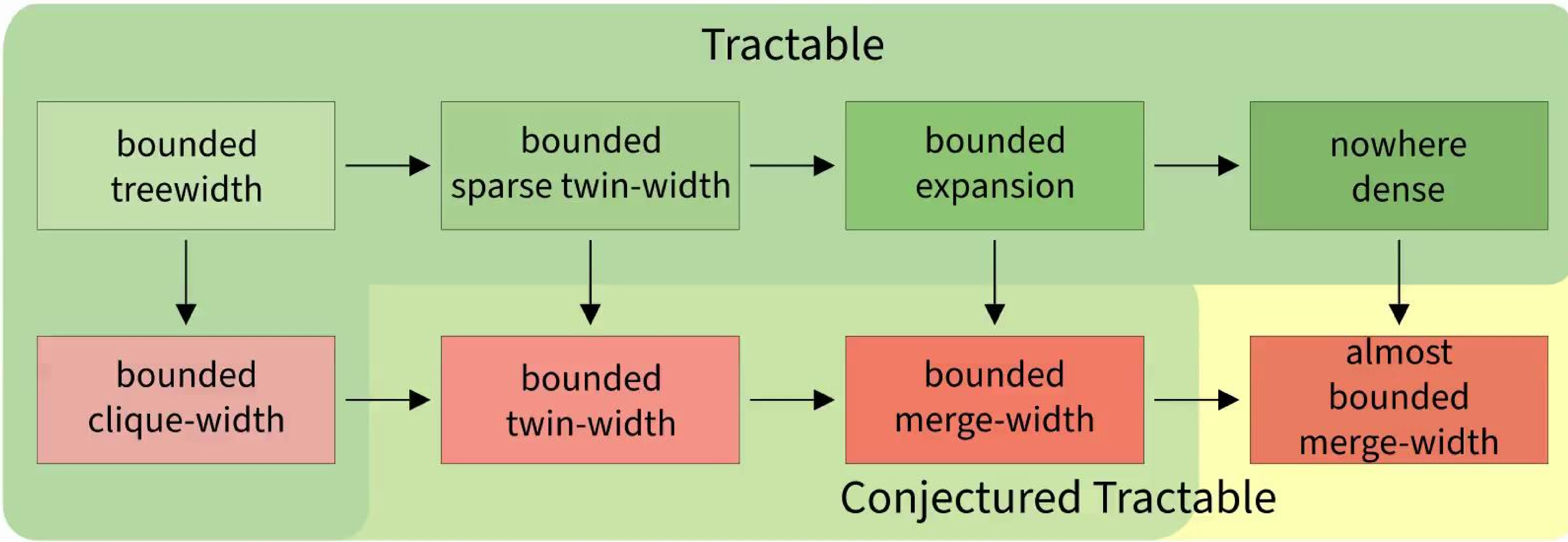

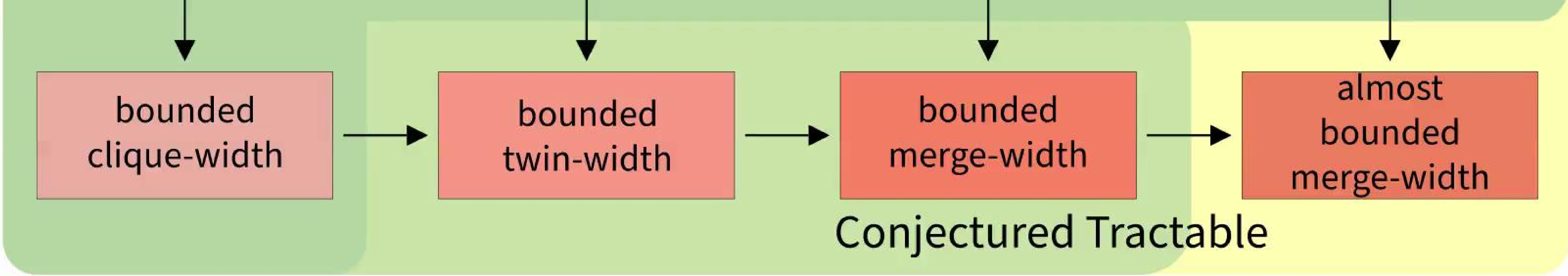

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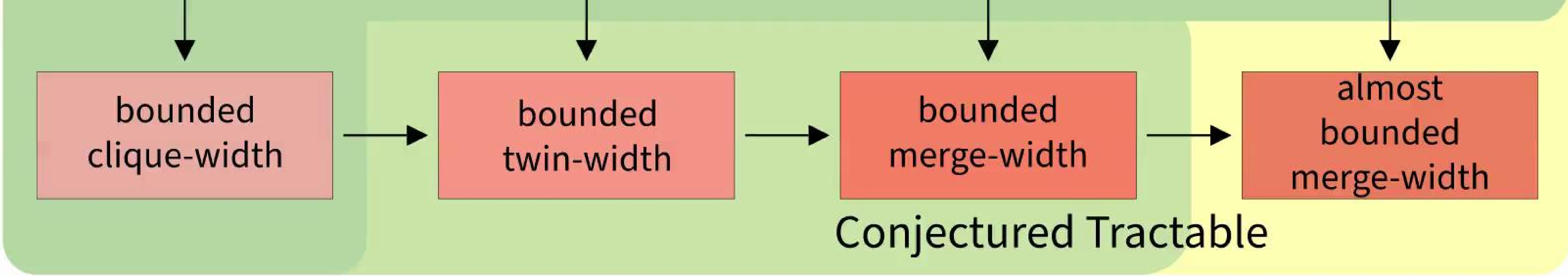

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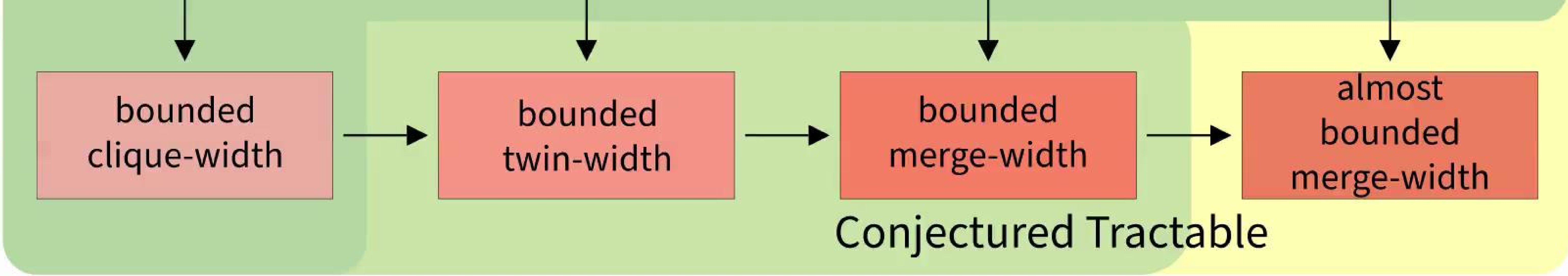

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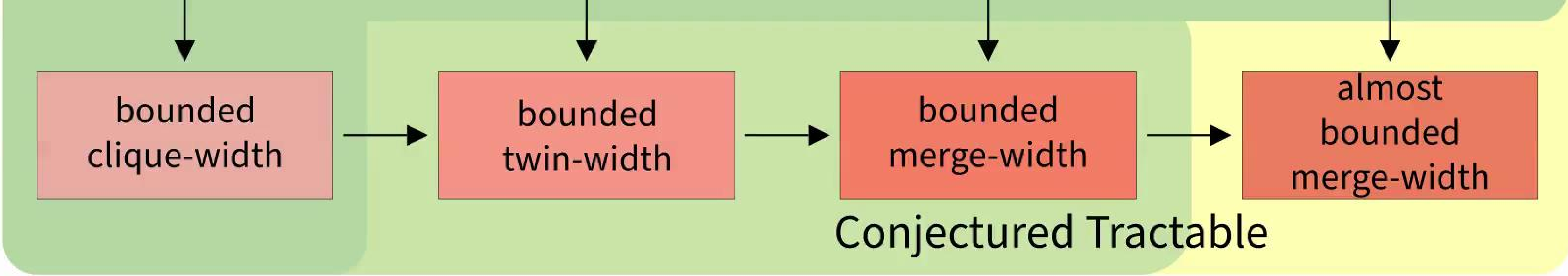

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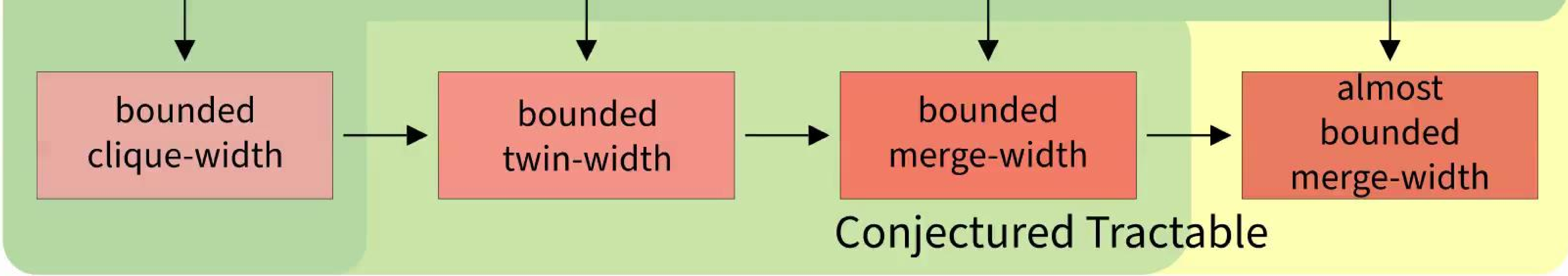

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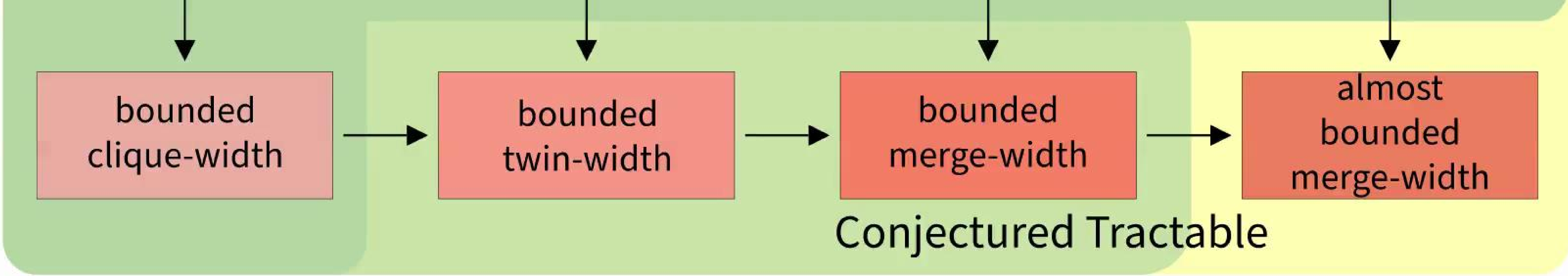

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- other algorithmic problems on classes of bounded merge-width?