Applied Logic

Lecture 1 - Propositional logic

Marcin Szczuka

Institute of Informatics, The University of Warsaw

Monographic lecture, Spring semester 2018/2019

Lecture plan

Language of propositional logic

The language of propositional logic consists of:

1. **Symbols**:
   - Infinite set of variables $VAR = \{p, q, r, ...\}$
   - Four (4) special symbols: $\neg, \vee, (,)$.
2. **Rules of composition:** We present a recursive definition of *properly constructed propositional formulæ* (or *propositional formulæ* for short)
   - Variables, i.e., elements of $VAR = \{p, q, r, ...\}$ are propositional formulæ.
   - If ϕ is a propositional formula then $\neg\phi$ is a propositional formula.
   - If ϕ and ψ are propositional formulæ then $(\phi \vee \psi)$ is a propositional formula.

The set of all properly constructed propositional formulæ we will denote by $FORM$.

Language of propositional logic

We call symbols $\neg$ and $\vee$ *logical connectives* (or *logical operators*). The parentheses (,) are introduced to determine the precedence of operations and can be dropped without lack of generality.
In order to simplify notation and support intuitive understanding we introduce additional symbols such as constants $\top, \bot$ and operators $\wedge$, $\Rightarrow$, $\Leftrightarrow$. They are defined as follows:

$$\begin{aligned}
\phi \wedge \psi &=_{def} \neg(\neg\phi \vee \neg\psi) \\
\phi \Rightarrow \psi &=_{def} \neg\phi \vee \psi \\
\phi \Leftrightarrow \psi &=_{def} (\neg\phi \vee \psi) \wedge (\neg\psi \vee \phi) \\
\top &=_{def} p \vee \neg p \text{ for any } p \in VAR \\
\bot &=_{def} p \wedge \neg p \text{ for any } p \in VAR
\end{aligned}$$

The meaning of logical operators and constants (which may be treated as operators with 0 arguments) will be explained further in this lecture.

Lecture plan

1. Language of propositional logic
2. Binary (two-value) semantics
3. Logical consequence, model
4. Syntax
5. Deduction property
6. Completeness
   - Soundness and adequacy
   - Consistency
   - Completeness proof
7. Compactness

Meaning (semantics) of symbols

Let $\mathcal{B} = \{0, 1\}$. We associate 0 with logical value *FALSE* and 1 with *TRUE*. Then, the logical operators are associated with corresponding functions on $\mathcal{B}$. Connective $\neg$ corresponds to function $\neg : \mathcal{B} \rightarrow \mathcal{B}$, such that $\neg(x) = 1 - x$. Two-argument connectives correspond to functions of the form

$$* : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$$

where $*$ is any operator in $\{\vee, \wedge, \rightarrow, \Leftrightarrow\}$. The following truth table defines common operators:

x	y	$x \vee y$	$x \wedge y$	$x \rightarrow y$	$x \Leftrightarrow y$
0	0	0	0	1	1
0	1	1	0	1	0
1	0	1	0	0	0
1	1	1	1	1	1

Intuitively, $\vee, \wedge, \rightarrow$ correspond to disjunction (alternative, ... *or* ...), conjunction (... *and* ...), and implication (*if* ... *then* ...) in natural language, respectively.

Valuation

Valuation is (any) a function $v : VAR \to \mathcal{B}$. For any valuation v we may define semantics, i.e., a function $[\![.]\!]_v : FORM \to \mathcal{B}$. The definition is recursive w.r.t. the structure of formula.

(Const) $[\![\top]\!]_v = 1; \quad [\![\bot]\!] = 0$

(Var) for any variable $p \in VAR$

$$[\![p]\!]_v = v(p)$$

($\neg$) for any formula $\phi \in FORM$

$$[\![\neg\phi]\!]_v = 1 - [\![\phi]\!]_v$$

($*$) for any formulæ $\phi, \psi \in FORM$

$$[\![\phi * \psi]\!]_v = [\![\phi]\!]_v * [\![\psi]\!]_v$$

where $*$ is an abitrary logical operator (e.g., $\vee, \wedge, \Rightarrow$, ...).

LOGICAL CONSEQUENCE, MODEL

MODEL IN PROPOSITIONAL LOGIC

Valuation v is a *model* of a formula ϕ iff

$$[\![\phi]\!]_v = 1.$$

Conversely, we say that ϕ is *true* for v.

In 1936, Tarski introduced the notion of "*logical consequence*" (also called "*semantic consequence relation*"). This relation is defined as follows:

LOGICAL (SEMANTIC) CONSEQUENCE

Formula ϕ is a *logical consequence* of the set of formulæ Φ if ϕ is true for every model of (all members of the set) Φ. We denote that by:

$$\Phi \models \phi$$

Tautologies, syntactic vs. sematic

Every formula ϕ for which relation

$$\models \phi$$

holds we call a *tautology*. Tautologies are formulaæ that are true for any model (any valuation).

The main objective of propositional calculus is to give syntactic description corresponding to the semantic consequence relation $\models$ by setting up an appropriate formal system (syntactic inference system). This can be done in a number of ways, but we will concentrate on one that is most convenient for later generalisation to modal situations.
We will describe this system, Hilbert style, in next slides.

Lecture plan

Propositional inference system

Every inference (deduction) system can be (due to Hilbert) set up by providing *axioms* and *rules of inference*.

Propositional inference system

- **Axioms.** Our axioms are all tautologies plus all formulæ of the form

$$(k)\quad \phi \to (\theta \to \phi)$$
$$(l)\quad (\theta \to (\psi \to \phi)) \to ((\theta \to \psi) \to (\theta \to \phi))$$

  together with enough axioms to control the other connectives.
- Single **rule of inference** called *modus ponens*:

$$\frac{\theta \quad , \quad \theta \to \phi}{\phi} \quad (Modus\ ponens)$$

The above system (axioms and rule of inference) will be used to generate a proof theoretic syntactic consequence (inference) relation $\vdash$.

Let Φ be a set of formulæ.

(A) A **witnessing deduction (proof) from** Φ is a sequence

$$\phi_0, \phi_1, ..., \phi_n$$

of formulæ such that for each formula ϕ_i of the sequence, at least one of the following holds:

1. $\phi_i \in \Phi$
2. ϕ_i is an axiom.
3. There exist formulæ ϕ_j, ϕ_k occurring earlier in the sequence (i.e., with $j, k < i$) such that $\phi_k = (\phi_j \to \phi_i)$

(B) For each formula ϕ, relation

$$\Phi \vdash \phi$$

holds if and only if there exists a witnessing deduction (a formal proof) from Φ ending with ϕ.

Lecture plan

Deduction property

Relation $\Phi \vdash \phi$ is the syntactic simulation of logical (semantic) consequence.

The introduced system, by definition, has the "*Deduction Property*":

Deduction Property

For each set of formulæ Φ and a pair of formulæ θ, ϕ the implication

$$\Phi, \theta \vdash \phi \quad \Rightarrow \quad \Phi \vdash (\theta \to \phi) \tag{1}$$

holds.

Unfortunately, this important property fails to hold for most of the modal logical systems.

Lecture plan

Lecture plan

Soundness and adequacy

Completeness of the logical system is the property that unifies syntactic and semantic consequences ($\models$ and $\vdash$). Normally, a proof of completeness for a given logical (formal) system consists of two steps:

1. Proof of soundness, i.e., proof that syntax is OK in the sense that it cannot lead to false conclusions given true axioms. Formally we write it as:

   $$\Phi \vdash \phi \Rightarrow \Phi \models \phi$$

   In our (propositional) case the soundness proof is done by a routine induction on the length of the witnessing (formal) deduction that justifies $\Phi \vdash \phi$.
2. Proof of adequacy, i.e., proof that all that is true under a set of assumptions can be formally deduced. This corresponds to showing that:

   $$\Phi \models \phi \Rightarrow \Phi \vdash \phi$$

Proof of adequacy is frequently treated as equivalent to the proof of completeness since soundness is typically quite easy to prove.

Lecture plan

1. Language of propositional logic
2. Binary (two-value) semantics
3. Logical consequence, model
4. Syntax
5. Deduction property
6. Completeness
   - Soundness and adequacy
   - Consistency
   - Completeness proof
7. Compactness

Consistency

Proving completeness means proving that

$$\Phi \models \phi \Leftrightarrow \Phi \vdash \phi$$

holds both ways. As stated before, proof of soundness is not interesting. We concentrate on proving adequacy and hence completeness. This can (again) be done in a number of ways, but we will use the notion of consistency in order to come out with proof that will be later easy to adjust to modal case.

Consistent set of fomulæ

The set of formulæ Φ is *consistent* if the relation

$$\Phi \vdash \bot$$

does not hold.

In plain language it means that one cannot falsify system using only consistent set of formulaæ, axioms and inference rule(s).

Consistent sets of formulæ

Let $\mathbf{CON}$ be the set of all consistent sets Φ. By design of formal system, $\mathbf{CON}$ has the following properties:

1. For each set of formulæ Φ we have $\Phi \in \mathbf{CON}$ iff $\Psi \in \mathbf{CON}$ for each **finite** subset $\Psi \subseteq \Phi$. This property is called **finite character**.
2. For each variable $p \in VAR$ we have $\{p, \neg p\} \notin \mathbf{CON}$ and, of course, $\{\bot\} \notin \mathbf{CON}$. This property is called **basic (or ground) consistency**.
3. For each set of formulæ $\Phi \in \mathbf{CON}$ an for each ϕ, θ we have:

$$\begin{aligned} (\phi \wedge \theta) \in \Phi &\Rightarrow \Phi \cup \{\phi, \theta\} \in \mathbf{CON} \\ \neg(\phi \vee \theta) \in \Phi &\Rightarrow \Phi \cup \{\neg\phi, \neg\theta\} \in \mathbf{CON} \\ \neg(\phi \to \theta) \in \Phi &\Rightarrow \Phi \cup \{\phi, \neg\theta\} \in \mathbf{CON} \end{aligned}$$

   This property is called **conjunctive preservation**.

4. For each set of formulæ $\Phi \in \mathbf{CON}$ an for each ϕ, θ we have:

$$\begin{array}{rcl} (\phi \vee \theta) \in \Phi & \Rightarrow & \Phi \cup \{\phi\} \in \mathbf{CON} \;\; or \;\; \Phi \cup \{\theta\} \in \mathbf{CON} \\ \neg(\phi \wedge \theta) \in \Phi & \Rightarrow & \Phi \cup \{\neg\phi\} \in \mathbf{CON} \;\; or \;\; \Phi \cup \{\neg\theta\} \in \mathbf{CON} \\ (\phi \rightarrow \theta) \in \Phi & \Rightarrow & \Phi \cup \{\neg\phi\} \in \mathbf{CON} \;\; or \;\; \Phi \cup \{\theta\} \in \mathbf{CON} \end{array}$$

   This property is called **disjunctive preservation**.

5. For each set of formulæ $\Phi \in \mathbf{CON}$ an for each ϕ we have:

$$\neg\neg\phi \in \Phi \quad \Rightarrow \quad \Phi \cup \{\phi\} \in \mathbf{CON}$$

   This property is called **negation preserving**.

Maximally consistent sets of formulæ

By $\mathbf{MAXCON}$ we denote the family of maximally consistent sets of formulæ, i.e., a family of those $\Psi \in \mathbf{CON}$, which for all $\Phi \in \mathbf{CON}$ satisfy the rule

$$\text{IF } \Psi \subseteq \Phi \quad \text{THEN} \quad \Psi = \Phi$$

The next lemma, showing the validity of $\mathbf{MAXCON}$ definition, is the cornerstone of the completeness proof.

Lemma - basic existence result

For each $\Phi \in \mathbf{CON}$ there exists $\Sigma \in \mathbf{MAXCON}$ such that $\Phi \subseteq \Sigma$. In other words, the notion of maximally consistent set of formulæ is non-trivial and each consistent set of formulæ can be extended to maximal one.

Proof of basic existence result

Let $\{\psi_i : i < \omega\}$ be the enumeration of all formulæ. We define a sequence of sets of formulæ $\{\Delta_r : r < \omega\}$ as follows:

$$\begin{array}{rcl} \Delta_0 & = & \Phi \\ \Delta_{r+1} & = & \left\{ \begin{array}{ll} \Delta_r \cup \{\psi_r\} & \text{if this set is in } \textsf{\textbf{CON}}, \\ \Delta_r & \text{otherwise.} \end{array} \right. \end{array}$$

Note, that $\Delta \in \mathbf{CON}$ for all $r < \omega$, and hence

$$\Sigma = \bigcup\{\Delta_r : r < \omega\} \in \mathbf{CON}$$

since Σ defined in this way is an upper limit of this family. By construction we also get that $\Sigma \in \mathbf{MAXCON}$ since no element of $\mathbf{CON}$ can contain the upper limit of the family in a non-trivial way.

Lecture plan

Theorem – existence of the model

Each consistent set of formulæ $\Phi \in \mathbf{CON}$ has a model.

Proof:

Let Σ be the maximally consistent set of formulæ such that $\Phi \subseteq \Sigma$. We define valuation $v_\Sigma : VAR \to \{0,1\}$ as follows:

$$v_\Sigma(p) = \begin{cases} 1 & \text{if } p \in \Sigma, \\ 0 & \text{if } p \notin \Sigma. \end{cases}$$

By simple induction we can show that valuation v_Σ is a model for (all formulæ in) Σ, and hence Φ. In the induction step we make use of the fact that implications in all preservation properties

Completeness theorem

For each set of formulæ Φ and each formula ϕ

$$\Phi \vdash \phi \quad \Leftrightarrow \quad \Phi \models \phi$$

Proof: Implication $(\Rightarrow)$ is soundness, so it suffices to prove $(\Leftarrow)$. Suppose $\Phi \models \phi$, then $\Phi \cup \{\neg\phi\}$ has no model. Hence existence theorem gives

$$\Phi \cup \{\neg\phi\} \notin \mathbf{CON}$$

thus

$$\Phi, \{\neg\phi\} \vdash \bot$$

hence from Deduction Property

$$\Phi \vdash (\neg\phi \rightarrow \bot)$$

which (with use of appropriate axiom) gives

$$\Phi \vdash (\neg\bot \rightarrow \phi).$$

But, we know that $\Phi \vdash \neg\bot$ always holds which gives $\Phi \vdash \phi$. (Q.E.D.)

Lecture plan

Let us recall, that a set of formulæ Φ is consistent if the relation $\Phi \vdash \bot$ does not hold. This, according to completeness theorem, means that it has a model, i.e., there exists valuation such that every formula in Φ is true. Due to this assertion we may refer to consistent sets as *satisfiable sets*. Let us consider a decision problem problem known as *satisfiability problem* or SAT.

> We have to decide if a given set of formulæ Φ is satisfiable, i.e., if there exists a satisfying valuation for all members of Φ.

In light of completeness theorem we may attempt to prove/disprove satisfiability in one of two ways:

1. Experimentally: check by trial-and-error if we can find, among all possible valuations, a model (satisfying valuation) for Φ.
2. Theoretically: check if we can produce a witnessing deduction that yields $\bot$ from Φ ($\Phi \vdash \bot$).

Compactness

Here we will showcase yet another, third way for checking/proving the satisfiability of a set of formulæ.
We will show that the logical system that we have crafted for propositional calculus possesses the Compactness Property.

Theorem – compactness of propositional system

In propositional calculus, if every finite subset of a set of formulæ Φ has a model, then Φ also has a model.

Proof:
Assumption: Let Φ be an arbitrary set of formulæ such that it satisfies preposition of the compactness theorem, i.e., each of its finite subsets has a model.
In order to prove theorem it suffices to show that Φ is consistent.
Let us apagogically assume that Φ is inconsistent. If Φ is inconsistent, then there exists a witnessing deduction

$$W = \langle \phi_0, \phi_1, ... \phi_n, \bot \rangle$$

for the relation $\Phi \vdash \bot$. But, in such case there exists a finite set of formulæ $\Psi \subseteq \Phi$ inside which W is a proper witnessing deduction as well. Hence, by definition, $\Psi \vdash \bot$, which means that Ψ has no model. Thus, we have directly contradicted the **Assumption** which proves the compactness.(Q.E.D.)