Limited Set quantifiers over Countable Linear Orderings

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Abstract. In this paper, we study several sublogics of monadic second-order logic over countable linear orderings, such as first-order logic, first-order logic on cuts, weak monadic second-order logic, weak monadic second-ordered logic with cuts, as well as fragments of monadic second-order logic in which sets have to be well ordered or scattered. We give decidable algebraic characterizations of all these logics and compare their respective expressive power.

1 Introduction

Monadic second-order logic (*i.e.*, first-order logic extended with set quantifiers) is a concise and expressive logic that retains good decidability properties (though with a bad complexity). In particular, since the seminal works of Büchi [3], Rabin [11] and Shelah [13], it is known to be decidable over infinite linear orderings with countably many elements, such as $(\mathbb{Q}, <)$ [5,7]. A breakthrough result of Shelah (also in [13]) states that over general linear orderings (*i.e.*, not necessarily countable), or simply over $(\mathbb{R}, <)$, this logic is not decidable anymore. There is also a long line of research focusing on the analysis of the expressive power and decidability status of temporal logics, which, for most of them are equivalent in expressiveness to first-order logic (but much more tractable), and can be decided on some non-countable linear orderings.

Such studies are interesting for themselves, *i.e.*, for the techniques involved in their resolution and the understanding of the logics it requires for doing so. Such studies are also interesting since infinite linear orderings offer a natural model of continuous linear time.

Recently, another step in our understanding of monadic second-order logic over countable linear orderings has been made. An algebraic model, $\circ$-monoids, was proposed [4], yielding among other results the first known quantifier collapse of monadic second-order logic (to the one alternation fragment over set quantifiers), the resolution of a conjecture of Gurevich and Rabinovich [8] concerning the use of cuts "in the background" [6]. Algebraic recognizers give us a much deeper understanding of the expressive power of monadic second-order logic.

The next natural step is to follow the footprints of Schützenberger, who characterized algebraically first-order logic over finite words as languages that are recognized by aperiodic monoids [12] (in fact, the first-order logic terminology is in combination with McNaughton and Papert [10]) as these languages that are recognized by aperiodic monoids. Now that a suitable algebraic model is known for understanding monadic second-order logic, a similar study can be performed in this more general context. There exist already results of this kind, but these are so far restricted to the case of scattered linear orderings (*i.e.*, without any dense sub-ordering). In this context, first-order logic and first-order logic on cuts have been algebraically characterized [1], as well as weak monadic second-order logic [2]. Simple decision procedures are derived in all these situations.

In this paper, we perform a systematic analysis of sublogics of monadic second-order logic on countable linear orderings depending on the kind of sets over which set quantifiers range. If such sets are just singletons, we have exactly first-order logic (FO). If such sets are Dedekind cuts, we obtain first-order logic on cuts (FO[cut]). If finite sets only are allowed, this is weak monadic second-order logic (WMSO). If it is possible to quantify over both finite sets and cuts, we obtain weak monadic second-order on cuts (MSO[finite,cut]). We consider also MSO[ordinal] in which quantified sets need to be well-ordered. Finally MSO[scattered] corresponds to the case where quantified sets are required to be scattered. Our contribution is to compare the expressive power of all these logics (all are distinct but for MSO[finite,cut] which coincide with MSO[ordinal]), and characterize each of them by decidable algebraic means.

Structure of the paper In Section 2, we introduce linear orderings, words, and the logics we are interested in. In Section 3 we provide sufficient material concerning the algebraic framework of $\circ$-monoids, state the main characterization theorem, Theorem 2, and show the separation result, Theorem 3. Section 4 is devoted to the description of some ideas concerning one direction of the proof of Theorem 2. Section 5 concludes the paper.

2 Preliminaries

In this preliminary section, we introduce the notion of linear orderings (Section 2.1), (countable) words (Section 2.2) and the studied logics (Section 2.3).

2.1 Linear orderings

A *linear ordering* $\alpha = (X, <)$ is a non-empty set X equipped with a total order $<$. A linear ordering α is *dense* if it contains at least two elements and for all $x < y \in \alpha$, there exists a z such that $x < z < y$. It is *scattered* if no subset of X induces a dense ordering. A *well ordering* is a linear ordering such that every non-empty subset has a minimal element. A subset of a linear ordering is well ordered (*resp.* scattered) if the linear ordering restricted to it is a well ordering (*resp.* scattered).

Given an element x, its *successor* (resp. *predecessor*) (if it exists) is the only $y > x$ (*resp.* $y < x$) such that there is no z such that $x < z < y$ (*resp.* $y < z < x$). A subset $I \subseteq \alpha$ of a linear ordering is *convex* if whenever $x, y \in I$ and $x < z < y$, $z \in I$. A *condensation* of a linear ordering is an equivalence relation $\sim$ such that all equivalence classes are convex. For a linear ordering α and a condensation $\sim$, we denote by $\alpha/\sim$, the *condensed linear ordering*: its elements are the equivalence classes for $\sim$, and the ordering is obtained by projection of the original ordering. Two convex subsets I, J of a linear ordering are *consecutive* if I and J are disjoint and their union is convex. Using the notations for elements, if $I < J$, then I is the predecessor of J, while J is the successor of I.

Given linear orderings $(\beta_i)_{i\in\alpha}$ (assumed disjoint up to isomorphism) indexed with a linear ordering α, their generalized sum $\sum_{i\in\alpha}\beta_i$ is the linear ordering over the (disjoint) union of the sets of the β_i's, with the order defined by $x < y$ if either $x \in \beta_i$ and $y \in \beta_j$ with $i < j$, or $x, y \in \beta_i$ for some i, and $x < y$ in β_i.

Given elements x, y, we denote by $[x, y)$ the set $\{z \mid x \leq z < y\}$, and similarly $[x, y]$, $(x, y]$ and (x, y). We also denote as $(-\infty, x)$, $(-\infty, x]$, $(x, +\infty)$ and $[x, +\infty)$ the intervals that are unlimited to the left or to the right. Usually Dedekind cuts are defined as ordered pairs of sets (L, R) such that $L < R$. Here, we define a *Dedekind cut* (or simply a cut) as a left-closed subset X of a linear ordering, *i.e.*, for all $x < y$ with $y \in X$, then $x \in X$.

2.2 Infinite words

Given a linear ordering α and a finite *alphabet* A, a *word* over A of *domain* α is a mapping $w : \alpha \to A$. The domain of a word is denoted $dom(w)$. In this work, all words are assumed of countable domain. The set of all *words of countable domain* is denoted by A°. A *language* is a subset of A°.

Given a convex set $X \subseteq dom(w)$ of word w, w_X denotes the word w *restricted* to X, *i.e.*, the word of domain X that coincides with w over X. A *factor* of a word w is any restriction of w to one of the convex subsets of its domain.

Given two words $u : \alpha \to A$ and $v : \beta \to A$ (where α and β are disjoint), we denote by uv the word over domain $\alpha + \beta$ such that each position $x \in \alpha$ (similarly $x \in \beta$) is labelled by $u(x)$ (by $v(x)$). The *generalized concatenation* of the words w_i (supposed of disjoint domain) indexed by a linear ordering α is

$$\prod_{i\in\alpha} w_i \ ,$$

and denotes the word of domain $\sum_i dom(w_i)$ which coincides with each w_i over $dom(w_i)$ for all $i \in \alpha$.

Some words will play an important role in the paper. The *empty word* ε, which is the only word of empty domain. The words denoted "$aaa\ldots$" and "$\ldots aaa$" are the words over the single letter a, and of respective domain $\omega = (\mathbb{N}, <)$ and $\omega^* = (\mathbb{N}, >)$. Finally, *perfectshuffle*(A) for A, a non-empty finite set of letters, is a word of domain $(\mathbb{Q}, <)$ in which all non-empty intervals (x, y) contain at least once each letter of A. This word is unique up to isomorphism.

2.3 First-order logic, monadic second-order logic, and between

We use logics for expressing properties of linear orderings or words. All of the several logics we study are all restrictions of monadic second-order logic (MSO).

We very succinctly recall the basics of this logic here. The reader can refer to many other works on the subject, *e.g.*, [14]. We only consider word models.

Monadic second-order logic (MSO for short) is a logic with the following characteristics. It is possible to use *first-order variables* $x, y, z, \ldots$, ranging over positions of the word, and quantify over them thanks to $\exists x$ or $\forall y$. It is possible to use *monadic variables* $X, Y, \ldots$ (traditionally typeset in capital letters), that range over sets of positions of the word, and quantify over them using $\exists X$, $\forall Y$. Three atomic predicates can be used. The predicate $a(x)$, for a a letter, and x a position, holds if the letter carried at position x in the word is an a. The predicate $x < y$ for x, y, first-order variables denotes the order of the domain of the word. The membership predicate $x \in Y$ tests the membership of (the valuation of) a first-order variable x in (the valuation of) a monadic variable Y. All the Boolean connectives are also allowed. *First-order logic* (FO of short) is the fragment of this logic in which monadic variables, as well as quantifiers over them, are not allowed.

In this study, we are interested in the expressive power of logics weaker than MSO. There is a long tradition of such researches, initiated by the seminal work of Schützenberger. For instance, it is classical to study first-order logic and its fragments when the quantifier alternation or the number of variables are restricted. In our case, our goal is to investigate the intricate relationship between the expressive power of the logic, and the infinite/dense nature of the linear orderings/words under study. The only parameter that we use for modifying the power of the logic is to change the range of monadic variables. By default, such variables range over any set of positions. We introduce now several *restricted set quantifiers* and the corresponding logics. Our simplest logic is first-order logic. The logic obtained by allowing monadic quantifiers restricted to Dedekind cuts is denoted *FO[cut]*. Another situation is when monadic second-order variables range over finite set, yielding *weak monadic second-order logic* (WMSO for short). We are also interested in the fragment in which it is possible to quantify both over finite sets and Dedekind cuts. We denote this logic *MSO[finite,cut]*. Then come logics in which monadic variables range over "infinite but small", sets of positions. We consider the case in which it is possible to quantify over well ordered sets, or scattered sets. We denote these logics *MSO[ordinal]* and *MSO[scattered]*.

We formally denote these restricted quantifiers as $\exists^V$ and $\forall^V$, where $V \subseteq \{\in, \notin\}^\circ$. A set belongs to the range of the quantifier $\exists^V$ or $\forall^V$ if its characteristic map (as a labelling of the domain by $\in, \notin$) is in V.

Given one of the above logics $\mathcal{L}$, a formula $\varphi \in \mathcal{L}$ and a countable word w we denote by $w \models \varphi$, the fact that the formula is true over w. We say that w is a *model* of φ. A language $L \subseteq A^\circ$ is *definable* in $\mathcal{L}$ if there exists a formula φ in $\mathcal{L}$ such that for all words $w \in A^\circ$, $w \in L$ if and only if $w \models \varphi$.

Remark 1. Some dependencies between these logics are simple to establish:

$$\text{FO} \begin{array}{c} \nearrow \text{FO[cut]} \searrow \\ \searrow \text{WMSO} \nearrow \end{array} \text{MSO[finite,cut]} \text{ — } \text{MSO[ordinal]} - \text{MSO[scattered]} - \text{MSO}$$

Indeed, FO[cut] is an extension of FO. Also WMSO extend FO since "being a singleton" is definable in WMSO. Similarly, MSO[finite,cut] is clearly an extension of both WMSO and FO[cut]. MSO[ordinal] can express finiteness, and represent cuts (as the left closure of a well ordered subset), and hence contains MSO[finite,cut]. In the same way, since being well ordered is expressible in MSO[scattered], MSO[scattered] contains MSO[ordinal]. Similarly, scatteredness being expressible in MSO, MSO[scattered] is a sublogic of MSO. In fact, all these logics are separated (Theorem 3), but for MSO[finite,cut] and MSO[ordinal] which happen to coincide (see Theorem 2).

The goal of this paper is to compare the expressive power of all these logics and be able to characterize them effectively.

3 The algebraic presentation: $\circ$-monoids

We now introduce the equivalent algebraic presentation of definable languages. We first describe the $\circ$-monoids in Section 3.1, and then the derived operations in Section 3.2, before presenting the theorems of characterization and separation in Section 3.3.

3.1 $\circ$-monoids, syntactic $\circ$-monoids and recognizability

As in the seminal work of Schützenberger, we use algebraic acceptors for describing regular languages of countable words: $\circ$-monoids. A *$\circ$-monoid* is a set M equipped with an operation π, called the *product*, from M° to M, that satisfies $\pi(a) = a$ for all $a \in M$, and the *generalized associativity* property: for every words u_i over M° with i ranging over a countable linear ordering α,

$$\pi\left(\prod_{i\in\alpha} u_i\right) = \pi\left(\prod_{i\in\alpha} \pi(u_i)\right) \ .$$

Of course, an instance of $\circ$-monoids is the set of words over some alphabet A equipped with the generalized concatenation $\prod$, *i.e.*, $(A^\circ, \prod)$. It is even the *free $\circ$-monoid* generated by A. A *$\circ$-monoid morphism* from $\mathbf{M}$ to $\mathbf{N}$ ($\circ$-monoids) is a map γ from M to N such that $\gamma(\prod_{i\in\alpha} a_i) = \pi(\prod_{i\in\alpha} \gamma(a_i))$.

Example 1. **Sing**$= (\{1, s, 0\}, \pi)$ where π is defined for all $u \in \{1, s, 0\}^\circ$ as:

$$\pi(u) = \begin{cases} 1 & \text{if } u \in \{1\}^\circ, \\ s & \text{otherwise if } u \text{ contains no } 0\text{, and exactly one } s, \\ 0 & \text{otherwise,} \end{cases}$$

is a ∘-monoid (checking generalized associativity requires a case by case study).

By slightly modifying the example, we obtain the ∘-monoid **Fin** in which the second line in the definition of π is changed into "s if u contains no 0, and finitely many s's". The ∘-monoid **Ord** is when $\pi(u)$ evaluates to "s if u contains no 0, and a well ordered set of s's". Finally, the ∘-monoid **Scat** is when $\pi(u)$ evaluates to "s if u contains no 0, and a scattered set of s's". Once more, checking generalized associativity is by case analysis.

The element $\pi(\varepsilon)$ is called the *unit*, and it is customary to denote it 1 as done above. A *zero* (that does not necessarily exist) is an absorbing element, *i.e.*, an element such that $\pi(u0v) = 0$ whatever are u and v. It is denoted by convention 0 as in the above examples. An *idempotent* is an element e such that $\pi(ee) = e$.

A ∘-monoid can be used to recognize languages as follows. Consider a ∘-monoid $\mathbf{M} = (M, \pi)$, a map h from an alphabet A to M and a set $F \subseteq M$, then $(\mathbf{M}, h, F)$ *recognizes* the language $L = \{u \in A^\circ \mid \pi(h(u)) \in F\}$ (where h has been extended implicitly into a map from A° to M°). Said differently, L is the inverse image of F under the ∘-monoid morphism $\pi \circ h$. From [4], being recognizable by a ∘-monoid is equivalent to be definable in MSO.

Furthermore, when a language is recognizable by a finite ∘-monoid, then there is a minimal one called the *syntactic ∘-monoid*. It is minimal in the algebraic sense: all ∘-monoids that would recognize this language can be trimmed and quotiented yielding the syntactic one. We do not develop this aspect more in this short abstract.

Example 2. Coming back to the above examples, with $h(\in) = s$ and $h(\notin) = 1$, then $(\mathbf{Sing}, h, \{s\})$ recognizes the language L_{Sing} over the alphabet $\{\in, \notin\}$ of words that contain exactly one occurrence of $\in$. Similarly, $(\mathbf{Fin}, h, \{1, s\})$, $(\mathbf{Ord}, h, \{1, s\})$, and $(\mathbf{Scat}, h, \{1, s\})$ recognize the languages L_{Finite}, L_{Ord} and L_{Scat} respectively, of words that contain "finitely many $\in$'s", "a well ordered set of $\in$'s", and "a scattered set of $\in$'s" respectively.

Let us note that these languages are the one used in the restricted quantifiers $\exists^V$ and $\forall^V$ for defining the logics (cuts are omitted for space considerations).

3.2 The derived operations

The product operation π is infinite, even in a finite ∘-monoid $\mathbf{M} = (M, \pi)$. Hence, π is *a priori* not representable in finite space (it has uncountably many possible inputs). This problems is resolved using derived operations.

The *operations derived* from π are the following:

- 1 is the unit constant $\pi(\varepsilon)$,
- $\cdot$: $M \times M \to M$ is defined for $a, b \in M$ as $a \cdot b = \pi(ab)$,
- ω: $M \to M$ is defined for all $a \in M$ as $a^\omega = \pi(aaa\dots)$,
- $\omega*$: $M \to M$ is defined for all $a \in M$ as $a^{\omega*} = \pi(\dots aaa)$,
- η: $\mathcal{P}(M) \setminus \{\emptyset\} \to M$ is defined as $E^\eta = \pi(\mathit{perfectshuffle}(E))$ for $E \subseteq M$ non-empty.

Note that from the definitions, using generalized associativity, the unit element satisfies $1 \cdot 1 = 1^\omega = 1^{\omega *} = \{1\}^\eta = 1$, $a \cdot 1 = 1 \cdot a = a$, and $(E \cup \{1\})^\eta = E^\eta$ for all $a \in M$ and all non-empty $E \subseteq M$. Similarly, if there is a zero 0 then it satisfies $0 \cdot a = a \cdot 0 = 0^\omega = 0^{\omega *} = (E \cup \{0\})^\eta = 0$ for all $a \in M$ and $E \subseteq M$. This is why we usually do not mention these elements when describing derived operations.

Example 3. The derived operation of the above examples are entirely determined by the following table:

	$s \cdot s$	s^ω	$s^{\omega *}$	$\{s\}^\eta$
Sing	0	0	0	0
Fin	s	0	0	0
Ord	s	s	0	0
Scat	s	s	s	0

Though not essential in this short abstract, let us emphasize that the derived operations determine entirely the product π, as shown now.

Theorem 1. *There exists a set of equalities (A) involving the derived operations*[1]*, such that:*

- *The operations derived from a* $\circ$*-monoid satisfy all the equations from (A).*
- *If* $1, \cdot, \omega, \omega^*, \eta$ *are maps of correct type over a finite set* M *that satisfy the equalities of (A), then there exists one and only one product over* M *from which* $1, \cdot, \omega, \omega^*, \eta$ *are derived.*

3.3 The core theorem

We state in this section our main results, Theorem 2 and 3. All $\circ$-monoids are assumed finite from now. We first refine our understanding of idempotents:

- A *gap insensitive* idempotent e is an idempotent such that $e^\omega \cdot e^{\omega *} = e$.
- An *ordinal idempotent* e is an idempotent such that $e^\omega = e$. The name comes from the fact that in such a case, all words $u \in \{e\}^\circ$ that have a well ordered (*i.e.*, isomorphic to an ordinal) non-empty domain satisfy $\pi(u) = e$.
- Symmetrically, an *ordinal* idempotent* e is an idempotent such that $e^{\omega *} = e$.
- A *scattered idempotent* e is an idempotent which is at the same time an ordinal and an ordinal* idempotent. For such idempotents, all words $u \in \{e\}^\circ$ that have a scattered non-empty domain satisfy $\pi(u) = e$.
- A *shuffle idempotent* e is an idempotent such that $\{e\}^\eta = e$.
- A shuffle idempotent e is *shuffle simple* if for all $K \subseteq M$ such that $e \cdot a \cdot e = e$ for all $a \in K$, $(\{e\} \cup K)^\eta = e$.

The following Table gives these definitions.

[1] These are variants of associativity, such as $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, $1 \cdot x = x \cdot 1 = x$, $(a^n)^\omega = a^\omega$, and so on. A complete list is known [4], but of no use here.

idempotent	$e^2 = e$
gap insensitive idempotent	$e^\omega e^{\omega^*} = e$
ordinal idempotent	$e^\omega = e$
ordinal* idempotent	$e^{\omega *} = e$
scattered idempotent	$e^{\omega^*} = e = e^{\omega^*}$
shuffle idempotent	$\{e\}^\eta = e$
shuffle simple	$\forall\ K \subseteq M\ ((\forall a \in K, eae = e) \implies (\{e\} \cup K)^\eta = e)$

Note that since in every $\circ$-monoid $(\{e\}^\eta)^\omega = (\{e\}^\eta)^{\omega *} = \{e\}^\eta$, every shuffle idempotent is a scattered idempotent. Note also that every scattered idempotent is by definition an ordinal idempotent and an ordinal* idempotent. Also, every scattered idempotent is obviously gap insensitive.

We define now the following properties of a $\circ$-monoid $\mathbf{M} = (M, \pi)$:

- *aperiodic* if for all $a \in \mathbf{M}$ there exists n such that $a^n = a^{n+1}$,
- `i→gi` if all idempotents are gap insensitive,
- `oi→gi` if all ordinal idempotents are gap insensitive,
- `o*i→gi` if all ordinal* idempotents are gap insensitive,
- `sc→sh` if all scattered idempotents are shuffle idempotent,
- `sh→ss` if all shuffle idempotents are shuffle simple.

The following Table gives these definitions.

	For all e
aperiodic	$\exists n \in \mathbb{N}, e^n = e$
`i→gi`	$e^2 = e \implies e^\omega e^{\omega *} = e$
`oi→gi`	$e^\omega = e \implies e^\omega e^{\omega *} = e$
`o*i→gi`	$e^{\omega *} = e \implies e^\omega e^{\omega *} = e$
`sc→sh`	$e^\omega = e = e^{\omega *} \implies e^\eta = e$
`sh→ss`	$e^\eta = e \implies$ shuffle simple

It is clear by definition that `oi→gi` (as well as `o*i→gi`) imply `i→gi`. There is in fact another, slightly less direct, implication to mention:

Lemma 1. `i→gi` *implies aperiodic.*

Proof. Let a be an element of a finite $\circ$-monoid M. There exists n such that a^n is idempotent. We compute $a^n = (a^n)^\omega \cdot (a^n)^{\omega *} = a \cdot (a^n)^\omega \cdot (a^n)^{\omega *} = a^{n+1}$. □

We are now ready to state our core theorem.

Theorem 2. *Let* $\mathbf{M}$ *be the syntactic* $\circ$*-monoid of a language* $L \subseteq A^\circ$*, then:*

- *L is definable in FO iff* $\mathbf{M}$ *satisfies* `i→gi`, `sc→sh` *and* `sh→ss`.
- *L is definable in FO[cut] iff* $\mathbf{M}$ *satisfies aperiodic,* `sc→sh` *and* `sh→ss`.
- *L is definable in WMSO iff* $\mathbf{M}$ *satisfies* `oi→gi`, `o*i→gi`, `sc→sh` *and* `sh→ss`.
- *L is definable in MSO[finite,cut] iff it is definable in MSO[ordinal] iff* $\mathbf{M}$ *satisfies* `sc→sh` *and* `sh→ss`.

– *L is definable in MSO[scattered] iff* $\mathbf{M}$ *satisfies* `sh`$\to$`ss`*.*

And as a consequence, these classes are decidable.

Example 4. Let us apply these characterizations to the ∘-monoids of Example 3:

	aper.	`i`$\to$`gi`	`oi`$\to$`gi`	`o`*`i`$\to$`gi`	`sc`$\to$`sh`	`sh`$\to$`ss`	definable in
Sing	yes	yes	yes	yes	yes	yes	FO
Fin	yes	no	yes	yes	yes	yes	WMSO, FO[cut], not FO
Ord	yes	no	no	yes	yes	yes	FO[cut], not WMSO
Scat	yes	yes	yes	yes	no	yes	MSO[scattered], not MSO[ordinal]

Remark 2. One aspect of Theorem 2 is that MSO[finite,cut] and MSO[ordinal] are equivalent. If we apply this fact to the domain ω, then cuts can be eliminated easily, and MSO[finite,cut] coincide with WMSO. Still over ω, MSO[ordinal] obviously coincide with MSO. Hence Theorem 2 implies that WMSO and MSO coincide over ω (in fact, the same argument is valid over any well ordered countable word). This non-trivial fact is usually established using the deep result of determinization of McNaughton [9] (other proofs involve weak alternating automata or algebra).

Theorem 3. *There are languages separating all situations not covered by Theorem 2.*

Proof (sketch). In fact, two among the five separating languages were given in Example 4: L_{Ord} ∈FO[cut]\WMSO and L_{Scat} ∈MSO[scattered]\MSO[ordinal].

WMSO\FO[cut]$\neq\emptyset$: The witnessing language is "the domain is of even finite length". It is the classical example of non-aperiodicity over finite words, and it works as well in this case.

MSO[ordinal]\(FO[cut]∪WMSO) $\neq\emptyset$: For this, it is sufficient to take the disjoint union (for instance using disjoint alphabets) of a language in WMSO\FO[cut] and a language in FO[cut]\WMSO.

MSO\MSO[scattered]$\neq\emptyset$: Call a set X *perfectly dense* if all elements $x<y<z$ with $y\in X$ are such that (x,y) and (y,z) both intersect X. Said differently, all elements in X are limits from the left of elements from X, and symmetrically from the right. The language "there exists a set X of a-labelled positions which is perfectly dense" is obviously definable in MSO. Computing its syntactic ∘-monoid would yield four elements $1,a,b,0$ with derived operations defined by $a\cdot a=a^\omega=a^{\omega*}=b\cdot b=b\cdot a=a\cdot b=b^\omega=b^{\omega*}=\{b\}^\eta=b$ and $\{a\}^\eta=\{a,b\}^\eta=0$. The morphism sends a to a and b to b, and the accepting set is $\{0\}$. However, this language is not definable in MSO[scattered]: b is a shuffle idempotent which is not shuffle simple since $\{b\}^\eta=b=b\cdot a\cdot b$ and $\{a,b\}^\eta\neq b$. □

4 From logics to ∘-monoids

In this section, we show some of the results of the form "if a language $L\subseteq A^\circ$ is definable in logic $\mathcal{L}$, then its syntactic ∘-monoid satisfies property P" for suitable choices of $\mathcal{L}$ and P. The standard approach for such results is to use the technique of Ehrenfeucht-Fraïssé games. We adopt a different presentation here, making use of our fine understanding of $\circ$-monoids.

Let us first recall that all the logics we work with differ by their use of restricted set quantifiers. These restricted quantifiers are parameterized by a language $V \subseteq \{\in, \notin\}^\circ$. The quantifier $\exists^V X$ signifies "there exists a set of positions X which, when written as a labelling of the linear ordering yields a word in V". We have seen the language L_{Sing}, L_{Finite}, L_{Ord}, L_{Scat} that correspond to the quantifiers over singletons, finite sets, well ordered sets, and scattered sets.

Thus, the core step in each of these proofs consists in showing that the operation of restricted set quantifier preserves the property we are interested in when done at the level of $\circ$-monoids. Essentially, this looks as follows: "assume that L_ϕ is recognized by a $\circ$-monoid that has property P' then $L_{\exists^V X\phi}$ also has property P". Thus, we start by describing how $\exists^V$ behaves.

Let us just mention here that the existential quantifier is the crux of the problem, and that the other constructions involved (atomic predicates and boolean connectives) have also to be treated, but do not involve interesting arguments. We also have to verify the closure of the properties we are interested in under quotient of $\circ$-monoids. This last step is usually not necessary, but, since we chose not to present the properties as identities, it has to be done explicitly.

4.1 Restricted quantifiers over $\circ$-monoids

Let us first recall how the existential set quantifier is implemented, from a language and algebraic theoretic point of view, and then refine this for restricted set quantifier.

Consider language $L \in (A \times \{\in, \notin\})^\circ$. A word over this alphabet can be seen as a usual word over the alphabet A, enriched with the characteristic map of some set X: if a position belongs to X, then the second component is $\in$, otherwise it is $\notin$. The operation equivalent to existential set quantifier over such languages is $\mathit{Proj}(L)$ defined as:

$$\mathit{Proj}(L) = \left\{u_{|1} \in A^\circ \mid \text{for some } u \in L\right\} ,$$

where $u_{|1}$ denotes the word obtained by projecting each letter of u to its first component (similarly for $u_{|2}$). If furthermore L is recognized by some $\mathbf{M} = (M, \pi), h, F$, we define the new $\circ$-monoid $\mathcal{P}(\mathbf{M})$ to be $(\mathcal{P}(M), \pi)$, where

$$\text{for all } U \in (\mathcal{P}(M))^\circ, \quad \pi(U) = \{\pi(u) \mid u \in U\} ,$$

in which $u \in U$ holds if $\mathit{dom}(u) = \mathit{dom}(U)$ and for all $i \in \mathit{dom}(u)$, $u(i) \in U(i)$.

This construction is known to (1) produce a valid $\circ$-monoid, and (2) be such that $\mathcal{P}(\mathbf{M}), h', F'$ recognizes $\mathit{Proj}(L)$ for $h'(a) = \{h(a, \in), h(a, \notin)\}$ and $F' = \{X \subseteq M \mid X \cap F \neq \emptyset\}$.

We present now a refinement of this construction, which furthermore restricts the range of the projection. Given a language $V \subseteq \{\in, \notin\}^\circ$ that represents the range of a restricted set quantifier, we define the *restricted projection* of L as:

$$Proj^V(L) = \{u_{|1} \in A^\circ \mid \text{for some } u \in L \text{ such that } u_{|2} \in V\} \ .$$

This operation is the language theoretic counterpart to the logical restricted quantifier $\exists^V$. Let us assume furthermore that V is recognized by some $\mathbf{V}, g, E$. We assume (and this will always be the case) that $\mathbf{V}$ has a zero 0, and that $0 \notin E$. We define the new $\circ$-monoid $\mathcal{P}_{\mathbf{V}}(\mathbf{M})$ to be (N, π), where

$$\text{for all } U \in (\mathcal{P}(M \times V))^\circ, \quad \pi(U) = \{(\pi(u_{|1}), \pi(u_{|2})) \mid u \in U\} \setminus (M \times \{0\}) \ ,$$
$$\text{and} \quad N = \{\pi(U) \mid U \in \{\{(h(a, \in), g(\in)), (h(a, \notin), g(\notin))\} \mid a \in A\}^\circ\} \ .$$

We can recognize in this construction the above powerset construction, applied to the $\circ$-monoid $\mathbf{M} \times \mathbf{V}$, from which all occurrences of the zero of $\mathbf{V}$ are removed as well all all non-reachable elements.

Lemma 2. *$\mathcal{P}_{\mathbf{V}}(\mathbf{M})$ is a $\circ$-monoid.*
If L is recognized by $\mathbf{M}, h, F$, then $Proj^V(L)$ is recognized by $\mathcal{P}_{\mathbf{V}}(\mathbf{M}), h', F'$ where $h'(a) = \{(h(a, \in), g(\in)), (h(a, \notin), g(\notin))\}$ and $F' = \{A \mid A \cap (F \times E) \neq \emptyset\}$.

4.2 Establishing invariants

The core result in the translation from logics to $\circ$-monoids is the following.

Lemma 3. *Let $\mathbf{M}$ be a $\circ$-monoid.*

1. *If $\mathbf{M}$ satisfies* `i→gi` *then $\mathcal{P}_{\mathbf{Sing}}(\mathbf{M})$ satisfies* `i→gi`.
2. *If $\mathbf{M}$ satisfies aperiodic then $\mathcal{P}_{\mathbf{Cut}}(\mathbf{M})$ satisfies aperiodic*[2].
3. *If $\mathbf{M}$ satisfies* `oi→gi` *then $\mathcal{P}_{\mathbf{Fin}}(\mathbf{M})$ satisfies* `oi→gi` *(resp.* `o*i→gi`*).*
4. *If $\mathbf{M}$ satisfies* `sc→sh` *then $\mathcal{P}_{\mathbf{Ord}}(\mathbf{M})$ satisfies* `sc→sh`.
5. *If $\mathbf{M}$ satisfies* `sh→ss` *then $\mathcal{P}_{\mathbf{Scat}}(\mathbf{M})$ satisfies* `sh→ss`.

Let us give some ideas about its proof. Let $\mathbf{N}$ be $\mathcal{P}_{\mathbf{V}}(\mathbf{M})$ where $\mathbf{V}$ is one of **Sing**, **Fin**, **Ord** or **Scat** (unfortunately, **Cut** having a different structure, it has to be treated separately).

Lemma 4. *There exists a $\circ$-monoid morphism ρ from $\mathbf{N}$ to $\mathbf{M}$ such that for all $A \in N$, $(x, 1) \in A$ if and only if $x = \rho(A)$.*

Proof. Essentially, the point is to prove that for all $A \in N$, there is one and only one $\rho(A)$ such that $(\rho(A), 1) \in A$. The fact that this ρ is a $\circ$-monoid morphism is then straightforward. For proving it, it is sufficient to do it for the neutral element $\{(1, 1)\}$, the image of each letter 'a' which happens to be $\{(h(a), 1), (h(a), s)\}$, and then show the preservation of the property under $\cdot, \omega, \omega^*$ and η. □

Let us show the simplest case of Lemma 3, the one for $\mathcal{P}_{\mathbf{Sing}}(\mathbf{M})$:

Lemma 5. *If a $\circ$-monoid $\mathbf{M}$ satisfies* `i→gi` *then $\mathcal{P}_{\mathbf{Sing}}(\mathbf{M})$ also does.*

[2] **Cut** is a $\circ$-monoid recognizing "cuts" that we omitted here for space reasons.

Proof. Let E be an idempotent in $\mathbf{N} = \mathcal{P}_{\mathbf{Sing}}(\mathbf{M})$. Our goal is to show that it is gap insensitive.

Let $(x, y) \in E$. Since $E = E \cdot E$, there exists $(x_1, y_1), (x_2, y_2) \in E$ such that $x_1 \cdot x_2 = x$ and $y_1 \cdot y_2 = y$. Since $y \neq 0$, at least one among y_1, y_2 is equal to 1. Without loss of generality, let us assume it is y_1. In this case, according to Lemma 4, $x_1 = \rho(E)$. In particular, since ρ is a morphism, this means that x_1 is an idempotent. Thus we can use the assumption that $\mathbf{M}$ satisfies $\mathtt{i}{\to}\mathtt{gi}$ on it, and get that $x_1^\omega \cdot x_1^{\omega*} = x_1$. It follows that the word

$$\overbrace{(x_1, 1)(x_1, 1)\dots}^{\text{of domain } \omega}\ \overbrace{\dots(x_1, 1)(x_1, 1)}^{\text{of domain } \omega*}(x_2, y_2)$$

has also value (x, y) under π (componentwise), and as a consequence $(x, y) \in E^\omega \cdot E^{\omega*}$. We have proved $E \subseteq E^\omega \cdot E^{\omega*}$.

Conversely, consider some $(x, y) \in E^\omega \cdot E^{\omega*}$. This means that there exists a word u of the form

$$\overbrace{(x_1, y_1)(x_2, y_2)\dots}^{\text{of domain } \omega}\ \overbrace{\dots(x'_2, y'_2)(x'_1, y'_1)}^{\text{of domain } \omega*}$$

which evaluates (componentwise) to (x, y), with (x_i, y_i) and $(x'_i, y'_i) \in E$ for all $i \in \mathbb{N}$. If all $y = 1$, then its clear. Otherwise, there is at most one among the y_i's and the y'_i's which is not equal to 1. Without loss of generality (by symmetry), we can assume that it is y_j. According to Lemma 4, $x_i = \rho(E)$ for all $i \neq j$ and $x'_i = \rho(E)$ for all i. Since ρ is a morphism, $\rho(E)$ is also an idempotent. Thus we can use the assumption that $\mathbf{M}$ satisfies $\mathtt{i}{\to}\mathtt{gi}$. We obtain that $\rho(E)^\omega \cdot \rho(E)^{\omega*} = \rho(E)$. Thus, u evaluates to $(\rho(E), 1) \cdot (x_j, y_j) \cdot (\rho(E), 1) \in E^3 = E$. Hence $E^\omega \cdot E^{\omega*} \subseteq E$.

This terminates the proof that $\mathbf{N}$ satisfies $\mathtt{i}{\to}\mathtt{gi}$. □

5 Conclusion

In this paper we have characterized algebraically and effectively several natural sublogics of MSO. Unfortunately the most involved arguments, namely the translation from algebra to logic, were not addressed in this short abstract. These can be found in the appendix.

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A General properties of $\circ$-monoids

Though the goal of this paper is to introduce suitable equations restricting the expressive power of $\circ$-monoid, we start our investigation by disclosing some facts about $\circ$-monoid in their general form. Let us fix from now a finite $\circ$-monoid $\mathbf{M}$ and a morphism $\gamma : \mathbf{M}^\circ \to \mathbf{M}$.

A.1 Some definitions

For an element $a \in \mathbf{M}$, we denote by $\overrightarrow{a}$ the language:

$$\overrightarrow{a} = \{w = \prod_{i \in \mathbb{N}} w_i \mid \text{ for all } i \in \mathbb{N},\ \gamma(w_i) = a\}$$

We can give a symmetric definition for $\overleftarrow{a}$, which consists of all left convergent condensations. Note that for any word $w \in \overrightarrow{a}$, we have $\gamma(w) = a^\omega$.

We denote by $\mathtt{E}(\mathbf{M})$ the set of all idempotents of $\mathbf{M}$.

Given sets $A, B \subseteq \mathbf{M}^\circ$, we denote by $AB = \{u \cdot v \mid u \in A, v \in B\}$, the set of all concatenations of words from each set.

We denote $w[x, y]$, $w(x, y]$, $w[x, y)$ and $w(x, y)$ respectively the word restricted to the domain $[x, y]$, $(x, y]$, $[x, y)$ and (x, y) respectively. We also denote by $w[p]$ the singleton element $w[p, p]$. We denote by [] the set of non-empty words that have a minimal and a maximal point, by [) the set of non-empty words that have a minimal but no maximal point,by (] the set of non-empty words that have a maximal but no minimal point, and by () the set of non-empty words that have neither a maximal nor a minimal point.

A.2 Green's relations

Green's relations are related to the theory of ideals of monoids. These play a crucial role in the construction of this paper. It is a very rich topic that we do not intend to cover here. We just give a quick recap of the notations we use.

Given two element a, b in a monoid, we define the following relation between them (*Green's relations*):

$$\begin{aligned}
a \leq_{\mathcal{R}} b \quad & \text{if } a = b \cdot x \text{ for some } x \in \mathbf{M},\\
a\mathcal{R}b \quad & \text{if } a \leq_{\mathcal{R}} b \quad \text{and} \quad b \leq_{\mathcal{R}} a,\\
a \leq_{\mathcal{L}} b \quad & \text{if } a = x \cdot b \text{ for some } x \in \mathbf{M},\\
a\mathcal{L}b \quad & \text{if } a \leq_{\mathcal{L}} b \quad \text{and} \quad b \leq_{\mathcal{L}} a,\\
a \leq_{\mathcal{J}} b \quad & \text{if } a = x \cdot b \cdot y \text{ for some } x, y \in \mathbf{M},\\
a\mathcal{J}b \quad & \text{if } a \leq_{\mathcal{J}} b \quad \text{and} \quad b \leq_{\mathcal{J}} a.
\end{aligned}$$

Finally we define $\mathcal{J}(b) = \{a \mid a\mathcal{J}b\}$, for all $b \in \mathbf{M}$, to be the $\mathcal{J}$-class of b. Similarly, we denote by $\mathcal{R}(b), \mathcal{L}(b), \mathcal{H}(b)$, the $\mathcal{R}$-class of b, the $\mathcal{L}$-class of b, and the $\mathcal{H}$-class of b respectively.

The best way to view the $\mathcal{J}$-classes in a finite monoid is the "egg-box" view. The $\mathcal{R}$-classes and $\mathcal{L}$-classes form rows and columns in the $\mathcal{J}$-class.

A.3 Properties of $\circ$-monoids

Let us look at a few properties of monoids in general.

Lemma 6. *1. Let $u, t, v \in \mathbf{M}^\circ$ and $a \in \mathbf{M}$ be such that $\gamma(ut), \gamma(tv) \geq_{\mathcal{J}} a$ and $\gamma(utv) \not\geq_{\mathcal{J}} a$. Then $\gamma(t) >_{\mathcal{J}} a$.*
2. Let $a\mathcal{R}b$. Then $m \cdot a <_{\mathcal{J}} a \implies m \cdot b <_{\mathcal{J}} a$.
3. In a finite monoid, $b \leq_{\mathcal{R}} a$ and $a\mathcal{J}b$ implies $a \;\mathcal{R}\; b$.

Proof. 1. Let us assume that $\gamma(t)\mathcal{J}a$. Then $\gamma(ut)\mathcal{J}\gamma(t)\mathcal{J}a$. This implies $\gamma(ut) \;\mathcal{L}\; \gamma(t)$, and therefore $\gamma(utv) \;\mathcal{L}\; \gamma(tv)$. Hence $\gamma(utv)\mathcal{J}a$, which is a contradiction. $\square$

We will be mostly be interested in $\overrightarrow{a}$, when a is an idempotent. The following lemmas explains why.

Lemma 7. *1. Let $a \in M$. Then $a^\omega = e^\omega$, for an idempotent e.*
2. If $a\mathcal{R}b$ and $a^\omega\mathcal{J}b^\omega\mathcal{J}a$, then $a^\omega = b^\omega$.
3. If $a\mathcal{J}a^\omega\mathcal{J}b^\omega\mathcal{J}b$, then $a^\omega\mathcal{L}b^\omega$.
4. Let $a\mathcal{L}b$. Then $a\mathcal{R}a^\omega$ implies $b^\omega\mathcal{L}a^\omega$.

The following lemmas talk about the interaction between η, ω and concatenation operators, for general $\circ$-monoids.

Lemma 8. *The following holds for $a \in M$ and $R, S \subseteq M$, where M is a $\circ$-semigroup.*

1. *If $S^\eta\mathcal{J}R^\eta$, then $S^\eta = R^\eta$*
2. *$S^\eta = S^\eta \cdot a \cdot S^\eta$, for an $a \in S$.*
3. *If $a^\eta = a$. Then $a^\omega = a$ and $a^{\omega^*} = a$. (The other direction need not hold in general)*
4. *If J is a $\mathcal{J}$-class that contains a non-trivial group, then for all a in J, $a^\omega, a^{\omega^*}, a^\eta$ are not in J.*

An element a is *regular* if $a \cdot x \cdot a = a$ of some x.

Lemma 9 (regular $\mathcal{J}$-classes). *In a $\mathcal{J}$-class J, the following properties are equivalent:*

- *there is one regular element.*
- *all elements are regular.*
- *there exists $a, b \in J$ with $a \cdot b \in J$.*
- *there is an idempotent,*
- *all $\mathcal{L}$-classes in J contain an idempotent,*
- *all $\mathcal{R}$-classes in J contain an idempotent.*

A $\mathcal{J}$-class satisfying one of the conditions of the above lemma is *regular*.

Lemma 10. *Let J be a regular $\mathcal{J}$-class. Then the following are equivalent.*

1. *J contains an ordinal idempotent.*

2. J *contains an idempotent* e*, such that* $e^\omega \in J$.
3. *Every* $\mathcal{R}$ *class in* J *contains an ordinal idempotent.*
4. *Every* $\mathcal{R}$ *class in* J *contains an idempotent* e *such that* $e^\omega \in J$.

A $\mathcal{J}$-class satisfying one of the clauses of the previous lemma is said *ordinal regular* (*resp.* ordinal* regular).

We say that a $\mathcal{J}$-class J is shuffle regular if $e^\eta = e$ for some $e \in J$.

Here is a connection between the shuffle operator and omega operator.

Lemma 11. *Let* $e^\eta = e$ *and let* $u = \prod_{i \in \mathbb{Q}} u_i$*, such that* $\gamma(u_i) = e$*. Then* $u \in \overrightarrow{e}$ *and* $u \in \overleftarrow{e}$.

Proof. Consider the following words. Let $v_0 = u[\infty, 0)$ and for all $j \in \mathbb{N}$, $v_j = u[j-1, j)$. Observe that $\gamma(v_0) = e$ and for all $j > 1$, $\gamma(v_j) = e \cdot \gamma(u(j-1, j)) = e \cdot e^\eta = e \cdot e = e$. Therefore $u \in \overrightarrow{e}$. A similar arguments shows that $u \in \overleftarrow{e}$. □

Lemma 12. *Let* J *be a* $\mathcal{J}$*-class.*

1. *If* J *is shuffle regular, then* J *is scattered regular.*
2. *If* J *is ordinal regular (or ordinal* regular), then* J *is [regular* $\mathcal{J}$*-class]regular.*

Lemma 13. *Let* J *be a scattered regular* $\mathcal{J}$*-class then* $e^\omega \cdot e^{\omega^*} = e$ *for all idempotents* $e \in J$.

Proof. Since J is scattered regular, there exists an $f \in J$ such that $f^{\omega^*} = f = f^\omega$. Let e be an idempotent in J. Then $g = e^{\omega^*} \cdot e^\omega \in J$. We see that $g\mathcal{R}f^{\omega^*}$ and $g\mathcal{L}f^\omega$. Since J is a group free regular $\mathcal{J}$-class, we have that $f = g$. Therefore $g \cdot g \in J$ and hence $h = e^\omega \cdot e^{\omega^*} \in J$. Therefore $e\mathcal{L}h\mathcal{R}e$, which because J is group free, implies $h = e$. □

Lemma 14. *For an idempotent* e*, let* $e\mathcal{J}e^\omega \cdot e^{\omega^*}$*. Then* $e = e^\omega \cdot e^{\omega^*}$.

Proof. Since $e\mathcal{J}e^\omega e^{\omega^*}$, $e\mathcal{L}e^\omega e^{\omega^*}$ and $e\mathcal{R}e^\omega e^{\omega^*}$. The claim holds, since $\mathcal{H}(e)$ contains only one element. □

Lemma 15. *Let* J *be a* $\mathcal{J}$*-class, which is aperiodic but not scattered regular, such that* $\mathcal{J}(a) = J$*. Then* $\forall g \mathcal{J} a$, $g^\omega g^{\omega^*} \not\geq_{\mathcal{J}} a$.

Proof. Assume not. That is, there exists a $g\mathcal{J}a$, such that $h = g^\omega g^{\omega^*} \in J$. Then $h\mathcal{L}g\mathcal{R}h$. Since J is aperiodic, $h = g$. Therefore J is regular, since $g^\omega \in J$. Now consider the element $e = g^{\omega^*} g^\omega$. Therefore $e.e = (g^{\omega^*} g^\omega)(g^{\omega^*} g^\omega) = g^{\omega^*}(g^\omega g^{\omega^*})g^\omega = g^{\omega^*} g g^\omega = e$. Since e is an idempotent and J is regular, $e^\omega \in J$ and since $e\mathcal{L}g^\omega$, we have $e^\omega = e$. Similarly we get $e^{\omega^*} = e$. Therefore J is scattered regular. This is a contradiction. □

Lemma 16. *Let* J *be a shuffle regular* $\mathcal{J}$*-class. Then* J *satisfies the equations* $(e^{\omega^*} e^\omega)^\eta = e^{\omega^*} e^\omega$, $e^\omega e^{\omega^*} = e$*, for all idempotents* e.

Proof. Since J is shuffle regular, there exists an $f \in J$ such that $f^\eta = f$. Clearly this implies $f^{\omega^*} = f = f^\omega$. Let e be an idempotent in J. Then $g = e^{\omega^*} e^\omega \in J$. We see that $g\mathcal{R}f^{\omega^*}$ and $g\mathcal{L}f^\omega$. Since J is a group free regular class, we have that $f = g$ or $g^\eta = g$.

To show $e^\omega e^{\omega^*} = e$, we note that shuffle regular implies scattered regular. Then it follows from previous Lemma. □

A.4 Types of $\mathcal{J}$-classes and their interactions

The following figure 1 shows the relationship between different $\mathcal{J}$-classes. We note that a $\mathcal{J}$-class containing a shuffle simple idempotent has a scattered idempotent. Similarly groups are in regular $\mathcal{J}$-classes, but do not contain any ω or ω^* operation.

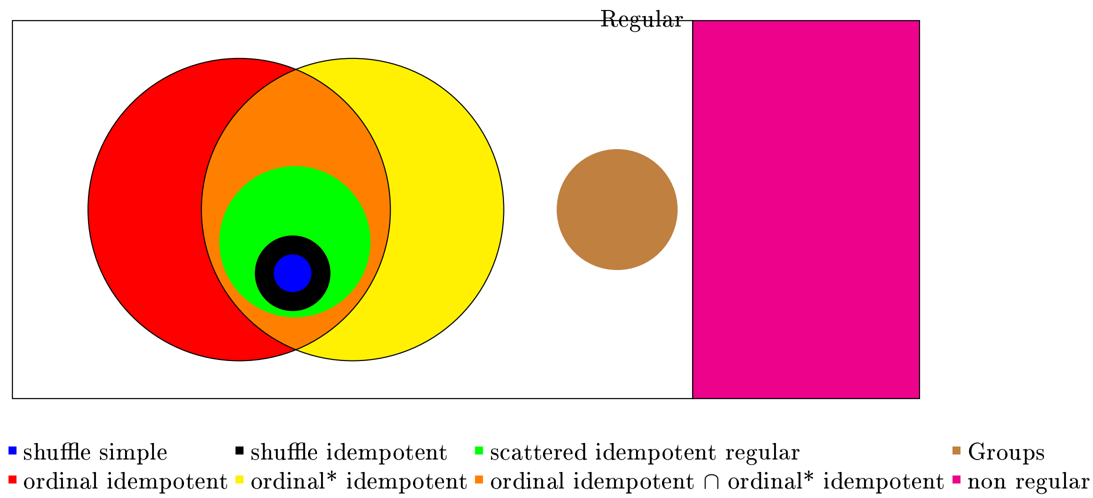
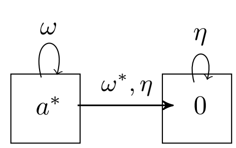


Fig. 1. Relationship between different $\mathcal{J}$-classes

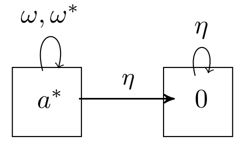
A.5 Examples

We will see a few examples now to understand that $a^\eta, a^\omega, a^{\omega *}$ all need not be in the same $\mathcal{J}$-class.

Example 5. Consider the language of all words over $\{a\}^\circ$, which do not have a left limit. Then $a = a^\omega >_{\mathcal{J}} a^\eta = a^{\omega *}$.



Example 6. Consider the language of all words over $\{a\}^\circ$, which are sparse. Then $a = a^\omega = a^{\omega^*} >_{\mathcal{J}} a^\eta$.


ω, ω^*
η
a^*
η
0

B The translation from logics to ∘-monoids

In this section, we add some complementary informations concerning the translation from logics to algebra. We have emphasized in the main part of the paper restricted restricted quantifiers. In this appendix, we complete the picture, and provide missing proofs.

As it is classical in this kind of translations from logics to algebra, compliment is for free, and in order to implement the disjunction and conjunction, the closure under union and intersection of languages accepted by ∘-monoids satisfying one of the properties has to be done. For this, the product of ∘-monoid should be shown to preserve the properties we are interest in. Also, in order to establish that if a language recognized by a ∘-monoid has some property of interest, then its syntactic ∘-monoid also has this property, we need to show that our properties are preserved under taking substructures and quotient. These invariants will be the subject of the first section, Section B.1.

We then show in the successive sections the preservation of aperiodic, `i`$\to$`gi`, `oi`$\to$`gi`, `sc`$\to$`sh` and`sh`$\to$`ss` when suitable restricted quantifiers are applied.

B.1 Closure under quotient substructure and product

Usually, in similar results of algebraic characterizations of logics, it is not necessary to establish the closure under quotient. Indeed, the standard way to define a class of monoids is thanks to the use of "identities". When a monoid (or some algebraic structure) satisfies an identity, then all its quotients also do. We made the choice in the body of the paper to not use this terminology for simplicity. The price to pay is to prove the closure under quotient explicitly. We do it here in a way that merely mimics what is usually done with identities.

An *implicit operation* f of arity k is a collection of maps $(f_{\mathbf{M}})$ indexed by ∘-monoids such that for all ∘-monoid $\mathbf{M}$, $f_{\mathbf{M}}$ is a function $\mathbf{M}^k$ to $\mathbf{M}$, and that satisfies the property that whenever there is a ∘-monoid morphism γ from $\mathbf{M}$ to $\mathbf{N}$, and for all $x_1, \ldots, x_k \in M$,

$$f_{\mathbf{N}}(\gamma(x_1), \ldots, \gamma(x_k)) = \gamma(f_{\mathbf{M}}(x_1), \ldots, f_{\mathbf{M}}(x_1)) \ .$$

When the ∘-monoid in which it is applied is clear from the context, we omit it in the notations.

Note that the derived operations can naturally be seen as implicit operations, and that operations constructed from implicit operations are also implicit operations.

An *identity* is an equality of the form

$$f(x_1, \ldots, x_k) = g(x_1, \ldots, x_k)$$

where f and g are implicit operations of arity k. A ∘-monoid $\mathbf{M}$ satisfies this identity if for all $a_1, \ldots, a_k$ in $\mathbf{M}$, $f_{\mathbf{M}}(a_1, \ldots, a_k) = f_{\mathbf{M}}(a_1, \ldots, a_k)$.

Lemma 17. *If* **M** *satisfies an identity, and* **N** *is a sub-*$\circ$*-monoid of it , then* **N** *satisfies the identity.*

If **M** *satisfies an identity, and there exists a surjective morphism of* **M** *onto* **N***, then* **N** *satisfies the identity.*

If **M** *and* **N** *satisfy an identity, then so does* $\mathbf{M}\times\mathbf{N}$ *(with the obvious definition).*

Proof. The first statement is obvious. For the second, assume that **M** satisfies $f(x_1,\ldots,x_k)=g(x_1,\ldots,x_k)$, and that γ is a morphism from **M** onto **N**. Let $b_1,\ldots,b_k\in N$. using the surjectivity assumption of γ, there exist $a_1,\ldots,a_k\in M$ such that $\gamma(a_1)=b_1$,...,$\gamma(a_k)=b_k$. We now have:

$$\begin{aligned} f_{\mathbf{N}}(b_1,\ldots,b_k) &= f_{\mathbf{N}}(\gamma(a_1),\ldots,\gamma(a_k))\\ &= \gamma(f_{\mathbf{M}}(a_1,\ldots,a_k))\\ &= \gamma(g_{\mathbf{M}}(a_1,\ldots,a_k)) && \text{(since } \mathbf{M} \text{ satisfies the identity)}\\ &= g_{\mathbf{N}}(\gamma(a_1),\ldots,\gamma(a_k))\\ &= g_{\mathbf{N}}(b_1,\ldots,b_k)\ . \end{aligned}$$

Thus **N** satisfies $f(x_1,\ldots,x_k)=g(x_1,\ldots,x_k)$.

The product result is also straightforward. □

The consequence of the above lemma is that it is sufficient for us to rephrose the properties aperiodic, `i`$\to$`gi`, `oi`$\to$`gi`, `o`*`i`$\to$`gi`, `sc`$\to$`sh` and `sh`$\to$`ss` into identities for having the closure under quotient.

The following lemma shows that every notions used in our characterization results can be interpreted in terms of implicit operations.

Lemma 18.
- *there exists an implicit operation "i" such that for all elements* a *in some* $\circ$*-monoid,* a^i *is an idempotent and if* e *is an idempotent, then* $e^i=e$ *(this operation is usually denoted* $^\omega$*, or* $^\pi$*).*
- *there exists an implicit operation "oi" such that for all elements* a *in some* $\circ$*-monoid,* a^{oi} *is an ordinal idempotent and if* e *is an ordinal idempotent, then* $e^{oi}=e$*.*
- *there exists an implicit operation "sc" such that for all elements* a *in some* $\circ$*-monoid,* a^{sc} *is an scattered idempotent and if* e *is a scattered idempotent, then* $e^{sc}=e$*. (and by symmetry, there is a similar operation "o***i")*
- *there exists an implicit operation "sh" of arity* $k+1$ *such that* $sh(e,a_1,\ldots,a_k)$ *is a shuffle idempotent* f *such that* $f\cdot a_n\cdot f=f$ *for all* $n=1\ldots k$*. Furthermore, if* e *is a shuffle idempotent such that* $e\cdot a_n\cdot e=e$ *for all* $n=1\ldots k$*, then* $sh(e,a_1,\ldots,a_k)=e$*.*

Proof. Operation "i": Let a be an element in a finite $\circ$-monoid. Consider the sequence $a^{1!}$, $a^{2!}$, ... It is ultimately constant. Let a^i be this element. Such definitions as limits always yield valid implicit operations: Let γ be a morphism from **M** to **N**, the sequence $\gamma(a^{1!})=\gamma(a)^{1!}$, $\gamma(a^{2!})=\gamma(a)^{2!}$, ... is also ultimately constant, and its limit is $\gamma(a^i)$. Also it is clear that if e is an idempotent, $e^i=e$.

Operation "oi": Let a be an element in a finite $\circ$-monoid $\mathbf{M}$. Consider the sequence defined by $a_0 = a$ and $a_{n+1} = (a_n)^\omega$. Clearly, this sequence can only go down in the $\mathcal{J}$-classes. It has to stabilize to some $\mathcal{J}$-class at some point. This means that $a_n \mathcal{J} a_{n+1} \mathcal{J} a_{n+2}$ for some n. Since $a_{n+1} = a_n^\omega$, $a \geq_\mathcal{J} a_{n+1}$. Using Lemma 6, we obtain $a_n \mathcal{R} a_{n+1}$. Now using Lemma 7, we obtain $a_n^\omega = a_{n+1}^\omega$, *i.e.*, $a_{n+1} = a_{n+2}$. This means that the sequence is ultimately constant. Let a^{oi} be this limit value. Once more, as defined by an ultimately constant sequence, it is an implicit operation. Furthermore, if e is an ordinal idempotent, *i.e.*, clearly $e^\omega = e$ which meant that $e^{oi} = e$.

Operation "sc": Let a be an element in a finite $\circ$-monoid $\mathbf{M}$. Consider the sequence defined by $a_0 = a$ and $a_{n+1} = (a_n)^{\omega*}(a_n)^\omega$. Clearly, this sequence can only go down in the $\mathcal{J}$-classes. It has to stabilize to some $\mathcal{J}$-class at some point. This means that $a_n \mathcal{J} a_{n+1} \mathcal{J} a_{n+2}$ for some n. Using Lemma 6, we obtain $a_n \mathcal{R} a_n^\omega$, and $a_n \mathcal{L} a_n^{\omega*}$. Then, using Lemma 7 we obtain $a_{n+1} \mathcal{L} a_{n+2}$ and $a_{n+1} \mathcal{R} a_{n+2}$. Hence (because this is an $\mathcal{H}$-trivial class) $a_{n+1} = a_{n+2}$. This means that the sequence is ultimately constant. Let a^{sc} be this limit value. Once more, as defined by an ultimately constant sequence, it is an implicit operation. Furthermore, if e is an scattered idempotent, *i.e.*, $e^\omega = e = e^{\omega*}$, we clearly have $e^{sc} = e$.

Operation "sh": Finally, given $e, a_1, \ldots, a_k$, consider the sequence defined by $e_0 = e$, and

$$\begin{aligned} f_n &= e_n^{\omega*} \cdot e_n^\omega \cdot a_1 \cdot e_n^{\omega*} \cdot e_n^\omega \cdot a_2 \cdots a_k \cdot e_n^{\omega*} \cdot e_n^\omega \ , \\ e_{n+1} &= \{f_n\}^\eta \ . \end{aligned}$$

Once more this sequence can only go down in the $\mathcal{J}$-classes and ultimately stabilizes in this $\mathcal{J}$-class. For the exact same reason as above, this means that is is ultimately constant. ☐

Once this implicit operations are known, it is easy to recast the various properties we are interested in into equivalent identities.

Lemma 19. – *aperiodic is equivalent to* $x^i = x^i \cdot x$.
- `i→gi` *is equivalent to* $(x^i)^\omega \cdot (x^i)^{\omega*}$.
- `oi→gi` *is equivalent to* $x^{oi} \cdot x^{\omega*} = x^{oi}$.
- `sc→sh` *is equivalent to* $\{x^{sc}\}^\eta = x^{sc}$.
- `sh→ss` *is equivalent to* $sh(x, y_1, \ldots, y_k) = (\{sh(x, y_1, \ldots, y_k)\} \cup \{y_1, \ldots, y_k\})^\eta$.

Corollary 1. *The properties aperiodic,* `i→gi`*,* `oi→gi`*,* `sc→sh` *and* `sh→ss` *are preserved under quotient.*

B.2 Preservation of aperiodicity under cut restricted quantifiers

Let us assume that a $\circ$-monoid $\mathbf{M}$ is aperiodic. Let $\mathbf{N} = \mathcal{P}_{\mathbf{Cut}}(\mathbf{M})$. Our goal is to show that $\mathbf{N}$ is aperiodic. This is achieved by Lemma 21.

We have to give first the definition of **Cut** which was not introduced in the main part of the paper. It has the five elements $1, l, r, m, 0$. Let us first describe the morphism γ from $\{\in, \notin\}^\circ$ to it. Given a word $u \in \{\in, \notin\}^\circ$,

$$\gamma(u) = \begin{cases} 1 & \text{if } u = \varepsilon, \\ l & \text{if } u \text{ contains only the letter } \in, \\ r & \text{if } u \text{ contains only the letter } \notin, \\ m & \text{if in } u \text{ all occurrences of } \in \text{ are to the left of all occurrences of } \notin, \\ 0 & \text{otherwise.} \end{cases}$$

The letter $\in$ is sent to l ("l" stands for "left of the cut"), and the letter $\notin$ is sent to r ("r" stands for "right of the cut"). The derived operations are the following (unit and zero are omitted, and $*$ stands for "anything"):

$\cdot$	l	r	m
l	l	m	m
r	0	r	0
m	0	m	0

$$l^\omega = l^{\omega *} = \{l\}^\eta = l$$
$$r^\omega = r^{\omega *} = \{r\}^\eta = r$$
$$m^\omega = m^{\omega *} = \{m, *\}^\eta = \{l, r, *\}^\eta = 0$$

Let $\mathbf{N}$ be $\mathcal{P}_{\mathbf{Cut}}(\mathbf{M})$. For simplicity, we replace the unit of $\mathbf{N}$, which is $1_{\mathbf{N}} = \{(1, 1)\}$ by the element $\{(1, l), (1, r)\}$. This changes nothing to the language accepted by $\mathbf{N}$ but makes the following lemma more uniform.

Lemma 20. *There exist two morphisms $\rho_L, \rho_R : \mathbf{N} \to \mathbf{M}$ such that for all $A \in \mathbf{N}$, $(x, l) \in A$ if and only if $x = \rho_L(A)$, and $(y, r) \in A$ if and only if $y = \rho_R(A)$.*

Proof. By case analysis.

Lemma 21. *Given an aperiodic $\circ$-monoid $\mathbf{M}$, then $\mathcal{P}_{\mathbf{Cut}}(\mathbf{M})$ is also aperiodic.*

Proof. Let A be some reachable element of $\mathbf{N}$. By assumption, there exists n such that $\rho_L(A)^n = \rho_L(A)^{n+1}$, and $\rho_R(A)^n = \rho_R(A)^{n+1}$. Let us consider now the constant $m = 2n + 1$, and the element $A^m \in \mathbf{N}$. Let $(x, y) \in A^m$. This means that there exists $(x_1, y_1), \ldots, (x_m, y_m) \in A$ such that $\pi(x_1 \ldots x_m) = x$ and $\pi(y_1, \ldots, y_m)$. Since $y \neq 0$, this means that there is ℓ such that $y_1 = y_2 = \cdots = y_{\ell-1} = l$, $y_\ell \in \{l, r\}$, and $y_{\ell+1}, \ldots, y_m = r$. By the above lemma we hence get $x_1 = x_2 = \cdots = x_{\ell-1} = \rho_L(A)$, and $x_{\ell+1} = \cdots = y_m = \rho_R(A)$.

Two cases may occur: either $\ell \leq m$, or $\ell > m$. Case $\ell > m$: we construct the word

$$\overbrace{(\rho_L(A), l) \ldots (\rho_L(A), l)}^{n+1 \text{ times}} (x_\ell, y_\ell) \overbrace{(\rho_R(A), r) \ldots (\rho_R(A), r)}^{n \text{ times}} .$$

Using the aperiodicity of $\mathbf{M}$, this word evaluates to (x, y) (componentwise). Furthermore, this word belongs to $\overbrace{A \ldots A}^{m+1 \text{ times}}$. Hence $(x, y) \in A^{m+1}$. The case $\ell \leq m$ is symmetric (this time adding an element in the right part of the word). We get once more $(x, y) \in A^{m+1}$. Overall we have $A^m \subseteq A^{m+1}$.

Using the same argument, this time removing elements, yields the other inclusion, and we have $A^m = A^{m+1}$. This shows that $\mathbf{N}$ is aperiodic. $\square$

B.3 Preservation $\mathtt{i}{\to}\mathtt{gi}$ under first-order quantifiers

Let us assume that a $\circ$-monoid $\mathbf{M}$ satisfies the property $\mathtt{i}{\to}\mathtt{gi}$. Let $\mathbf{N} = \mathcal{P}_{\mathbf{Sing}}(\mathbf{M})$. Our goal is to show that $\mathbf{N}$ also satisfies $\mathtt{i}{\to}\mathtt{gi}$. This is achieved by Lemma 22.

Lemma 22. *Given a $\circ$-monoid $\mathbf{M}$ satisfying $\mathtt{i}{\to}\mathtt{gi}$, then $\mathcal{P}_{\mathbf{Sing}}(\mathbf{M})$ also satisfies $\mathtt{i}{\to}\mathtt{gi}$.*

Proof. Let E be an idempotent in $\mathbf{N}$. Our goal is to show that it is gap insensitive.

Let $(x, y) \in E$. Since $E = E \cdot E$, there exists $(x_1, y_1), (x_2, y_2) \in E$ such that $x_1 \cdot x_2 = x$ and $y_1 \cdot y_2 = y$. Since $y \neq 0$, at least one among y_1, y_2 is equal to 1. Without loss of generality, let us assume it is y_1. In this case, according to Lemma 4, $x_1 = \rho(E)$. In particular, since ρ is a morphism, this means that x_1 is an idempotent. Thus we can use the assumption that $\mathbf{M}$ satisfies $\mathtt{i}{\to}\mathtt{gi}$ on it, and get that $x_1^{\omega} \cdot x_1^{\omega*} = x_1$. It follows that the word

$$\overbrace{(x_1,1)(x_1,1)\ldots}^{\text{of domain } \omega}\ \overbrace{\ldots(x_1,1)(x_1,1)}^{\text{of domain } \omega*}(x_2,y_2)$$

has also value (x, y) under π (componentwise), and as a consequence $(x, y) \in E^{\omega} \cdot E^{\omega*}$. We have proved $E \subseteq E^{\omega} \cdot E^{\omega*}$.

Conversely, consider some $(x, y) \in E^{\omega} \cdot E^{\omega*}$. This means that there exists a word u of the form

$$\overbrace{(x_1,y_1)(x_2,y_2)\ldots}^{\text{of domain } \omega}\ \overbrace{\ldots(x'_2,y'_2)(x'_1,y'_1)}^{\text{of domain } \omega*}$$

which evaluates (componentwise) to (x, y), with (x_i, y_i) and $(x'_i, y'_i) \in E$ for all $i \in \mathbb{N}$. According to the definition of **Sing**, there is at most one among the y_i's and the y'_i's which is not equal to 1. Without loss of generality (by symmetry), we can assume that it is y_j. According to Lemma 4, $x_i = \rho(E)$ for all $i \neq j$ and $x'_i = \rho(E)$ for all i. Since ρ is a morphism, $\rho(E)$ is also an idempotent. Thus we can use the assumption that $\mathbf{M}$ satisfies $\mathtt{i}{\to}\mathtt{gi}$. We obtain that $\rho(E)^{\omega} \cdot \rho(E)^{\omega*} = E$. Thus, u evaluates to $(\rho(E), 1) \cdot (x_j, y_j) \cdot (\rho(E), 1) \in E^3 = E$. Hence $E^{\omega} \cdot E^{\omega*} \subseteq E$.

This terminates the proof that $\mathbf{N}$ satisfies $\mathtt{i}{\to}\mathtt{gi}$.

B.4 Preservation of $\mathtt{oi}{\to}\mathtt{gi}$ and $\mathtt{o^*i}{\to}\mathtt{gi}$ under weak set quantifiers

Let us assume that a $\circ$-monoid $\mathbf{M}$ satisfies the property $\mathtt{oi}{\to}\mathtt{gi}$ (the $\mathtt{o^*i}{\to}\mathtt{gi}$ is obviously symetric). Let $\mathbf{N} = \mathcal{P}_{\mathbf{Fin}}(\mathbf{M})$. Our goal is to show that $\mathbf{N}$ also satisfies $\mathtt{oi}{\to}\mathtt{gi}$. This is achieved by Lemma 23.

Lemma 23. *Given a $\circ$-monoid $\mathbf{M}$ satisfying $\mathtt{oi}{\to}\mathtt{gi}$, then $\mathcal{P}_{\mathbf{Fin}}(\mathbf{M})$ also satisfies $\mathtt{oi}{\to}\mathtt{gi}$.*

Proof. Let E be an ordinal idempotent in $\mathbf{N}$. Our goal is to show that it is gap insensitive.

Let $(x,y)\in E$. Since $E=E^\omega$, there exists a word

$$(x_1,y_1)(x_2,y_2)\ldots\in\overbrace{EE\ldots}^{\text{domain }\omega}$$

of value under π (componentwise) (x,y). Since $y\neq 0$, and by definition of **Fin**, $y_i=1$ for all i's but finitely many. This means that there exists j such that $y_i=1$ for all $i\geq j$. According to Lemma 4, this means $x_i=\rho(E)$ for all $i\geq j$. Since ρ is a morphism, x_1 is also an ordinal idempotent. Thus we can use the assumption that $\mathbf{M}$ satisfies `oi`$\to$`gi` on it. It follows that the word

$$(x_1,y_1)\ldots(x_j,y_j)\overbrace{(\rho(E),1)(\rho(E),1)\ldots}^{\text{domain }\omega}\ \overbrace{\ldots(\rho(E),1)(\rho(E),1)}^{\text{domain }\omega*}$$

has also value (x,y) under π (componentwise), and as a consequence $(x,y)\in E^\omega\cdot E^{\omega*}$. We have proved $E\subseteq E^\omega\cdot E^{\omega*}$.

Conversely, consider some $(x,y)\in E^\omega\cdot E^{\omega*}$. This means that there exists a word u of the form

$$\overbrace{(x_1,y_1)(x_2,y_2)\ldots}^{\text{of domain }\omega}\ \overbrace{\ldots(x_2',y_2')(x_1',y_1')}^{\text{of domain }\omega*}$$

which evaluates (componentwise) to (x,y), with (x_i,y_i) and $(x_i',y_i')\in E$ for all $i\in\mathbb{N}$. According to the definition of **Fin**, there $y_i=1$ and the $y_i'=1$ for all but finitely many i's. Let j be such that $y_i=1$ and the $y_i'=1$ for all $i>j$. According to Lemma 4, $x_i=x_i'=\rho(E)$ for all $i>j$ Since ρ is a morphism, $\rho(E)$ is also an idempotent. Thus we can use the assumption that $\mathbf{M}$ satisfies `oi`$\to$`gi`. We obtain that

$$(x_1,y_1)\ldots(x_i,y_j)\,(\rho(E),1)\,(x_j',y_j')\ldots(x_1',y_1')$$

also evaluates to (x,y). Furtermore it belongs to $E^{2j+1}=E$. Hence $E^\omega\cdot E^{\omega*}\subseteq E$.

This terminates the proof that $\mathbf{N}$ satisfies `oi`$\to$`gi`.

B.5 Preservation of `sc`$\to$`sh` under ordinal quantifiers

Let us assume that a $\circ$-monoid $\mathbf{M}$ satisfies the property `sc`$\to$`sh`. Let $\mathbf{N}=\mathcal{P}_{\mathbf{Ord}}(\mathbf{M})$. Our goal is to show that $\mathbf{N}$ also satisfies `sc`$\to$`sh`. This is achieved by Lemma 24.

Lemma 24. *Given a $\circ$-monoid $\mathbf{M}$ satisfying* `sc`$\to$`sh`*, then $\mathcal{P}_{\mathbf{Ord}}(\mathbf{M})$ also satisfies* `sc`$\to$`sh`*.*

Let E be a scattered idempotent in $\mathbf{N}$. Our goal is to show that it is a shuffle idempotent, *i.e.*, that $\{E\}^\eta=E$.

We claim first $(\star)$ that if $(x,y)\in E$, then $x=\rho(E)\cdot x$ (for ρ from Lemma 4). Let $(x,y)\in E$. Since $E=E^{\omega*}$, there exists a word

$$\ldots(x_2,y_2)(x_1,y_1)\in\overbrace{\ldots EE}^{\text{domain }\omega^*}$$

of value (x, y) under π (componentwise). Since $y \neq 0$, and by definition of **Ord**, the set of indices i such that $y_i = s$ is well ordered. In our case, as a subset of ω^*, this means that it is finite. Hence there exists j such that $y_i = 1$ for all $i \geq j$. According to Lemma 4, this means $x_i = \rho(E)$ for all $i \geq j$. Since ρ is a morphism, x_1 is also an scattered idempotent. It follows that u can be written

$$\overbrace{\ldots(\rho(E), 1)(\rho(E), 1)}^{\text{domain } \omega^*}(x_j, y_j)\ldots(x_1, y_1)$$

and hence has value $(\rho(E) \cdot x_j \cdots x_1, y_j \cdots x_1)$ under π (componentwise). As a consequence $x = \rho(E) \cdot x$. The claim is established.

We can now prove that if $(x, y) \in E$, then $(x, y) \in \{E\}^\eta$. Note first that since $E = E^\omega$,

$$(x, y) = \pi(\overbrace{(x_1, y_1)(x_2, y_2)\ldots}^{\in EE\ldots}).$$

We know that $\rho(E)$ is a scattered idempotent since E is one. Hence, using the assumption it is a shuffle idempotent: $\{\rho(E)\}^\eta = \rho(E)$. It follows in combination with claim $(\star)$ that $(x_i, y_i) = (\rho(E), 1)\}^\eta(x_i, y_i)$ for all i. We obtain

$$(x, y) = \pi(\underbrace{\mathit{perfectshuffle}(\{(\rho(E), 1)\})(x_1, y_1)\mathit{perfectshuffle}(\{(\rho(E), 1)\})(x_2, y_2)\ldots}_{\in \mathit{perfectshuffle}(\{E\})}),$$

and by consequence $(x, y) \in \{E\}^\eta$.

Conversely, let $(x, y) \in \{E\}^\eta$. This means that there is a word u of domain $(\mathbb{Q}, <)$ over the alphabet E such that $\pi(u) = (x, y)$. Since $y \neq 0$, and by definition of **Ord**, the set of indices $I = \{i \in \mathbb{Q} \mid y_i = s\}$ is well ordered. Let us define the equivalence relation $\sim$ over $\mathbb{Q}$ defined for $i < j$ by $i \sim j$ if there is i', j' such that $[i, j] \subseteq (i', j')$ and $(i', j') \cap I = \emptyset$.

It is easy to check that $\sim$ is a condensation. Consider now some non-singleton condensation class C. By definition $y_i = 1$ for all $i \in C$. According to Lemma 4, this implies $x_i = \rho(E)$ for all $i \in C$. Furthermore, C has to be infinite and dense. (as a non-singleton interval of the rationals). Furthermore, C does not have a minimal nor a maximal point. All this together means that $u|_C$ is isomorphic to *perfectshuffle*$(\{(\rho(E), 1)\})$. Since $\rho(E)$ is a shuffle idempotent, this means $\pi(u|_C) = \{(\rho(E), 1)\}^\eta = (\rho(E), 1) \in E$.

Note that all intervals (i, j) that intersects to distinct classes also intersects I. Indeed, by contradiction, assume that $(i, j) \cap I = \emptyset$, then, by definition, all points in (i, j) would be $\sim$-equivalent. A contradiction.

Let us show that the condensed order $dom(u)/_\sim$ is scattered. For the sake of contradiction, assume contains a dense subset X. Take two condensation classes $A < B$, and $i \in A$ and $j \in B$. By density there are at least two other classes in between A and B. It follows by the above remark that (i, j) intersects I. It follows that I also contains a dense subset. A contradiction.

Overall, this condensation witnesses the fact that $\pi(u)$ can be rewritten as $\pi(v)$ with v in E° of scattered domain. Since E is a scattered idempotent, this implies that $(x, y) = \pi(u) \in E$. Overall $\{E\}^\eta = E$.

This terminates the proof that $\mathbf{N}$ satisfies $\texttt{sc}\to\texttt{sh}$.

B.6 Preservation of $\texttt{sh}\to\texttt{ss}$ under scattered set quantifier

Let us assume that a $\circ$-monoid $\mathbf{M}$ satisfies the property $\texttt{sh}\to\texttt{ss}$. Let $\mathbf{N} = \mathcal{P}_{\mathbf{Scat}}(\mathbf{M})$. Our goal is to show that $\mathbf{N}$ also satisfies $\texttt{sh}\to\texttt{ss}$. This is achieved by Lemma 26.

The following standard lemma has to be stated before.

Lemma 25. *In any $\circ$-monoid, let e be a shuffle idempotent, and K be a set of elements such that $e \cdot a \cdot e = e$ for all $a \in K$, then every word over $K \cup \{e\}$ such that:*

1. *u is non-empty and scattered,*
2. *every non-trivial factor (of at least two letters) contains a letter e,*
3. *the minimal position, if it exists, is labelled e,*
4. *the maximal position, if it exists, is labelled e,*

is such that $\pi(u) = e$.

We can now state our preservation result.

Lemma 26. *Given a $\circ$-monoid $\mathbf{M}$ satisfying $\texttt{sh}\to\texttt{ss}$, then $\mathcal{P}_{\mathbf{Scat}}(\mathbf{M})$ also satisfies $\texttt{sh}\to\texttt{ss}$.*

Proof. Let E be some shuffle idempotent. As usual $\rho(E)$ is a also a shuffle idempotent, which, by assumption is furthermore shuffle simple. Let also $K \subseteq \mathbf{N}$ be a set of elements such that $E \cdot A \cdot E = E$ for all $A \in K$. Without loss of generality, we can assume $E \in K$. Let us denote also $\rho(K)$ for $\{\rho(A) \mid A \in K\}$. Note that since ρ is a morphism, $\rho(E) \cdot a \cdot \rho(E) = \rho(E)$ for all $a \in \rho(K)$.

Let $(x, y) \in E$. Since $E = \{E\}^\eta$, there exists a word $u \in \mathit{perfectshuffle}(\{E\})$ such that $\pi(u) = (x, y)$. Since $y \neq 0$, and by definition of $\mathbf{Scat}$, the set $I = \{i \in \mathit{dom}(u) \mid y_i = s\}$ is scattered. Let us define the equivalence relation $\sim$ over $\mathbb{Q}$ defined for $i < j$ by $i \sim j$ if there is i', j' such that $[i, j] \subseteq (i', j')$ and $(i', j') \cap I = \emptyset$. Using the arguments as in the previous section we have:

- The condensed order by $\sim$ is scattered.
- For all non-singleton condensation class C, $u|_C$ is isomorphic to $\mathit{perfectshuffle}(\{(\rho(E), 1)\})$.

Let us now construct the word v which coincide with u over singleton classes, and for which the labels of each non-singleton class is set to $\mathit{perfectshuffle}(\{\rho(a) \mid a \in K\})$ (this is possible since the domains are isomorphic). This amounts to substitute to factors of values $\{\rho(E), 1\}^\eta = (\rho(E), 1)$ under π factors of value $\{\rho(a) \mid a \in K\}^\eta = (\rho(E), 1)$. Thus, $\pi(v) = \pi(u) = (x, y)$.

Since furthermore $\rho(E)$ is shuffle simple, we even have $\pi(u|_C) = \{(\rho(A), 1) \mid A \in K\}^\eta$. Furthermore, it is routine to check that $v \in \mathit{perfectshuffle}(K)$. Hence $(x, y) \in K^\eta$.

Conversely, consider some $(x, y) \in K^\eta$. This means that there exists a word

$$u \in \mathit{perfectshuffle}(K) \ ,$$

such that $\pi(u) = (x, y)$. Since $y \neq 0$ and by definition of **Scat**, the set $I = \{i \in \mathit{dom}(u) \mid y_i = s\}$ is scattered. Again, we use the condensation $\sim$, and obtain:

- The condensed order by $\sim$ is scattered.
- For all non-singleton condensation class C, $u|_C$ is isomorphic to $\mathit{perfectshuffle}(\{(\rho(A), 1) \mid A \in K\})$. This means, since $\rho(E)$ is shuffle simple by assumption, that $\pi(u|_C) = \rho(E)$.

We obtain the word v from u by substituting to each condensation class C a single letter word $(\rho(E), 1)$. Since we have substituted factors in the word by factors of same value under π, $\pi(v) = \pi(u) = (x, y)$.

Furthermore, this word is (a) scattered, (b) all non-trivial factors of it contain a letter $(\rho(E), 1)$, (c) its minimal element, if it exists, is $(\rho(E), 1)$, and (c) its maximal element, if it exists, is $(\rho(E), 1)$.

By lifting the letters to their origin sets, there exists a word:

$$v \in V \ ,$$

over the alphabet K such that: it is (a) scattered, (b) all non-trivial factors of it contain a letter E, (c) its minimal element, if it exists, is E, and (c) its maximal element, if it exists, is E.

Then, from lemma 25, $\pi(V) = E$. This means that $(x, y) \in E$.

Everall, we have proved $K^\eta = E$, and hence `sh`$\to$`ss`. □

C Properties of words over M

In this section, we identify properties of words over $\mathbf{M}$ which will be used later in the proof. The main idea which we introduce here is the notion of witness. Our proof for going from monoid to logic, goes via induction on the $\mathcal{J}$-class. An important requirement to do this induction is to understand when a word fall to a lower $\mathcal{J}$-class. The notion of the witness captures the idea.

This section also contains statement of useful Lemmas like Ramsey theorem, Shalah' theorem, Zorn's lemma etc.

C.1 Standard Lemmas on words and linear order

The following standard lemmas helps to understand linear orders and $\circ$-monoids better.

Lemma 27 (Ramsey). *Given a countable linear ordering α with a minimum element x_0 and no maximum element, and given an additive labelling $f : \alpha^2 \rightarrow M$, there exists an ω sequence $x_0 < x_1 < x_2 < ...$ of points in α and two elements $a, e \in M$ such that*

- *For all $y \in \alpha$, there exists an $x_j > y$*
- *$f(x_0, x_1) = a$*
- *$f(x_i, x_j) = e$, for all $j > i > 0$*

Here is an interesting consequence of Ramsey theorem.

Lemma 28. *Let $u \in \overrightarrow{e}$, for an idempotent e. Then, there exists an $f\mathcal{L}e$ such that $u = r\prod_{i\in\mathbb{N}} u_i$, where for all $i \in \mathbb{N}$, $u_i \in []$ and $\gamma(u_i) = f$.*

Proof. Let $u \in \overrightarrow{e}$. From Ramsey's theorem (Lemma 27), we get that there exists points $x_0 < x_1 < x_2 < \dots$ such that $\gamma(w[-\infty, x_0]) = r$, for some $r \in \mathbf{M}$ and there exists an $f \in \mathbf{M}$, such that for all $i \geq 0$, $\gamma(w[x_i, x_{i+1}]) = f$. The u_i required in the proof is $w[x_i, x_{i+1}]$. We are now left to prove that $e\mathcal{J}f$. Note that $e^c\mathcal{J}f$, for some $c \in \mathbb{N}$. But since e is an idempotent, $e\mathcal{J}f$. Since $e = r \cdot f$, it follows that $f\mathcal{L}e$. □

Lemma 29 (Shelah). *Every word indexed by a non-singular countable dense linear ordering contains a perfect shuffle.*

Lemma 30 (Zorn). *If P is a poset in which every well-ordered subset has an upper bound, and if x is any element of P, then P has a maximal element that is greater than or equal to x. That is, there is a maximal element which is comparable to x.*

C.2 Witness

Consider a word $w \in \mathbf{M}^\circ$, such that $\gamma(w) \not\geq_{\mathcal{J}} a$. We need to understand the property which makes w fall below the $\mathcal{J}$-class of a. It can be that, a factor of w is responsible for this fall. We call this factor a witness. Its formal definition is given below. We also identify the different kinds of witnesses. The transformation of monoids to logic, requires understanding of witnesses and its properties.

Definition 1. *Witness*
Let $w \in \mathbf{M}^\circ$ be such that $\gamma(w) \not\geq_{\mathcal{J}} a$. Then we say that $t \in \mathbf{M}^\circ$ is a witness of w if $w \in \mathbf{M}^\circ t \mathbf{M}^\circ$ and $\gamma(t) \not\geq_{\mathcal{J}} a$ and one of the following conditions hold

- *Letter witness: $t \in \mathbf{M}$*
- *Concatenation witness: $t = u \cdot v$ such that $\gamma(u) \geq_{\mathcal{J}} a, \gamma(v) \geq_{\mathcal{J}} a$*
- *Omega witness: $t \in \overrightarrow{e}$, where e is an idempotent and $\gamma(e) \geq_{\mathcal{J}} a$*
- *Omega* witness: $t \in \overleftarrow{e}$, where e is an idempotent and $\gamma(e) \geq_{\mathcal{J}} a$*
- *Shuffle witness: t is a perfect shuffle over $B \subset \mathbf{M}^\circ$ where for all $u \in B$, $\gamma(u) \geq_{\mathcal{J}} a$.*

We now show that if a word w is such that $\gamma(w) \not\geq_{\mathcal{J}} a$, then w contains atleast one of the above witness.

Theorem 4. *Let $w \in \mathbf{M}^\circ$. If $\gamma(w) \not\geq_{\mathcal{J}} a$ then w contains a witness.*

Proof. Let us assume that w does not contain any witness. We will show that this implies $\gamma(w) \geq_{\mathcal{J}} a$, which proves the claim. We will denote by α the domain of w and let $\mathcal{I}$ be the set of all intervals of α such that for all $I \in \mathcal{I}$, we have $\gamma(w_I) \geq_{\mathcal{J}} a$ (recall that w_I is the factor of w restricted to the domain I). We say that $I_1, I_2 \in \mathcal{I}$ are consecutive intervals if $I_1 \cup I_2$ is an interval and $I_1 \cap I_2 = \emptyset$. First let us look at some properties of words.

Claim. If $I_1, I_2 \in \mathcal{I}$ are consecutive intervals, then $I_1 \cup I_2 \in \mathcal{I}$.

The proof of the claim is as follows. $\gamma(w_{I_1}) \geq_{\mathcal{J}} a, \gamma(w_{I_2}) \geq_{\mathcal{J}} a$, since $I_1, I_2 \in \mathcal{I}$. Since w does not contain witness of type (2) (by our assumption w does not contain any witness) $\gamma(w_{I_1} \cdot w_{I_2}) \geq_{\mathcal{J}} a$ and hence $I_1 \cup I_2 \in \mathcal{I}$.

Claim. Let $(J_k)_{k\in\omega}$ be such that for all $i \in \omega$, $J_i \in \mathcal{I}$ and J_i, J_{i+1} are consecutive intervals. Then $\bigcup_{k\in\omega} J_k \in \mathcal{I}$.

Let $J = \bigcup_{k\in\omega} J_k$ and $J_{i,k} = \bigcup_{l=i}^{k} J_l$. Ramsey theorem (Lemma 27) gives that there exists an idempotent e and $i_0 = 1 < i_1 < i_2 < \ldots$ such that $\gamma(w_{J_{i_k,i_{k+1}}}) = e$ for all $k \geq 0$. We observe that $e \geq_{\mathcal{J}} a$, since $e \geq_{\mathcal{J}} \gamma(w_{J_{1,i_1}}) \geq_{\mathcal{J}} \gamma(w_{I_{i_1}}) \geq_{\mathcal{J}} a$. Therefore $\gamma(w_J) = e^\omega$ and since there is no witness of form (3) we have $\gamma(w_J) \geq_{\mathcal{J}} a$.

A similar argument, but using the fact that there is no witness of form (4), shows that

Claim. Let $(J_k)_{k\in\omega^*}$ be such that for all $i \in \omega^*$, $J_i \in \mathcal{I}$ and J_i, J_{i-1} are consecutive intervals. Then $\bigcup_{k\in\omega^*} J_k \in \mathcal{I}$.

We are now in a position to prove the theorem. We use Zorn's lemma 30 on the inclusion ordering on $\mathcal{I}$ in our argument. We need to first show that every chain has an upper bound in $\mathcal{I}$. Let (I_k) be a chain where all the sets coincide on the left. We show that the limit $I = \bigcup_k I_k$ also belongs to $\mathcal{I}$. There are two cases to consider depending on whether the chain length is finite or not. If the chain length is finite then there exists a k such that $I_k = I$. Therefore $I \in \mathcal{I}$. We now look at the case when the chain length is not finite. We can assume that the chain is of ω length since the base linear order is countable and any chain can be covered by a countable chain. Consider the sets $J_k = I_k \backslash I_{k-1}$, for all $k > 1$ and $J_1 = I_1$. That is J_k consists of those elements in I_k which are not in any set before it. Clearly $\bigcup_k J_k = I$ and all $J_k \in \mathcal{I}$ since $\pi(w_{J_k}) \geq_{\mathcal{J}} \pi(w_{I_k}) \geq_{\mathcal{J}} a$. We can now apply Claim C.2 to get that $I \in \mathcal{I}$. A similar argument but using Claim C.2 gives us that ascending chains which coincide on the right also have an upper bound in $\mathcal{I}$. We are now left with showing the case when the ascending chain do not coincide either on the left or right. Any such sequence I_k can be split into two sequences I'_k and I''_k such that $I_k = I'_k \cup I''_K$ and the chain I'_k coincide on the right and the chain I''_k coincide on the left. Using Claim C.2 and the fact that I'_k, I''_k are consecutive intervals, we get $I \in \mathcal{I}$. Thus we have shown that any ascending chain has an upperbound in $\mathcal{I}$.

The set $\mathcal{I}$ satisfies the preconditions for Zorn's Lemma. Applying the strong form of Zorn's lemma gives us a set of maximal condensations. Let C be these maximal elements. There are 4 cases to consider: C contains one maximal element, C contains intersecting sets, C contains consecutive intervals, C is a dense linear order. We show that only the first condition is possible. First we note that if there is only one maximal element, that is $C = \{\alpha\}$. Then $\gamma(w) \geq_{\mathcal{J}} a$. The second condition is that there are sets $C_1, C_2 \in C$ such that $C_1 \cap C_2 \neq \emptyset$. Then $C_1 \cap C_2$ does not have one unique maximal element which is a contradiction of Zorn's lemma. So let us consider the case where there are two consecutive maximal elements C_1 and C_2. Due to Claim C.2 we have $C_1 \cup C_2 \in \mathcal{I}$ which is a contradiction. So we are left with the case when the maximal elements form a dense linear order. Observe that this is not possible in the scattered case. Consider the word $w = \prod_{I \in C} w_I$. From Shellah's theorem (Lemma 29) it follows that there exist a factor u of w which is a perfect shuffle over the domain $C' \subseteq C$. Let $J = \bigcup_{I \in C'} I$. Then $\gamma(w_J) \geq_{\mathcal{J}} a$ because witness condition (5) does not hold. Therefore $J \in \mathcal{I}$ which is a contradiction. So the only case possible is that there exists only one maximal element. Therefore $\gamma(w_\alpha) \geq_{\mathcal{J}} a$. $\square$

C.3 Nice concatenation witness

In the previous section, we introduced the notion of a concatenation witness. Here, we introduce *nice concatenation witness*, which is a special concatenation witness. Nice concatenation witnesses make it easier for logic to recognize the witness. Let us look at the properties.

Definition 2 (Nice concatenation witness).
We say that (u, t, v) is a nice witness of a word w, where $\gamma(w) \not\geq_{\mathcal{J}} a$ if

– $w \in \mathbf{M}^\circ utv \mathbf{M}^\circ$ *and* $\gamma(utv) \not\geq_{\mathcal{J}} a$
–
  - $u \in \mathbf{M}$ *or*
  - $u \in [) \cap \overrightarrow{e}$ *or*
  - $u \in (] \cap \overleftarrow{e}$ *and there is no strict suffix,* u' *of* u *such that* $\gamma(u'tv) \not\geq_{\mathcal{J}} a$
–
  - $v \in \mathbf{M}$ *or*
  - $v \in (] \cap \overleftarrow{e}$ *or*
  - $v \in [) \cap \overrightarrow{e}$ *and there is no strict prefix,* v' *of* v *such that* $\gamma(utv') \not\geq_{\mathcal{J}} a$
– $\gamma(ut), \gamma(tv) \geq_{\mathcal{J}} a$
– $\gamma(t) >_{\mathcal{J}} a$*, if* t *is not empty*

Later, we will find that the properties of the nice concatenation witness helps us in writing the logic formulas easier. But first we show that if a word contains a concatenation witness, then it always contains a nice concatenation witness

Lemma 31 (nice concatenation witness). *Let* $w \in \mathbf{M}^\circ$ *such that* w *has a concatenation witness. Then it has a nice concatenation witness. Moreover, for any concatenation witness, there exists a factor which is a nice concatenation witness.*

Proof. Let $p \cdot q$ be a concatenation witness of w. We give strings $u, t_1, t_2, v \in \mathbf{M}^\circ$ such that $(u, t_1 \cdot t_2, v)$ is a nice concatenation witness. First, we define the following: q' is the largest prefix of q such that any strict prefix of $p \cdot q'$ is $\geq_{\mathcal{J}} a$. That is

$$I = \bigcup \{i \mid \gamma(p \cdot q(-\infty, i]) \geq_{\mathcal{J}} a\}$$

and $q' = q_I$. Two cases can happen.
Case 1: $\gamma(p \cdot q') \not\geq_{\mathcal{J}} a$. Since all strict prefixes of pq' is $\geq_{\mathcal{J}} a$, we can rewrite q' as $t_2 \cdot v$, where $v \in \overrightarrow{e}$, for an $e \geq_{\mathcal{J}} a$. and $t_2 \in \mathbf{M}^\circ$.
Case 2: $\gamma(p \cdot q') \geq_{\mathcal{J}} a$. Then there exists a $v \in \mathbf{M}$ or $v \in \overleftarrow{e}$ for an $e \geq_{\mathcal{J}} a$, such that $q \in q' \cdot v \cdot \mathbf{M}^\circ$ and $\gamma(pq'v) \not\geq_{\mathcal{J}} a$. In this case take, $t_2 = q'$.
Note that in both the above cases we have $\gamma(pt_2) \geq_{\mathcal{J}} a$ and $\gamma(pt_2v) \not\geq_{\mathcal{J}} a$.

Now we define p' to be the largest suffix of p such that any strict suffix of $p't_2v$ is $\geq_{\mathcal{J}} a$. Again two cases arise.
Case 1: $\gamma(p't_2v) \not\geq_{\mathcal{J}} a$. Then $p' = u \cdot t_1$ such that $u \in \overleftarrow{e}$ for an $e \geq_{\mathcal{J}} a$ and $t_1 \in \mathbf{M}^\circ$.
Case 2: $\gamma(p't_2v) \geq_{\mathcal{J}} a$. Then there exists a $u \in \mathbf{M}$ and $u \in \overrightarrow{e}$ for an $e \geq_{\mathcal{J}} a$, such that $p \in \mathbf{M}^\circ \cdot u \cdot p'$. Let us assign $t_1 = p'$.
Note that $\gamma(t_1t_2v) \geq_{\mathcal{J}} a$ and $\gamma(ut_1t_2v) \not\geq_{\mathcal{J}} a$.

We now assign $t = t_1 \cdot t_2$ and let us assume t is not empty. We claim the following $\gamma(ut), \gamma(tv) \geq_{\mathcal{J}} a$ and $\gamma(t) >_{\mathcal{J}} a$. We first show that $\gamma(ut), \gamma(tv), \gamma(t) \geq_{\mathcal{J}} a$. This is because, $\gamma(ut_1t_2) \geq_{\mathcal{J}} \gamma(pt_2) \geq_{\mathcal{J}} a$. As shown above we also have that $\gamma(tv) \geq_{\mathcal{J}} a$ and therefore $\gamma(t) \geq_{\mathcal{J}} a$. It then follows from Lemma 6 that $\gamma(t) >_{\mathcal{J}} a$. □

When the monoid $\mathbf{M}$ satisfy certain properties, then the nice concatenation witness will satisfy more conditions. Let us consider the case when $\mathbf{M}$ satisfy the property `oi→gi` and `o*i→gi`. This extra conditions enable a first-order formula to detect a nice concatenation witness. Note that, the monoids are weaker than $\mathtt{i}{\to}\mathtt{gi}$. Yet, we will show that first-order logic is sufficient to detect the concatenation witness.

Lemma 32. *Let $w \in \mathbf{M}^\circ$ such that $\mathbf{M}$ satisfy $\mathtt{oi}{\to}\mathtt{gi}$ and $\mathtt{o}^*\mathtt{i}{\to}\mathtt{gi}$ and w contains a concatenation witness. Then there exist $\alpha, u, t, v, \beta \in \mathbf{M}^\circ$, such that (u, t, v) is a nice concatenation witness and the following additional properties hold.*

1. $w = \alpha u t v \beta$
2. $(\alpha, u), (u, t), (t, v), (v, \beta) \in \mathcal{D}_a(w)$
3. (a) *u is either a letter*
   (b) *or $u \in [)$ and $u \in \overrightarrow{e}$, for some $e \geq_{\mathcal{J}} a$*
   (c) *or $u \in \overleftarrow{e} \cap (]$, where $e >_{\mathcal{J}} a$ and $e^{\omega^*} \geq_{\mathcal{J}} a$.*
4. *v has same (but symmetric) properties.*
5. *If t is empty, $(u, v) \in \mathcal{D}_a(w)$.*

Proof. Let $w = \alpha u t v \beta$, where (u, t, v) is a nice concatenation witness, given by Lemma 31.

First we will show property (3) to hold. Let $u \in \overrightarrow{f}$, for some $f \geq_{\mathcal{J}} a$. Then from Lemma 28, it follows the existence of an e and an $u' \in [)$, such that $\gamma(u'tv) \ngeq_{\mathcal{J}} a \Leftrightarrow \gamma(utv) \ngeq_{\mathcal{J}} a$. Therefore, without loss of generality, we can assume $u = u'$. Now, let us assume $u \in \overleftarrow{e}$. Let us assume that $e^{\omega^*} \mathcal{J} e$ and $u \in \overleftarrow{e}$. Since $\mathbf{M}$ satisfy $\mathtt{o}^*\mathtt{i}{\to}\mathtt{gi}$, $e^\omega e^{\omega^*} = e$. Therefore $\gamma(e^{\omega^*} tv) \ngeq_{\mathcal{J}} a \Leftrightarrow \gamma(etv) \ngeq_{\mathcal{J}} a$. This violates the condition of the nice concatenation witness.

We now show property (2) holds. Let us assume $(u, t) \notin \mathcal{D}_a(w)$. That is, $u \in \overrightarrow{e}$ and $t \in \overleftarrow{e} \cdot t'$, for some idempotent $e \geq_{\mathcal{J}} a$ and some $t' \in \mathbf{M}^\circ$. Then $\gamma(ut) = \gamma(e^\omega e^{\omega^*} t') = \gamma(et') = \gamma(t)$. From the properties of concatenation witness, $\gamma(tv) \geq_{\mathcal{J}} a$ and therefore $\gamma(utv) = \gamma(tv) \geq_{\mathcal{J}} a$. This is a contradiction. Therefore $(u, t) \in \mathcal{D}_a(w)$. Similarly $(t, v) \in \mathcal{D}_a(w)$.

Let us now consider the cut (α, u). Since property (3) holds, it is clear if $u \in \mathbf{M}$ or $u \in \overrightarrow{e}$ or $u \in \overleftarrow{e}$ and $e >_{\mathcal{J}} e^{\omega^*}$, we have $(\alpha, u) \in \mathcal{D}_a(w)$. Therefore, let us assume $u \in \overleftarrow{(}e)$ and $e \geq_{\mathcal{J}} e^{\omega^*}$. Since $\mathbf{M}$ satisfy $\mathtt{o}^*\mathtt{i}{\to}\mathtt{gi}$, we have $e^\omega e^{\omega^*} = e$ and hence $\gamma(e^{\omega^*} tv) \ngeq_{\mathcal{J}} a \Leftrightarrow \gamma(etv) \ngeq_{\mathcal{J}} a$. This violates the condition of the nice concatenation witness.

Finally we show property (5) holds. Let t be empty. Therefore, there cannot be an idempotent $e \geq_{\mathcal{J}} a$, such that $u \in \overrightarrow{e}$ and $v \in \overleftarrow{e}$, since $\gamma(uv) \geq_{\mathcal{J}} a$ (because $e^\omega \cdot e^{\omega^*} = e \geq_{\mathcal{J}} a$). □

C.4 $\mathcal{R}$ class property

Here we try to understand, how a word $w \in \mathbf{M}^\circ$ looks like, if $\gamma(w) \mathcal{R} a$. Again, this property is used later to write formulas to detect the $\mathcal{R}$ class. Note that, the following lemma has a symmetric version (which looks at $\mathcal{L}$ classes).

Lemma 33. *Let $w \in \mathbf{M}^\circ$. Then $\gamma(w)\ \mathcal{R}\ a$ iff $\gamma(w) \mathcal{J} a$ and there exists $t, v \in \mathbf{M}^\circ$ such that they satisfy the following properties:*

1. $\gamma(tv) \leq_{\mathcal{R}} a$ *and if* $t \neq \epsilon$, $\gamma(t) >_{\mathcal{J}} a$
2. $v \in \mathbf{M}$ *or* $v \in \overleftarrow{e}$ *or* $v \in \overrightarrow{e}$ *for some* $e \in \mathbf{M}$.

Moreover $v \notin \overrightarrow{e}$ *if* $e^{\omega}e^{\omega^*} = e$.

Proof. Let w belong to the right hand side of the equation. Then clearly $\gamma(w)\mathcal{J}a$ and $\gamma(w) \leq_{\mathcal{R}} a$. Then by lemma 6 it follows that $\gamma(w)\ \mathcal{R}\ a$.

Now let us assume that $\gamma(w)\ \mathcal{R}\ a$. Then clearly $\gamma(w)\mathcal{J}a$. Let t' be the largest prefix of w such that any strict prefix of t' is $>_{\mathcal{J}} a$. Now two cases can happen. Case 1, $\gamma(t') >_{\mathcal{J}} a$: Then there exists a $v \in \mathbf{M}$ or $v \in \overleftarrow{e}$ for some $e \in \mathbf{M}$, such that $w \in t'v\mathbf{M}^{\circ}$ and $\gamma(t'v) \leq_{\mathcal{R}} a$. Take $t = t'$
Case 2, $\gamma(t') \leq_{\mathcal{J}} a$: Then we can split $t' = tv$, such that $t \in \mathbf{M}^{\circ}$ and $\gamma(t) >_{\mathcal{R}} a$ and $v \in \mathbf{M}$ or $v \in \overrightarrow{e}$ for some $e \in \mathbf{M}$.

Thus in both cases we have that $\gamma(t) >_{\mathcal{R}} a$ and $\gamma(tv) \leq_{\mathcal{R}} a$.

We are now left with proving the special case when $e^{\omega}e^{\omega^*} = e$. Then $\gamma(te^{\omega}) \leq_{\mathcal{J}} a \Leftrightarrow \gamma(te) \leq_{\mathcal{J}} a$, that is a strict subset of t' is not $>_{\mathcal{J}} a$. This is a contradiction. □

D First order logic and definable cuts

In this section, we introduce definable cuts.Informally, a definable cut is a cut which can be defined in first-order logic. To find the morphism of a factor t of a word w, both the left and right cuts of t has to be recognized. If the cuts are first-order recognizable, then we can relativize formulas to act only on that factor. Therefore, recognizing which cuts are first-order definable and which cuts are not, is crucial.

D.1 First order formulas

We state couple of properties first-order logic can verify.

Lemma 34. *Given a set* X*, there exists first-order formulas to check, each of these properties:*

- X *has an infimum*
- X *is a right limit sequence.*

D.2 Cuts and definable cuts

Given a word $w \in \mathbf{M}^{\circ}$, a cut is a pair of points $(u, v) \in \mathbf{M}^{\circ} \times \mathbf{M}^{\circ}$ such that $w = uv$. We define the set of all cuts of a word w as follows:

$$\mathcal{C}(w) = \{(u, v) \in \mathbf{M}^{\circ} \times \mathbf{M}^{\circ} \mid uv \text{ is a factor of } w\}$$

We will now see *definable cuts*, $\mathcal{D}(w)$. It is easier to introduce its complement.

$$\mathcal{D}'(w) = \{(u, v) \in \mathcal{C}(w) \mid u \in \mathbf{M}^{\circ}\overrightarrow{e} \text{ and } v \in \overleftarrow{e}\mathbf{M}^{\circ} \text{ st } e \in \mathtt{E}(M) \text{ and } e^{\omega}e^{\omega^*} = e\}$$

$$\mathcal{D}(w) = \mathcal{C}(w) \setminus \mathcal{D}'(w)$$

The following definable cuts are interesting: *unnatural cuts* of w are those cuts $(u, v) \in \mathcal{D}(w)$ such that u ends with a letter or v starts with a letter. That is:

$$\mathcal{U}(w) = \{(u, v) \in \mathcal{D}(w) \mid u \in \mathbf{M}^{\circ}\mathbf{M} \text{ or } v \in \mathbf{M}\mathbf{M}^{\circ}\}$$

All other cuts are called *natural cuts*. Let $a \in \mathbf{M}$.

$$\mathcal{D}_a(w) = \mathcal{U}(w) \cup \{(u, v) \in \mathcal{D}(w) \mid u \in \mathbf{M}^{\circ}\overrightarrow{e} \text{ or } v \in \overleftarrow{e}\mathbf{M}^{\circ} \text{ st } e \in \mathtt{E}(M), e >_{\mathcal{J}} a\}$$

We will soon see that, $\mathcal{D}_a(w)$ are first-order definable. But first, let us look at formulas which define cuts.

D.3 First order and definable cuts

Consider a formula $\alpha(y, x)$. We say that α defines a cut, if for all $w \in \mathbf{M}^{\circ}$, $w \models \alpha(b, i)$, then $w \models \alpha(b, j)$, for all $j > i$ where $b, i, j \in \mathit{dom}(w)$. That is α models all points ahead of the cut and does not model any point behind the cut. We say that α defines the cut $(u, v) \in \mathcal{D}(w)$, if there exists a $b \in \mathit{dom}(w)$ such that $w \models \alpha(b, i)$ for all $i > |u|$ and $w \not\models \alpha(b, j)$ for all $j \leq |u|$.

An example of a formula which defines a cut is: $y < x$, since once we assign a value to y, then x is true for all points to the right of y. For a word $w \in \mathbf{M}^{\circ}$, $y < x$ defines all cuts in $\{(u, v) \in \mathcal{D}(w) \mid u \in \mathbf{M}^{\circ}\mathbf{M}\}$. Similarly $y \leq x$ defines all cuts in $\{(u, v) \in \mathcal{D}(w) \mid v \in \mathbf{M}\mathbf{M}^{\circ}\}$.

Let $\mathrm{L}(y_1, x_1)$ and $\mathrm{R}(y_2, x_2)$ be formulas which defines cuts (u, w) and (w, v) respectively. Let β be an arbitrary formula. Then we can define a relativized formula $\beta[\geq \mathrm{L}(y_1, x_1), \leq \mathrm{R}(y_2, x_2)]$ such that

$$w \models \beta \Leftrightarrow uwv \models \beta[\geq \mathrm{L}(y_1, x_1), \leq \mathrm{R}(y_2, x_2)]$$

The next lemma, says that definable cuts are FO definable, provided we have formulas to detect the morphism for all $b >_{\mathcal{J}} a$.

Lemma 35. *Let for all $b >_{\mathcal{J}} a$, there be formulas $\mathtt{Product_b}$, such that for all $w \in \mathbf{M}^{\circ}$, $\gamma(w) = b \Leftrightarrow w \models \mathtt{Product_b}$. Then, there is a formula $\Lambda(x, y)$ (which uses additional existential quanifiers) such that*

$$(u, v) \in \mathcal{D}_a(w) \Leftrightarrow \Lambda \textit{ defines the cut } (u, v)$$

Proof. Let us assume $(u, v) \in \mathcal{U}(w)$. Earlier, we saw that formulas $(y < x)$ and $(y \leq x)$ defines all formulas in $\mathcal{U}(w)$. So, let us assume $(u, v) \in \mathcal{D}(w) \setminus \mathcal{U}(w)$. Let us, therefore assume $u \in \mathbf{M}^{\circ}\overrightarrow{e}$ and $v \in \overleftarrow{f}\mathbf{M}^{\circ}$, for idempotents $e, f \in \mathtt{E}(M)$ and such that $e^{\omega} f^{\omega^*} \notin \{e, f\}$. The following claim holds.

Claim. $e^{\omega} f^{\omega^*} <_{\mathcal{J}} e$ or $e^{\omega} f^{\omega^*} <_{\mathcal{J}} f$

We will first show the above claim. Let us assume the claim is false. Therefore $e^{\omega} f^{\omega^*} \mathcal{J} e \mathcal{J} f$. Lemma 7 says that, there exists a $g \in \mathbf{M}$ such that $f^{\omega^*} = e^{\omega^*} g$. Therefore $v \in \overleftarrow{e}\mathbf{M}^{\circ}$. Moreover $e^{\omega} e^{\omega^*} g \mathcal{J} e$ implies $e^{\omega} e^{\omega^*} \mathcal{J} e$, which by Lemma 14 implies $e^{\omega} e^{\omega^*} = e$. This is a contradiction, since $u \in \mathbf{M}^{\circ}\overrightarrow{e}$ and $v \in \overleftarrow{e}\mathbf{M}^{\circ}$ and $(u, v) \notin \mathcal{D}(w)$. This proves the claim.

Now let us assume that $e \geq_{\mathcal{J}} f$ (the case where $f \geq_{\mathcal{J}} e$ is symmetric). Therefore $e >_{\mathcal{J}} a$, since either e or f is $>_{\mathcal{J}} a$. From the claim, therefore, we have $e^{\omega} f^{\omega^*} <_{\mathcal{J}} e$. Using Lemma 28, without loss of generality, we can assume $u = \alpha \prod_{i \in \mathbb{N}} u_i$, $u_i \in []$ and $\gamma(u_i) = e$ for all $i \in \mathbb{N}$. Now consider the following. Let y point to a $u_i[0]$ for some $i \in I$ and x point to somewhere in v. The following properties hold.

Claim. 1. For every $y' > y$ such that $\gamma(w[y, y']) = e$, there exists a $y'' > y'$ such that $\gamma(w[y'', y]) = e$.
2. $\gamma(w[y, x]) <_{\mathcal{J}} e$.

The second claim holds because $e^{\omega} f^{\omega^*} <_{\mathcal{J}} e$. For the first claim, we can always choose y'' to point to $u_i[0]$ for some particular $i \in \mathbb{N}$. Therefore $\gamma(w[y'', y]) = e$, since y and y'' point to $u_i[0], u_j[0]$ for some $j > i$ respectively.

Using the above claims, we can write our formula as follows.

$$\begin{aligned} \Lambda(y,x) := & \bigwedge_{b \geq_{\mathcal{J}} e} \neg \, (\texttt{Product}_b(\geq y, \leq x)) \, \bigwedge \\ & \exists y' \, \Big((y < y' < x) \wedge \, \texttt{Product}_e(\geq y, \leq y')\Big) \, \bigwedge \\ & \forall y' > y \, \Big(\texttt{Product}_e(\geq y, \leq y') \implies (\exists y'' > y' \, \texttt{Product}_e(\geq y, \leq y''))\Big) \end{aligned}$$

The formula $\Lambda(y, x)$ defines all cuts in $(u, v) \in \mathcal{D}(w) \setminus \mathcal{U}(w)$. □

E The translation from monoids to logic

Consider a $\circ$-monoid $\mathbf{M}$ and language $L \subseteq \mathbf{M}^{\circ}$ recognized by $\mathbf{M}$ using the morphism $\gamma : \mathbf{M}^{\circ} \to \mathbf{M}$. In this section, we prove the following direction of Theorem 2.

- If $\mathbf{M}$ satisfies `i`$\to$`gi`, `sc`$\to$`sh` and `sh`$\to$`ss`, then L is definable in FO
- If $\mathbf{M}$ satisfies aperiodic, `sc`$\to$`sh` and `sh`$\to$`ss`, then L is definable in FO[cut]
- If $\mathbf{M}$ satisfies `oi`$\to$`gi`, `o`*`i`$\to$`gi`, `sc`$\to$`sh` and `sh`$\to$`ss`, then L is definable in WMSO
- If $\mathbf{M}$ satisfies `sc`$\to$`sh` and `sh`$\to$`ss`, then L is definable in MSO[finite,cut]
- If $\mathbf{M}$ satisfies `sh`$\to$`ss`, then L is definable in MSO[scattered]

The next subsection introduces the main proof technique. We will first give the general framework of the proof. Then we go through the different cases.

E.1 Main proof technique and induction hypothesis

Our proof is by inducting on the $\mathcal{J}$-class ordering. In this section, we will prove all the claims required in the general framework. The next sections consider the property satisfied by the monoid $\mathbf{M}$ and deal appropriately.

All the above logics can simulate existential quantifiers. Therefore we are free to use existential quantifiers regardless of the logic we consider. Let $a \in \mathbf{M}$ be an element in the monoid. Then, our *induction hypothesis* is as follows:

For all $b \geq_{\mathcal{J}} a$, there is a formula $\texttt{Product}_b$ such that, for all $w \in \mathbf{M}^{\circ}$

$$w \models \texttt{Product}_b \Leftrightarrow \gamma(w) = b$$

So, let us assume that the hypothesis is true for all $b >_{\mathcal{J}} a$. We need to give the formula $\texttt{Product}_a$ such that $w \models \texttt{Product}_a \Leftrightarrow \gamma(w) = a$. To prove this, we go through the following sequence of steps.

1. Identify all words w which are $\gamma(w) \not\geq_{\mathcal{J}} a$
2. Identify all words w which are $\gamma(w)\mathcal{J}a$
3. Identify all words w which are $\gamma(w)\mathcal{R}a$ (Similarly all w which are $\gamma(w)\mathcal{L}a$)
4. Identify all words w which are $\gamma(w)\mathcal{H}a$
5. Identify all words w which are equivalent to a

The next few subsections solves each of these sub problems.

E.2 Identifying words which are $\not\geq_{\mathcal{J}} a$

Recall the definition of witness and definable cuts. In this subsection, we identify all words $w \in \mathbf{M}^{\circ}$, which satisfy the condition $\gamma(w) \not\geq_{\mathcal{J}} a$, that is we give a formula $\texttt{Product}_{\not\geq_{\mathcal{J}} a}$, which will detect a witness. Theorem 4 states that if $\gamma(w) \not\geq_{\mathcal{J}} a$, then there exists one of these witnesses in w: letter witness, concatenation witness, omega witness, omega* witness or shuffle witness. We write formulas to detect each of these witnesses in different subsections. Note that depending on the type of $\mathcal{J}$-class we are looking at, the logic differs.

Before, we go into detecting witnesses, let us describe the main idea we use in the proof. The witnesses (except for letter witness) are factors which appear consecutively. We might have to check for finite number of consecutive factors, or omega, or scattered or even dense. In all these cases though, our aim is to find the morphism of these factors. Usually these morphisms are got from our assumption of induction hypothesis, because the factor will $>_{\mathcal{J}} a$ (sometimes the morphism will be $\mathcal{J}\ a$ and then, we may have to use other techniques). The first step before we can check for morphism of a factor, is to identify the cuts on the left and right side of the factor. If these cuts are not definable in the logic we are working with, then the factor cannot be separated. Therefore only those factors whose left and right cuts are definable in the logic we are working with matters. This is why we have introduced the notion of definable cuts earlier. Our first step, in the different cases of the proof, will be to show that the cuts are definable (usually in first-order logic). Once these cuts are identified, we check for properties of the factors.

Since we will be looking at $\mathcal{J}$-classes, we need to introduce some special types of $\mathcal{J}$-classes. Let J be a $\mathcal{J}$-class. We say that J is ordinal regular, if J contains an ordinal idempotent. Similarly, we say J is ordinal* regular, if J contains an ordinal* idempotent. We say that a J class is scattered regular if J contains a scattered idempotent and finally we say that J is shuffle regular, if J contains a shuffle idempotent.

Letter witnesses

Lemma 36. *There exists a formula* `letterWitness` *such that for all* $w \in \mathbf{M}^{\circ} c \mathbf{M}^{\circ}$ *where* $c \not\geq_{\mathcal{J}} a$*, we have* $w \models$ `letterWitness`.

Proof. The formula `letterWitness` can be written as $\exists x \ \bigvee_{c \not\geq_{\mathcal{J}} a} c(x)$. □

Concatenation witnesses To detect concatenation witness, we use the property of nice concatenation witness as given by Lemma 31. Depending on the property of monoid $\mathbf{M}$, we give formulas in different type of logic to detect the witness. We also assume that the word does not have a letter witness. This is fine, since if there is a letter witness, we would have detected it.

The general idea behind detecting the concatenation witness is as follows: A nice concatenation witness is a tuple (u, t, v) such that the product of them will be a witness. We show that the factors u, t and v can be separately identified, because the cuts are first-order definable. Then we find the morphisms of each of these factors using the induction hypothesis. In certain cases, $\gamma(u) \mathcal{J} a$ (similarly v also). In this case, we cannot use induction hypothesis. Therefore after assuming $u \mathcal{J} a$, we need to use Lemma 51 to detect the morphism. A detailed explanation of the procedure follows, for different logics.

Concatenation witness for FO and WMSO: Let $\mathbf{M}$ satisfy either `i`$\rightarrow$`gi` or (`oi`$\rightarrow$`gi` and `o`*`i`$\rightarrow$`gi`). We show that, in either of this case, first-order logic is enough to detect concatenation witness. Recall that, by assumption, we have formulas $\texttt{Product}_b$ for all $b >_{\mathcal{J}} a$, such that $w \models \texttt{Product}_b \Leftrightarrow \gamma(w) = b$. Also recall that `oi`$\rightarrow$`gi` and `o`*`i`$\rightarrow$`gi` implies that for all $e \mathcal{J} e^{\omega}$ or $e \mathcal{J} e^{\omega^*}$, implies $e^{\omega} e^{\omega^*} = e$

We use Lemma 32, to write an FO formula. The Lemma says that there is a nice concatenation witness (u, t, v) which satisfy certain properties. Moreover the left and right cuts of each of u, t and v are first-order definable and hence these factors can be isolated.

Lemma 37. *There exists a FO formula* `concatWitness` *such that both the following claims hold.*

1. *if* w *contains a concatenation witness then* $w \models$ `concatWitness`
2. *if* $w \models$ `concatWitness` *then* $\gamma(w) \not\geq_{\mathcal{J}} a$

Proof. Let w be a word which contains a concatenation witness. By Lemma 32, there exists, $\alpha, u, t, v, \beta \in \mathbf{M}^\circ$, such that (u, t, v) is a nice concatenation witness and $(\alpha, u), (u, t), (t, v), (v, \beta) \in \mathcal{D}_a(w)$. Our aim is to give a formula which can detect these factors and check whether they satisfy the properties of being a nice concatenation witness.

Since $(\alpha, u), (u, t), (t, v), (v, \beta) \in \mathcal{D}_a(w)$, there exists formulas $\mathrm{L}(y_0, x_0), \mathcal{F}_1(y_1, x_1), \mathcal{F}_2(y_2, x_2)$ and $\mathrm{R}(y_3, x_3)$, which defines these cuts respectively. Now we give formulas ($\mathtt{tVal}_b, \mathtt{uVal}_b, vVal_b$, for all $b \geq_{\mathcal{J}} a$) to compute the product of each of the factors t, u and v.

First, we detect t. Lemma 32 shows that $\gamma(t) >_{\mathcal{J}} a$. Therefore, by induction hypothesis, there exists formulas, $\mathtt{Product}_b$, such that $t \models \mathtt{Product}_b \Leftrightarrow \gamma(t) = b$. The formulas $\mathtt{Product}_b$s can be relativized by formulas $\mathcal{F}_1$ and $\mathcal{F}_2$ to check whether there exists a factor of w whose morphism is b. We defined the relativized formula, for all $b >_{\mathcal{J}} a$ as:

$$\mathtt{tVal}_b(y_1, y_2) := \ \mathtt{Product}_b(\geq \mathcal{F}_1(y_1, x_1), \leq \mathcal{F}_2(y_2, x_2))$$

We will now show how to detect u. If $u \in A$, then we define $\mathtt{uVal}_b$, for some $b \geq_{\mathcal{J}} a$ as follows.

$$\mathtt{uVal}_b(y_0) := \bigvee_{\substack{c \in \mathbf{M} \\ \gamma(c) = b \geq_{\mathcal{J}} a}} c(y_0)$$

If $u \in \overrightarrow{e}$, then there are two cases to consider: $\gamma(u) >_{\mathcal{J}} a$ and $\gamma(u) \mathcal{J} a$. First, the former condition. By induction hypothesis, there exists formulas $\mathtt{Product}_b$, for all $b \in \mathbf{M}$, such that $u \models \mathtt{Product}_b \Leftrightarrow \gamma(u) = b$. Since $u \in [)$, there exists a formula, $\mathrm{L}(y_0, x_0)$ to detect the left cut of u. The right cut of u is detected by the formula $\mathcal{F}_1$. The relativized formula, for all $b >_{\mathcal{J}} a$ is as follows:

$$\mathtt{uVal}_b(y_0, y_1) := \ \mathtt{Product}_b(\geq \mathrm{L}(y_0, x_0), \leq \mathcal{F}_1(y_1, x_1))$$

The second case is when $\gamma(u) \mathcal{J} a$. Since $e^\omega \mathcal{L} f^\omega$, for all $f^\omega \mathcal{J} e^\omega$, $\gamma(e^\omega tv) <_{\mathcal{J}} a \Leftrightarrow \gamma(f^\omega tv) <_{\mathcal{J}} a$. Hence to detect u, we need to check the following two conditions.

- u is a right limit sequence

$$\mathtt{rightSequence} := \neg \left(\exists z \ \forall x > z \ \mathcal{F}_1(y_1, x) \right)$$

- u is not equivalent to $\overrightarrow{e}$ for any $e >_{\mathcal{J}} a$. The following formula uses the fact: $e \mathcal{J} e^\omega$.

$$\mathtt{notJgreaterThan}_a := \neg \bigwedge_{b >_{\mathcal{J}} a} \mathtt{Product}_b(\geq \mathrm{L}(y_0, x_0), \leq \mathcal{F}_1(y_1, x_1))$$

Thus for all $b \mathcal{J} a$, we have the following formula.

$$\mathtt{uVal}_b(y_0, y_1) := \mathtt{rightSequence} \wedge \mathtt{notJgreaterThan}_a$$

Note the following claim.

Claim. $b \mathcal{J} a$, $u \models \mathtt{uVal}_b \Leftrightarrow \gamma(u) \leq_{\mathcal{J}} a$ and $u \in [)$.

If $u \in \overleftarrow{e}$, we know that $e >_{\mathcal{J}} a$. Therefore, we relativize the formula $\texttt{Product}_b$, for all $b >_{\mathcal{J}} a$ as:

$$\texttt{uVal}_b(y_0, y_1) := \texttt{Product}_b(\geq \mathrm{L}(y_0, x_0), \leq \mathcal{F}_1(y_1, x_1))$$

Finally we show how to detect v. This is symmetric to the case above and hence we know there exists formulas $vVal_b(y_2, y_3)$, for all $b \geq_{\mathcal{J}} a$.

The formula $\texttt{concatWitness}$ is got by combining all three formulas.

$$\bigvee_{\substack{e,f \geq_{\mathcal{J}} a,\ e,f \in \mathrm{E}(M) \\ b >_{\mathcal{J}} a, e^{\omega} b f^{\omega^*} <_{\mathcal{J}} a}} \exists y_0, y_1, y_2, y_3 \ \left(\texttt{uVal}_e(y_0, y_1) \wedge \texttt{tVal}_b(y_1, y_2) \wedge vVal_f(y_2, y_3)\right)$$

We now show that the formula satisfies the two conditions mentioned in Lemma. (1) : Let w contains a concatenation witness. Then, the formula $\texttt{concatWitness}$ detects correctly the witness (u, t, v).
(2) : Let $w \models \texttt{concatWitness}$. Then, there exists $e, b, f \in \mathbf{M}$, such that the formula identifies witness (u, t, v), where $\gamma(t) = b >_{\mathcal{J}} a$ and $u \models \texttt{uVal}_e$ and $v \models vVal_f$. Consider the following cases for u (symmetrically for v). If $e >_{\mathcal{J}} a$, then $\gamma(u) = e$. In the case, when $e \mathcal{J} a$, $\gamma(u) \leq_{\mathcal{J}} a$. If $\gamma(u) \mathcal{J} a$, then $\gamma(u) \mathcal{L} e^{\omega} \mathcal{J} a$. In both these cases $\gamma(utv) <_{\mathcal{J}} a \Leftrightarrow e^{\omega} b f^{\omega^*} <_{\mathcal{J}} a$. If $\gamma(u) <_{\mathcal{J}} a$, then $\gamma(w) <_{\mathcal{J}} a$ and therefore w has a witness. This proves the claim. □

Concatenation witness for FO[cut] We assume that the $\mathcal{J}$-classes satisfy the property oo*r->sc. That is, if there exists an idempotent e, such that $e^{\omega} = e = e^{\omega^*}$, then $e^{\eta} = e$. We will see that an FO[cut] formula can detect all concatenation witnesses.

Lemma 38. *There exists a FO[cut] formula* $\texttt{concatWitness}$ *such that both the following claims hold.*

1. *if w contains a concatenation witness then* $w \models \texttt{concatWitness}$
2. *if* $w \models \texttt{concatWitness}$ *then* $\gamma(w) \not\geq_{\mathcal{J}} a$

Proof. Let the concatenation witness be (u, t, v) and let $w = \alpha u t v \beta$. There exists FO[cut] formulas, $\mathcal{F}_0(y_0, x_0), \mathcal{F}_1(y_1, x_1), \mathcal{F}_2(y_2, x_2), \mathcal{F}_3(y_3, x_3)$ such that the cuts $(\alpha, u), (u, t), (t, v), (v, \beta)$ are defined by formulas $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ respectively. We now give formulas $\texttt{tVal}_b$, for all $b >_{\mathcal{J}} a$, and $\texttt{uVal}_b, vVal_b$ for all $b \geq_{\mathcal{J}} a$, which compute the morphism of the factors t, u, v respectively.

The formula $\texttt{tVal}_b$, as in Lemma 37 is given by relativizing the formula $\texttt{Product}_b$, got by inductive hypothesis.

$$\texttt{tVal}_b := \texttt{Product}_b[\geq \mathcal{F}_1, \leq \mathcal{F}_2]$$

We will see, how to give formulas $\texttt{uVal}_b$, for all $b \geq_{\mathcal{J}} a$. Following the arguments presented above, it is clear how to write the formulas, for all $b >_{\mathcal{J}} a$. Now let us assume $b \mathcal{J} a$. Lemma 51 gives formulas Γ_b which identifies correctly $\gamma(w)$ provided $\gamma(w) \mathcal{J} a$. Finally we get $\texttt{uVal}_b$ after relativizing Γ_b with $\mathcal{F}_1$ and $\mathcal{F}_2$.

Note that, it might happen that $u \models \texttt{uVal}_b$, for a $b\mathcal{J}a$ but $\gamma(u) <_{\mathcal{J}} a$. Therefore the claim holds.

A similar argument gives us formulas $vVal_b$ for all $b \geq_{\mathcal{J}} a$.

Finally we combine the formulas as in Lemma 37. We now show that the two conditions of the lemma holds.
(1) : If w contains a concatenation witness, it is clear $w \models \texttt{concatWitness}$.
(2) : Following the argument in Lemma 37 if $w \models \texttt{concatWitness}$ and $\gamma(u) \geq_{\mathcal{J}} a$ and $\gamma(v) \geq_{\mathcal{J}} a$, then $w \models \texttt{concatWitness} \Rightarrow w$ has a concatenation witness. On the other hand, if $w \models concatWitness$ and $\Gamma(u) <_{\mathcal{J}} a$ or $\Gamma(v) <_{\mathcal{J}} a$, then $\Gamma(w) \not\geq_{\mathcal{J}} a$. □

Omega witnesses (*resp.* omega* witnesses) Again, the first idea is to show that the factors are first-order definable. Then we use a mix of induction hypothesis and Lemma 51 to detect the witness. The main ideas in the proof are similar to the concatenation witness case above. We deal separately for each of the different type of monoid.

Omega witnesses for FO In this case $\texttt{i}{\rightarrow}\texttt{gi}$ or e is idempotent implies $e\mathcal{J}e^{\omega}$. Therefore, there is no omega witness for FO.

Omega witnesses for WMSO We assume that monoid $\mathbf{M}$ satisfy the property $\texttt{oi}{\rightarrow}\texttt{gi}$ and $\texttt{o}^*\texttt{i}{\rightarrow}\texttt{gi}$. That is, if $e\mathcal{J}e^{\omega}$ (or $e\mathcal{J}e^{\omega^*}$), then $e^{\omega}e^{\omega^*} = e$.

Let $t \in \overrightarrow{e}$ be an omega witness of w. Our goal is to show that the left and right cuts of t are first-order definable. Moreover we need to show that all the cuts between factors, which make e are also first-order definable. Once the cuts are identified, then we can check whether a factor is e or not by induction hypothesis (when $e >_{\mathcal{J}} a$). When $e\mathcal{J}a$, we use a slightly different technique. There we make use of the fact that e^{ω} is an omega witness if and only if e^{ω^*} is an omega* witness. Therefore, we need not separately identify e^{ω}, but check for infinite occurences of e. A weak MSO formula will be able to test this.

First, let us show that the factors can all be identified, because the left and right cuts of them are first-order definable.

Claim. If w contains an omega witness, then it contains a factor $u = \prod_{i\in\mathbb{N}} u_i$, such that $u_i \in []$ and $\gamma(u_i) = e$, for all $i \in \mathbb{N}$ and $e \geq_{\mathcal{J}} a$.

Proof. From Lemma 28 it follows that there exists an $f\mathcal{J}e$, with the desired properties. □

So henceforth, our omega witness, without loss of generality, will satisfy the properties of the above claim.

We will first see the case, when $e >_{\mathcal{J}} a$ and $e^{\omega} \not\geq_{\mathcal{J}} a$. Let $w = \alpha u \beta$, where $u \in \overrightarrow{e}$. We show that the cuts $(\alpha, u\beta), (\alpha u, \beta)$ are first-order definable. Then it is easy to check whether $u \in \overrightarrow{e}$.

Claim. $(\alpha, u\beta), (\alpha u, \beta) \in \mathcal{D}_a(w)$

Proof. The claim above, says that without loss of generality $u \in [)$. So let us assume $(\alpha u, \beta) \notin \mathcal{D}_a(w)$. Then $\beta \in \overleftarrow{e}\mathbf{M}^\circ$ and $e = \gamma(u)e^{\omega^*} = e^\omega e^{\omega^*}$. This is a contradiction, since $e^\omega \not\geq_{\mathcal{J}} a$ and $e >_{\mathcal{J}} a$. □

Lemma 39. *Let $e >_{\mathcal{J}} a$, such that $e^\omega \not\geq_{\mathcal{J}} a$. Then, there exists a formula* $\texttt{OmegaWitness}_\texttt{e}$ *such that:*

$$w \models \texttt{OmegaWitness}_\texttt{e} \quad \textit{if and only if} \quad w \textit{ contains an omega witness}$$

Proof. From the claim above, it follows that we need to check for a factor u whose left and right cuts are first-order definable. Then we need to see whether $u \in \overrightarrow{e}$. This is possible using the assumption that we have formulas $\texttt{Product}_b$ for all $b >_{\mathcal{J}} a$. □

Now we will consider the case, when $e\mathcal{J}a$ and $e^\omega \not\geq_{\mathcal{J}} a$. Note that, since $\mathbf{M}$ satisfy the properties oi→gi and o*i→gi, $e^\omega \not\geq_{\mathcal{J}} a \Leftrightarrow e^{\omega^*} \not\geq_{\mathcal{J}} a$. Therefore, any factor which is made up of an infinite sequence of e is a witness. Our first step is therefore to construct a set $X = \{x_0, x_1, \dots\}$, such that

1. $\gamma(w[x_i, x_{i+1}]) = e$, for all $i \geq 0$
2. for all j, where $x_i < j < x_{i+1}$, $\gamma(w[x_i, j]) \neq e$.

It now follows.

Claim. X is infinite iff there is an ω or omega* witness.

This is now a WMSO testable property.

Lemma 40. *Let $e\mathcal{J}a$, such that $e^\omega \not\geq_{\mathcal{J}} a$. Then, there exists a formula* $\texttt{OmegaWitness}_e$ *such that:*

1. *If w contains a factor from $\overrightarrow{e}$ or $\overleftarrow{e}$, then $w \models \texttt{OmegaWitness}_e$*
2. *$w \models \texttt{OmegaWitness}_e \implies \gamma(w) \not\geq_{\mathcal{J}} a$*

Proof. We say that there exists two points x, y (we are guessing the factor $w[x, y]$ which is the witness), such that there does not exist any finite set X with the 2 properties mentioned above. To check whether the morphism of a factor u_i is (or not) e, we

1. Check $\gamma(u_i) \not>_{\mathcal{J}} a$. This is verifiable because of our inductive hypothesis.
2. Then assuming $\gamma(u_i)\mathcal{J}a$, use Lemma 51 to compute the morphism correctly.

Thus, if w contains an ω or ω^* witness, then the formula models the word.
If on the other hand $w \models \texttt{OmegaWitness}_e$, then it means, there is an infinite set of points $x_0 < x_1 < \dots$ such that $\gamma(w[x_i, x_{i+1}]) \leq_{\mathcal{J}} a$ and the formula identifies (correctly or incorrectly) the morphisms of $w[x_i, x_{i+1}]$ to be e. Two cases can arise.
Case $\forall i, \gamma(w[x_i, x_{i+1}])\mathcal{J}a$: In this case Lemma 51 correctly identifies there are infinite set of morphisms mapping to e and hence there is a witness.
Case $\exists i, \gamma(w[x_i, x_{i+1}]) <_{\mathcal{J}} a$. Then, clearly $\gamma(w) \not\geq_{\mathcal{J}} a$ as required by the lemma. □

This completes the proof.

Omega witness for FO[cut]: We consider that the monoid $\mathbf{M}$ satisfy the property o*i→gi.

Lemma 41. *There is an FO[cut] formula* `OmegaWitness`*, such that*

1. *if w contains an omega witness then $w \models$* `OmegaWitness`
2. *$w \models$ `OmegaWitness` $\implies \gamma(w) \not\geq_{\mathcal{J}} a$.*

Proof. Let w be a word. If w has an omega witness, then there are two cuts x, y, such that $w[x, y] = \prod_{i \in \mathbb{N}+} u_i$, where $\gamma(u_i) = e$, for an $e \geq_{\mathcal{J}} a$. Two cases arise.
Case $e >_{\mathcal{J}} a$: By inductive hypothesis, we have formulas to check whether $\gamma(u_i) = e$. It is now easy to check whether $u \in \overrightarrow{e}$.
Case $e\mathcal{J}a$: In this case, we use inductive hypothesis to test that $\gamma(u_i) \leq_{\mathcal{J}} a$ and then assume that $\gamma(u_i)\mathcal{J}a$ and use Lemma 51 to compute the morphism correctly. If our assumption is correct always, then our formula will be correct. If our assumption is wrong, then for some i, $\gamma(u_i) <_{\mathcal{J}} a$ and therefore $\gamma(w) <_{\mathcal{J}} a$. ☐

Shuffle witnesses We assume in this section that the word does not have a concatenation witness or a letter witness. To detect shuffle witness, we need to check whether there is a dense ordering of condensed words. Again, the first step is to detect the left and right cuts of words which form the shuffle. This is possible whenever the factors are $>_{\mathcal{J}} a$. When the factor is $\mathcal{J}a$, it is not always possible to detect the cut. In this case, because all monoids satisfy the sh→ss property, we will be able to detect the witness, without explicity checking for a dense set.

Shuffle witnesses for FO and WMSO: Here we show that all shuffle witnesses for WMSO are detectable by a first-order formula (using induction hypothesis).

If there exists an $f \in \mathtt{E}(M)$ such that $f^\omega = f\mathcal{J}a$, then there exists an $e\mathcal{J}a$, where $e^\eta = e$. We leave the symmetric case of looking at the case when $f^{\omega^*} = f\mathcal{J}a$. In this section, we also assume that $\mathcal{J}(a)$ satisfy sc→sh and sh→ss also.

Now, let us look at shuffle witness. Let $t = \prod_{i \in \mathbb{Q}} u_i$ be the shuffle witness, such that $\gamma(u_i) \geq_{\mathcal{J}} a$ for all i and $\gamma(t) \not\geq_{\mathcal{J}} a$. We now separate the different cases which can occur.

1. $\mathcal{J}(a)$ is not ordinal regular.
2. $\gamma(u_i) >_{\mathcal{J}} a$, for all $i \in \mathbb{Q}$
3. $\exists\, j \in \mathbb{Q},\ (\gamma(u_j) \geq_{\mathcal{J}} a)$

Case 1 *and* 2*:* For all $j \in \mathbb{Q}$, consider the words $l_j = \prod_{i<j} u_i$ and $l'_j = \prod_{i>j} u_i$. We claim

Lemma 42. $(l_j, u_j), (u_j, l'_j) \in \mathcal{D}_a(w)$*, for all $j \in \mathbb{Q}$*

Proof. We will show that $(l_j, u_j) \in \mathcal{D}_a(w)$, for all $j \in \mathbb{Q}$. A similar (but symmetric) proof exists for showing $(u_j, l'_j) \in \mathcal{D}_a(w)$. Since $l_j = \prod_{i\in\mathbb{Q}} u_i$ by Lemma 11, we get $l_j \in \overrightarrow{e}$, for an $e \ngeq_{\mathcal{J}} a$. Now consider the two cases.

$\mathcal{J}(a)$ is regular and $\gamma(u_j) >_{\mathcal{J}} a$ for all $j \in \mathbb{Q}$: Since $\gamma(u_j) >_{\mathcal{J}} a$ and $l_j \in \overrightarrow{e}$, it follows $(l_j, u_j) \in \mathcal{D}_a(w)$.

$\mathcal{J}(a)$ is not ordinal regular: Therefore, $b^{\omega^*} \ngeq_{\mathcal{J}} a$, for all $b\mathcal{J}a$, since $b^{\omega^*} = f^{\omega^*}$, for some idempotent f and $f \notin \mathcal{J}(a)$. Therefore $u_j \notin \overleftarrow{b}\,\mathbf{M}^\circ$, for a $b\mathcal{J}a$. It follows, either $u \in \overleftarrow{b}\,\mathbf{M}^\circ$ for a $b >_{\mathcal{J}} a$ or u_j starts with a letter. Since $l_j \in \overrightarrow{e}$, $(l_j, u_j) \in \mathcal{D}_a(w)$. □

The above lemma means that, cuts before and after each of the subwords which form the shuffle witness are first-order definable. Using this observation, we can detect given two points x, y whether they point to different u_js or not.

Claim. There exists a formula $\texttt{diffuj}(x, y)$ such that

$$w \models \texttt{diffuj}(p, q) \Leftrightarrow p, q \text{ point to } u_i, u_j \text{ where } i \neq j$$

Proof. The formula has to state that between the two points there exists a $\overrightarrow{f}$ sequence, for an $f \leq_{\mathcal{J}} a$. This can be done in first-order logic. □

We next give formulas $\texttt{valOfuj}_b(x, y)$, for all $b \geq_{\mathcal{J}} a$, such that

Claim. There are formulas $\texttt{valOfuj}_b(x, y)$, for all $b \geq_{\mathcal{J}} a$, such that for two points p, q in the same u_j,

1. $(\gamma(u_j) = b) \implies w \models \texttt{valOfuj}_\texttt{b}(p, q)$
2. $w \models \texttt{valOfuj}_\texttt{b}(p, q) \implies (\gamma(u_j) = b)$ or $(\gamma(u_j) \ngeq_{\mathcal{J}} a)$

Proof. Note that the left and right cuts of u_j are first-order definable. When $b >_{\mathcal{J}} a$, we can use induction hypothesis to prove the claim. When $b\mathcal{J}a$, we assume that $\gamma(u_j)\mathcal{J}a$ and identify $\gamma(u_j)$ using Lemma 51. □

We are now in a position to write the formula $niceShuffle$ as follows

Lemma 43. *There exists a formula $niceShuffle$ such that*

1. *w contains a shuffle witness of Case* 1 *or* 2*, then* $w \models \texttt{niceShuffle}$.
2. *if* $w \models \texttt{niceShuffle}$ *then* $\gamma(w) \ngeq_{\mathcal{J}} a$

Proof. Let $S \subseteq \mathbf{M}$ such that for all $c \in S, c >_{\mathcal{J}} a$ and $S^\eta \ngeq_{\mathcal{J}} a$. We will first give a formula, $\texttt{checkuj}_c(p, q)$ which checks whether there is an u_j in between p, q such that $\gamma(u_j) = c$.

$$\exists x_c, y_c \left((p < x_c < y_c < q) \wedge \neg\texttt{diffuj}(x_c, y_c) \wedge \texttt{valOfuj}_c(x_c, y_c)\right)$$

We also have a formula, $\texttt{checkNotuj}_S(x, y)$ which verifies that there does not exist an u_j such that $\gamma(u_j) = c$, for a $c \notin S$.

$$\forall p, q \left(\left((x < p < q < y) \wedge \neg\texttt{diffuj}(p, q)\right) \implies \bigwedge_{c\notin S} \neg\texttt{valOfuj}_c(p, q)\right)$$

Finally, we give a formula $\texttt{checkWitness}_S(x, y)$ which does the following.

1. All u_is between x and y are such that $\gamma(u_i) \in S$.
2. We have to also check that the letters appear dense. That is, for all points p, q between x and y, where p and q are from different u_is, for each $c \in S$, there exists x_c, y_c pointing to the same u_j and such that $\gamma(u_j) = c$.

$$\texttt{checkNotuj}_S(x,y) \wedge \Big(\forall p, q \ \big((x < p < q < y) \wedge \texttt{diffuj}(p,q)\big) \implies (\bigwedge_{c \in S} \texttt{checkuj}_c(p,q))\Big)$$

The final formula is a disjunction over all formulas $\texttt{checkWitness}_S$, where S satisfies the required conditions.

$$\exists\, x, y \ \Big(\bigvee_{\substack{S \subseteq M, S^\eta \not\geq_{\mathcal{J}} a \\ \forall c \in S, c >_{\mathcal{J}} a}} \texttt{checkWitness}_S(x,y)\Big)$$

$\square$

Case 3*:* In this case, there exists an $e \in \mathcal{J}(a)$, such that $e^\eta = e$ and in the shuffle witness $t = \prod_{i \in \mathbb{Q}} u_i$, there exists u_js such that $\gamma(e \cdot u_j) \not\geq_{\mathcal{J}} a$ or $\gamma(u_j \cdot e) \not\geq_{\mathcal{J}} a$. Let us assume the existence of u_j such that $\gamma(e \cdot u_j) \not\geq_{\mathcal{J}} a$. A symmetric argument exists when $\gamma(u_j \cdot e) \not\geq_{\mathcal{J}} a$. We first show that a first-order logic formula can

1. Find the morphism of u_j.
2. Detect an $\overrightarrow{f}$ word to the left of u_j, for an $f \leq_{\mathcal{J}} a$
3. Check that the product of $e\gamma(u_j) \not\geq_{\mathcal{J}} a$.

We first show that the left cut of u_j is first-order definable. Consider the words, $l_j = \prod_{i<j} u_i$. We claim

Claim. $(l_j, u_j) \in \mathcal{D}_a(w)$.

Proof. Clearly, the claim holds, if $u_j \in \mathbf{M}\mathbf{M}^\circ$. If $u_j \in \overleftarrow{f}\,\mathbf{M}^\circ$, for an $f >_{\mathcal{J}} a$, then the claim holds, since $l_j \in \overrightarrow{g}$ for a $g \not>_{\mathcal{J}} a$.

So let us assume that $u_j \in \overleftarrow{f}\,\mathbf{M}^\circ$, for an $f \mathcal{J} a$. Since $f^{\omega^*} \mathcal{L} e^{\omega^*}$, $u_j \in \overleftarrow{e}\,\mathbf{M}^\circ$. Let $u_j \in \overleftarrow{e}\,u'_j$ for an $u'_j \in \mathbf{M}^\circ$. From our assumption $e \cdot \gamma(u_j) \not\geq_{\mathcal{J}} a$ and $\gamma(u_j) \geq_{\mathcal{J}} a$. Since $e^\eta = e$, $e^{\omega^*} = e$, and therefore $e \cdot \gamma(u_j) = e \cdot e^{\omega^*} \gamma(u'_j) = ee\gamma(u'_j) = e\gamma(u'_j) = e^{\omega^*}\gamma(u'_j) = \gamma(u_j)$. A contradiction and hence $u_j \notin \overleftarrow{f}\,\mathbf{M}^\circ$ for an $f \mathcal{J} a$. $\square$

The right cut of u_j may not be first-order definable. We therefore show that, if it is not definable, we can find a big enough prefix v_j of u_j, such that $e\gamma(v_j) \not\geq_{\mathcal{J}} a$.

Claim. One of the following holds

1. $(u_j, \prod_{i>j} u_i) \in \mathcal{D}_a(w)$
2. There exists a prefix v_j of u_j, such that $v_j \in (] \cup []$ and $e \cdot \gamma(v_j) \not\geq_{\mathcal{J}} a$.

Proof. Let $u_j \in \mathbf{M}^\circ \overrightarrow{f}$, for some $f >_{\mathcal{J}} a$ or $u_j \in \mathbf{M}^\circ\mathbf{M}$. Then clearly $(u_j, \prod_{i>j} u_i) \in \mathcal{D}_a(w)$. So, let us assume $u_j \in \mathbf{M}^\circ \overrightarrow{f}$, for some $f \mathcal{J} a$. Since $f^\omega f^{\omega^*} = f$, we have $f^\omega \mathcal{R} f$ and therefore, we can find a prefix v_j of u_j, such that $e \cdot \gamma(v_j) \not\geq_{\mathcal{J}} a$. $\square$

The above claims show that there is a word v_j, whose left and right cuts are first-order definable and $e \cdot v_j$ is a witness. Note that v_j can be u_j or a prefix, depending on the case.

We need to now guess the cuts of v_j, find the morphism of v_j and check that $\overrightarrow{f}$, for an $f \leq_{\mathcal{J}} a$, appears on the left of v_j. First, let us guess v_j and find its morphism.

Claim. For all $b \geq_{\mathcal{J}} a$, there are formulas $valOfvj_b(x, y)$, such that for two points p, q in v_j,

1. $(\gamma(v_j) = b) \implies w \models valOfvj_b(p, q)$
2. $w \models valOfvj_b(p, q) \implies \gamma(v_j) \leq_{\mathcal{J}} a$

Proof. If $b >_{\mathcal{J}} a$, we guess the cuts and use induction hypothesis to get the required formula.
If $b\mathcal{J}a$, we assume that $\gamma(v_j)\mathcal{J}a$ and find out the morphism (by Lemma 51). By the arguments used in previous proofs, the claim holds. □

We are now in a position to write the formula `nNiceShuffle`(x, y) as follows.

Lemma 44. *There exists a formula* `nNiceShuffle`(x, y) *such that*

1. *w contains a shuffle witness of Case* 3, *then* $w \models$ `nNiceShuffle`(x, y)
2. *if* $w \models$ `nNiceShuffle`(x, y) *then* $\gamma(w) \ngeq_{\mathcal{J}} a$

Proof. The formula guess the left and right cuts of v_j and use the formula $valOfvj_b$, from the previous claim to compute the morphism of v_j. Now we need to check whether there is a right limit sequence $\overrightarrow{f}$, for an $f \leq_{\mathcal{J}} a$, to the left of v_j. It is easy to write a formula `OmegaSequence` which does this. Finally we can mix these two formulas as follows.

$$\bigvee_{\substack{b \geq_{\mathcal{J}} a \\ eb <_{\mathcal{J}} a}} \exists x, y \ (OmegaSequence(x) \wedge valOfvj_b(x, y))$$

We follow the same arguments we saw earlier, for the correctness of the formula. □

Witness for FO[cut] Earlier, we saw that first-order logic can identify shuffle witnesses in either of the following $\mathcal{J}$-classes: shuffle regular or not ordinal regular and ordinal* regular. Here we show that FO[cut] formulas can detect shuffle witnesses for $\mathcal{J}$-classes which are ordinal regular or ordinal* regular or both but not scattered regular.

Let $t = \prod_{i \in \mathbb{Q}} u_i$ be a shuffle witness of w. $\gamma(u_j) >_{\mathcal{J}} a$ for all j, is symmetric to Case 2 of FO. Let $l_j = \prod_{i<j} u_i$ and $l'_j = \prod_{i>j} u_i$. We have now the following three cases:

1. $(l_j, u_j), (u_j, l'_j) \in \mathcal{D}_a(w)$ for all $j \in \mathbb{Q}$

2. $\exists k \in \mathbb{Q}\ u_k \in \overleftarrow{f}\mathbf{M}^\circ$ for an $f\mathcal{J}a$
3. $\exists k \in \mathbb{Q}\ u_k \in \mathbf{M}^\circ\overrightarrow{f}$ for an $f\mathcal{J}a$

The first case is similar to case 1 of FO. Since case 2 and 3 are symmetric, we will only consider case 2. We show that to identify this witness, we only have to check for a cut where an ω sequence and ω^* sequence meet.

Claim. For all $e\mathcal{J}f\mathcal{J}a$, $e^\omega f^{\omega^*} \not\geq_{\mathcal{J}} a$.

Proof. Assume not. Then there exists an $e\mathcal{J}a$, such that $e^\omega e^{\omega^*} = e$. Therefore (Lemma **??**), $\mathcal{J}(a)$ is scattered regular, which is a contradiction. □

It is easy to identify such a witness.

Lemma 45. *There is an FO[cut] formula* `shuffleWitness`*, such that*

1. *if w has a shuffle witness then $w \models$* `shuffleWitness`
2. *if $w \models$* `shuffleWitness` *then $\gamma(w) \not\geq_{\mathcal{J}} a$*

Proof. To check for case 2, we guess a cut and get formulas to see that sequences on the left and right side of the cuts are $\overrightarrow{e}$ and $\overleftarrow{f}$ for some $e, f\mathcal{J}a$. The formulas can only detect if the sequences are $\leq_{\mathcal{J}} a$. □

Witness for MSO[scattered] If there exists an element $e \in \mathcal{J}(a)$ which is shuffle regular, then e is also shuffle simple, since monoid $\mathbf{M}$ satify `sh`$\rightarrow$`ss`. This case has been considered by case 3 of first-order logic. Hence, let us look at the case when $\mathcal{J}(a)$ is not shuffle regular. Let $t = \prod_{i\in\mathbb{Q}} u_i$ be the shuffle witness. We show that the left and right cuts of u_i, for all $i \in \mathbb{Q}$, can be identified in MSO[scattered]. Then, we need to check whether, there is a perfect shuffle (that is, whether there exists a dense subset of elements). Let us first show that the left and right cuts of u_i are identifiable. Let $l_j = \prod_{i<j} u_i$ and $l'_j = \prod_{i>j} u_i$.

Claim. The cuts (l_j, u_j) and (u_j, l'_j) are definable in MSO[scattered].

Proof. We will show that (l_j, u_j) is definable in MSO[scattered] (the other case is symmetric). If $(l_j, u_j) \in \mathcal{D}_a(w)$, then it is clearly definable in MSO[scattered]. Therefore, let us assume $(l_j, u_j) \notin \mathcal{D}_a(w)$. We show that there is a scattered set to the right of the cut and there is no scattered set to the left of the cut, which satisfy certain properties.

First let us show the former. Assume u_j is of the form vw, where $v \in \overleftarrow{f}$, for an $f\mathcal{J}a$ and $w \in \mathbf{M}^\circ$. Condense all dense linear orderings in v to a single element. Since $\mathcal{J}(a)$ is not shuffle regular, this condensation would not give an element in $\mathcal{J}(a)$. Let S be the set of all cuts after all condensation has been done. Therefore, for all points $i, j \in S$, $\gamma(w[i,j]) >_{\mathcal{J}} a$. S also satisfy the property that, there is a left limit sequence towards the left cut of u_j.

Now for the second condition. It is easy to observe that there does not exist a scattered set S to the left of the cut, such that for all consecutive points $i, j \in S$, $\gamma(w[i,j]) >_{\mathcal{J}} a$ and there is a right limit sequence towards the cut (l_j, u_j).

Therefore, there exists MSO[scattered] formulas to detect the cuts. □

Once the cuts are identified, we need to see whether the u_js form a dense set. This can be done as in case 1 and 2 of FO.

Lemma 46. *There exists a formula* `shuffleWitness` *such that*

1. *w contains a shuffle witness* $\implies w \models \texttt{shuffleWitness}$
2. $w \models \texttt{shuffleWitness} \implies \gamma(w) \ngeq_{\mathcal{J}} a$.

E.3 Identifying words which are $\mathcal{J}$ equivalent to a

In the previous subsection we saw that, depending on the type of monoid, we have formulas in $\mathcal{L}$ to identify words, $w \in \mathbf{M}^\circ$ where $\gamma(w) \ngeq_{\mathcal{J}} a$. That is, there exists a formula $\texttt{Product}_{\ngeq_{\mathcal{J}} a} \in \mathcal{L}$, such that $w \models \texttt{Product}_{\ngeq_{\mathcal{J}} a} \Leftrightarrow \gamma(w) \ngeq_{\mathcal{J}} a$. Using this, we built the formula $\texttt{Product}_{\mathcal{J}(a)}$.

Lemma 47. *There exists formula* $\texttt{Product}_{\mathcal{J}(a)}$*, such that for all words* $w \in \mathbf{M}^\circ$*,*

$$\gamma(w)\mathcal{J}a \Leftrightarrow w \models \texttt{Product}_{\mathcal{J}(a)}$$

Proof. Clearly

$$\{w \in \mathbf{M}^\circ \mid \gamma(w)\mathcal{J}a\} = \{w \in \mathbf{M}^\circ \mid \gamma(w) \geq_{\mathcal{J}} a\} \cap \bigcap_{b >_{\mathcal{J}} a} \{w \in \mathbf{M}^\circ \mid \gamma(w) \not>_{\mathcal{J}} b\}$$

We define $\texttt{Product}_{\mathcal{J}(a)}$ as

$$\neg\ \texttt{Product}_{\ngeq_{\mathcal{J}} a} \ \wedge \bigwedge_{b >_{\mathcal{J}} a} \neg\ \texttt{Product}_b$$

It is clear that this formula will satisfy all words, w such that $\gamma(w)\mathcal{J}a$. □

E.4 Identifying words which are $\mathcal{R}$ (similarly $\mathcal{L}$) equivalent to a

Here, we show how to detect the $\mathcal{R}$ classes correctly, if we know the $\mathcal{J}$-class. A symmetric argument helps us detect the $\mathcal{L}$ class.

Lemma 48. *There exists formula* $\texttt{Product}_{\mathcal{R}(a)}$*, such that for all* $w \in \mathbf{M}^\circ$*, where* $\gamma(w)\mathcal{J}a$*,*

$$\gamma(w)\ \mathcal{R}\ a \Leftrightarrow w \models \texttt{Product}_{\mathcal{R}(a)}$$

Proof. The proof, depends on the property of the monoid. To detect $\mathcal{R}$ class, we need to use Lemma 33. The Lemma gives us a pair of consecutive factors of w, which needs to satisfy certain properties. These properties can be verified by following the proof of concatenation witness. □

E.5 Identifying words which are $\mathcal{H}$ equivalent to a

Here we identify all words which are $\mathcal{H}$ to a.

Lemma 49. *There exists a formula* $\texttt{Product}_{\mathcal{H}(a)}$*, such that for all words* w*, where* $\gamma(w)\mathcal{H}a$*, we have*

$$w \models \texttt{Product}_{\mathcal{H}(a)} \Leftrightarrow \gamma(w)\mathcal{H}a$$

Proof. Using Lemma 48, we can write the formula, $\texttt{Product}_{\mathcal{H}(a)}$ as:

$$\texttt{Product}_{\mathcal{L}(a)} \wedge \texttt{Product}_{\mathcal{R}(a)}$$

This accepts all words $w \in \mathbf{M}^\circ$ such that $\gamma(w)\ \mathcal{H}\ a$. □

E.6 Identifying words which are equivalent to a

Here we give a formula which detects all words, w such that $\gamma(w) = a$, provided $\gamma(w)\mathcal{J}a$. We use the fact that: $\mathcal{H}(a)$ is singleton if and only if $\mathcal{J}(a)$ is aperiodic. Therefore, for all aperiodic $\mathcal{J}(a)$ classes, identifying $\mathcal{H}$ class of the word w, is enough to detect the morphism, $\gamma(w)$. The non-trivial part is when $\mathcal{J}(a)$ is non-aperiodic. The next subsection will look into this.

Detecting a, when $\mathcal{H}(a)$ is not singleton When $\mathcal{J}(a)$ is not aperiodic, we have $\mathcal{H}(a)$ is not singleton. Let $w \in \mathbf{M}^\circ$, such that $\gamma(w)\mathcal{J}a$. Since $\mathcal{J}(a)$ is non-aperiodic, we know that there are no ω, ω^*, η operations on elements from $\mathcal{J}(a)$.

For the word w, consider X, a set of cuts, with the following properties.

1. For all $i, j \in X$, $\gamma(w[i,j])\mathcal{J}a$
2. For all consecutive $i, j \in X$, for all k such that $i < k < j$, either $\gamma(w[i,k])\not\mathcal{J}a$ or $\gamma(w[k,j])\not\mathcal{J}a$.
3. For consecutive $i, j \in X$, the sets $\{k < j \mid \gamma(w[i,k])\mathcal{J}a\}$ and $\{k > i \mid \gamma(w[k,j])\mathcal{J}a\}$ are either empty or infinite.

We claim that the set X will be finite and the properties of X can be checked in WMSO.

Claim. $\gamma(w)\mathcal{J}a \Rightarrow$ there exists a finite set X with the above properties.

Proof. First let us assume there exists a set X with the above properties. Then X cannot be infinite because, then there is either an ω or ω^* sequence of elements from $\mathcal{J}(a)$. This is not possible.

So, we need to now prove that there exists such a set. We construct the set X as follows. First, we add the left and right cuts of w to X. Consider any consecutive cuts $i, j \in X$. If there exists a cut k in between i, j such that property (2) is violated, then we add k also into X. Repeat this procedure. Since $\mathcal{J}(a)$ is non-aperiodic, we cannot continue doing this infinite number of times. □

Claim. There exists a set X with the above properties, which is definable in WMSO

Proof. Let X be a set with the above properties, but not definable in WMSO. Let (u_1, u_2) be a cut in X, such that $(u_1, u_2) \notin \mathcal{D}_a(w)$. Then $u_1 \in \mathbf{M}^\circ \overrightarrow{e}$ and $u_2 \in \overleftarrow{e}\mathbf{M}^\circ$ such that $e^\omega e^{\omega^*} = e$. Choosing a point i in either $\overrightarrow{e}$ or $\overleftarrow{e}$, rather than the cut (u_1, u_2) will preserve all the properties of X and also make the cut first-order definable. For all cuts in X which are not first-order definable, we can do the above procedure.

It is easy to see that the rest of the properties are WMSO expressible. □

Let i, j be consecutive elements in X. We need to now find $\gamma(w[i, j])$.

Claim. $w[i, j]$ is in one of the following forms.

1. For all $r, s \in \mathbf{M}^\circ$ such that $w[i, j] = r \cdot s$, we have $\gamma(r) >_{\mathcal{J}} a$ and $\gamma(s) >_{\mathcal{J}} a$
2. There exists $r, e \in \mathbf{M}^\circ$ such that $w[i, j] = r \cdot e^\omega$, $\gamma(r) >_{\mathcal{J}} a$ and $\gamma(e) >_{\mathcal{J}} a$
3. There exists $r, e \in \mathbf{M}^\circ$ such that $w[i, j] = e^{\omega^*} r$, $\gamma(r) >_{\mathcal{J}} a$ and $\gamma(e) >_{\mathcal{J}} a$

Proof. Assume $w[i, j]$ is not of form (1). Then there exists a k such that either $\gamma(w[i, k])\mathcal{J}a$ or $\gamma(w[k, j])\mathcal{J}a$ and not both. Let us assume $\gamma(w[i, k])\mathcal{J}a$. We will assign $r = w(k, j]$ and we know from the construction of set X that $\gamma(w(k, j]) >_{\mathcal{J}} a$. Now from property (3) of set X, we have that $\{l \mid \gamma(w[i, l])\mathcal{J}a\}$ is infinite. Therefore, $w[i, k] = e^{\omega^*}$ for some $e >_{\mathcal{J}} a$. This is of the form (3). □

Finally we make use of the above claims to write a formula to detect the morphism correctly.

Lemma 50. *There exist a formula Γ_a such that for all w, where $\gamma(w)\mathcal{J}a$*

$$w \models \Gamma_a \Leftrightarrow \gamma(w) = a$$

Proof. The formula, guesses a finite set X and verify all the properties specified in the claims above. We will have to use induction hypothesis to verify whether a factor u is such that $\gamma(u) >_{\mathcal{J}} a$.

We now need to verify that the product $\gamma(\prod_{i \in X} w[i-1, i]) = a$. This is a standard technique. For each element $g \in \mathcal{J}(a)$, we have a set $Y_g \subseteq X$, such that $j \in Y_g$ if and only if $\gamma(w[-\infty, j)) = g$. It is easy to verify the consistency between the sets, by observing that for two consecutive cuts $i, j \in X$, if $i \in Y_h$ and $j \in Y_g$ for $h, g \in \mathcal{J}(a)$, then $g = h\gamma(w[i, j))$. Finally $\gamma(w) = a$ if and only if $|w| \in Y_a$. □

Final formula to find the morphism, $\boldsymbol{\gamma(w)}$ We are now in a position to combine all the Lemmas we have seen.

Lemma 51. *There exists formula Γ_a, such that for all words w, where $\gamma(w)\mathcal{J}a$, we have*

$$w \models \Gamma_a \Leftrightarrow \gamma(w) = a$$

Proof. When $\mathbf{M}$ is an aperiodic monoid, there is no other element $\mathcal{H}$ equivalent to a and therefore Γ_a is equivalent to $\texttt{Product}_{\mathcal{H}(a)}$. When monoid is non-aperioidic, we use Lemma 50 to get the required formula. □

Finally we give the formula $\texttt{Product}_a$.

Lemma 52. *There exists a formula* $\texttt{Product}_a$ *such that* $\forall w \in \mathbf{M}^\circ$

$$\gamma(w) = a \Leftrightarrow w \models \texttt{Product}_a$$

Proof. Using Lemma 51 and Lemma 47 we get the formula, $\texttt{Product}_a$:

$$\Gamma_a \wedge \texttt{Product}_{\mathcal{J}(a)}$$

□