A Duality between Pawlak's Knowledge Representation Systems and BI-Consequence Systems
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A Duality Between Pawlak's Knowledge Representation Systems and BI-Consequence Systems

Abstract. A duality between Pawlak's knowledge representation systems and certain information systems of logical type, called bi-consequence systems is established. As an application a first-order characterization of some informational relations is given and a completeness theorem for the corresponding modal logic INF is proved. It is shown that INF possesses finite model property and hence is decidable.

Introduction

This work is a continuation of the research line of the papers [22], [23], [25], [26], [27], [28], concerning some problems of logical foundation of knowledge representation, initiated by Orlowska and Pawlak in [11] and [12]. The main topic in the above series of papers is a study of some concrete informational relations between objects in Pawlak's knowledge representation systems like indiscernibility, different kinds of similarities and informational orderings. Such informational relations are used as a semantic base of some modal logics, aimed to provide a formal account for reasoning about objects in knowledge representation systems. The main difficulty here is connected with the axiomatization of the corresponding modal logics. The essence is the following. There are standard methods in Modal Logic for axiomatizing modal operations interpreted in classes of relational systems, characterized by some abstract first-order conditions. The informational relations however are concretely defined relations between objects in knowledge representation systems, using the specific form of the represented information. So, one main step towards the problem of modal completeness theorems is to find and abstract characterization of the considered informational relations by some first-order conditions. Let us mention that there is no a unique universal method for this. In the above mentioned papers the reader can find several examples of such abstract characterizations, based on common ideas. The main aim of this paper is to present a new method, which is based on a duality-like connection systems between Pawlak's knowledge representation systems and some information system of logical type, called bi-consequence

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systems.

The information in Pawlak's knowledge representation systems is represented in terms of objects, attributes (like "color") and values of attributes (like "green") and consists of listing for each object x and attribute a the set $f(x, a)$ of all values of a possessed by x. I will call these systems attribute systems (A-systems for short). Several variations of A-systems under the name of information systems have been introduced by Pawlak in [13], [14]. In Orlowska and Pawlak [11], [12] they are studied under the name of knowledge representation systems, The notion of A-system is one of the fundamentals of the so called Rough Sets methodology in AI ([15], [20]). Since the notions of object, attribute and value of attribute have an ontological nature, we may say that A-systems are knowledge representation systems of ontological type.

The information in knowledge representation systems of logical type is represented by collections of sentences, equipped with some deductive mechanisms for yielding a new information from an old one. As a logical counterpart of the notion of A-system we introduce the notion of bi-consequence system (B-system for short) as an abstract system in the form $(Sen, \vdash, \succ\!\!-)$, where Sen is a nonempty set, whose elements are called sentences and $\vdash$ and $\succ\!\!-$ are two relations between finite set of sentences, called respectively strong and weak consequence relations and satisfying some axioms like structural rules in Gentzen systems. The first who introduced abstract syustems of a similar nature was Scott [17]. Other examples are given in [28]. One of the main results of the paper is a representation theorem for B-systems in A-systems, which generalizes the well-known Stone representation theorem for distributive lattices [21]. In this way we establish a duality-like connection between B-systems and A-systems, similar to the duality between Boolean algebras and set. This representation theorem extends a result from [28], where a representation theorem for simpler information systems of logical type was proven.

The representation theorem of B-systems in A-systems is applied to obtain a first-order characterization for some informational relations in A-systems: weak and strong versions of informational ordering, and positive and negative similarities. As a consequence a complete modal logic for these relations is introduced.

1. Attribute and bi-consequence systems

Attribute systems

By an attribute system, A-system for short, we mean any system of the form $S = (Ob, At, \{Val(a)/a \in At\}, f)$, where:

- $Ob \neq \emptyset$ is a set, whose elements are called objects,
- At is a set, whose elements are called attributes,
- for each $a \in At, Val(a)$ is a set, whose elements are called values of the attribute a,
- f is a two-argument total function, called information function, which assigns to each object $x \in Ob$ and attribute $a \in At$ a subset $f(x, a) \subseteq Val(a)$, called the information of x according to a.

The components of a given A-system S will be written with subscript S: $Ob_S, At_S, Val_S(a)$ and f_S.

An example of attribute is $a=$ "official language". Then the set of values of a is $Val(a) = \{$*English, German, French, Russian*$\}$. If x is a person who knows only English and German then the information of x according to a is $f(x, a) = \{$*English, German*$\}$. It is possible however for x to speak neither of the four official languages, then $f(x, a) = \emptyset$ and this is a very definite information for x.

Now we shall give a set-theoretical construction of A-systems. Since attributes can be considered as sets of properties and the set-theoretical analog of a property is a set of objects, then the set-theoretical analog of an attribute is a set of sets of objects. This leads to the following construction.

Let (W, V) be a pair with $W \neq \emptyset$ and $V \subseteq \mathbf{P}(\mathbf{P}(W))$, i. e. the elements of V are sets of subsets of W. Define an A-system $S = A(W, V)$ as follows: put $Ob_S = W, At_S = V$, for each $a \in At_S$ define $Val_S(a) = a$, and for each $x \in Ob_S$ and $a \in At_S$ put $f_S(x, a) = \{A \in a/x \in A\}$. The system $S = A(W, V)$ will be called the set-theoretical A-system over the pair (W, V).

Bi-consequence systems

First we shall introduce the notion of a Consequence system (C-system) (see [28]) which is more intuitive. This is an abstract system of the form $S = (Sen, \vdash)$ where $Sen \neq \emptyset$ is a set whose elements are called sentences and $\vdash$ is a binary relation between finite subsets $P_{fin}(Sen)$ of Sen, called Scott consequence relation, satisfying the following axioms coming from classical logic.

- (Refl) If $A \cap B \neq \emptyset$ then $A \vdash B$,
- (Mono) If $A \vdash B, A \subseteq A'$ and $B \subseteq B'$ then $A' \vdash B'$,
- (Cut) If $A \vdash B \cup \{x\}$ and $\{x\} \cup A \vdash B$ then $A \vdash B$.

S is called non-trivial C-system if $\emptyset \not\vdash \emptyset$, otherwise S is called trivial.

If S is a C-system then the components of S sometimes will be denoted by Sen_S and $\vdash_S$.

Following the usual practice, instead of $\{a_1, \ldots, a_n\} \vdash \{b_1, \ldots, b_m\}$ and $A \cup \{x_1, \ldots, x_m\} \vdash B \cup \{y_1, \ldots, y_n\}$ we will write $a_1, \ldots, a_n \vdash b_1, \ldots, b_m$ and $A, x_1, \ldots, x_m, \vdash B, y_1, \ldots, y_n$. We assume that for $n = 0$ we have $\{x_1, \ldots, x_n\} = \emptyset$.

For a set $A \in \mathbf{P}_{fin}(Sen_S)$ and $x, y \in Sen_S$ we say that:

- x implies y in S iff $x \vdash_S y$,
- A is inconsistent (contradictory) in S iff $A \vdash_S \emptyset$,
- A is consistent in S iff $A \not\vdash_S \emptyset$,
- A is complete (tautological) in S iff $\emptyset \vdash A$,
- A is incomplete in S iff $\emptyset \not\vdash_S A$.

The axioms (Refl), (Mono) and (Cut) are known under the names of Reflexivity, Monotonicity and Cut.

Typical example of a C-system is a logical theory L, based on the classical logic. In such an example the elements of Sen are real sentences and the relation $a_1, \ldots, a_n \vdash b_1, \ldots, b_m$ holds if the implication $(a_1, \wedge \ldots \wedge a_n) \Rightarrow (b_1 \vee \ldots \vee b_m)$ is true in L.

Let S be a C-system. A subset $x \subseteq Sen_S$ is called a prime filter in S if the following condition is satisfied

$$(\forall A, B \in P_{fin}(Sen))(A \vdash_S B \& \ A \subseteq x \rightarrow B \cap x \neq \emptyset)$$

The set of all prime filters of S will be denoted by $PrFil(S)$. The following lemma is proved in [28].

LEMMA 1. 1. (Separation lemma for C-systems) *Let S be a C-system and for $A, B \in P_{fin}(Sen_S)$ we have $A \not\vdash_S B$. Then there exists a prime filter x of S such that $A \subseteq x$ and $B \cap x = \emptyset$.*

Now we can give the definition of bi-consequence system.

By a *bi-consequence system*, B-*system* for short, we will mean a system S of the following form $S = (Sen, \vdash, \succ\!\!-)$, where:

- $(Sen, \vdash)$ is a C-system and $\vdash$ is now called a strong consequence relation

- $\succ$ is a binary relation in the set $P_{fin}(Sen)$, called weak consequence relation and satisfying the following axioms for any $A, B, A', B' \in P_{fin}(Sen)$:

(Refl $\succ$)	If $A \cap B \neq \emptyset$ then $A \succ$,
(Mono $\succ$)	If $A \succ B, A \subseteq A'$ and $B \subseteq B'$ then $A' \succ B'$,
(Cut $\succ$1)	If $A \vdash x, B$ and $A, x \succ B$ then $A \succ B$,
(Cut $\succ$2)	If $A, x \vdash B$ and $A \succ x, B$ then $A \succ B$,
(Incl)	If $A \vdash B$ then $A \succ B$.

Let us note that the axiom (Incl) is equivalent on the base of the remaining axioms to the following more simple axiom

(Incl 0) If $\emptyset \vdash \emptyset$ then $\emptyset \succ \emptyset$.

The following example of a B-system will give the main intuition of this notion. Let $S_i = (Sen, \vdash_i)$ $i \in I$ be a non-empty set of C-systems with one and the same set of sentences Sen. Define the relations $\vdash$ and $\succ$ in $\mathbf{P}_{fin}(Sen)$ as follows:

$$A \vdash B \text{ iff } \forall i \in I \ A \vdash_i B, A \succ B \text{ iff } \exists i \in I A \succ_i B.$$

Then it is easy to see that the system $(Sen, \vdash, \succ)$ is a B-system. It is now clear why $\vdash$ and $\succ$ are called strong and weak consequence relations respectively.

LEMMA 1. 2. *Let S be a B-system. Then the following Cut condition is true in S for any $X, Y, A, B \in P_{fin}(Sen_S)$*

($Cut \succ$) If $X \succ_S Y$ and for any $x \in X$ and $y \in Y$ we have $A \vdash_S x, B$ and $A, y \vdash_S B$ then $A \succ_S B$.

PROOF. Let $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$. Then prove by induction that for any i and j, $0 \leq i \leq m$, $0 \leq j \leq m$ the following is true: $x_1, \ldots, x_i, A \succ_S y, \ldots, y_j, B$. Then the assertion is obtained from $i = j = 0$. ∎

The next example of a B-system is connected with A-systems and will be of a great importance.

Let S be an A-system. We shall construct a system $B(S)$, called the B-system over S, in the following way. Put $Sen_{B(S)} = Ob_S$ and for any finite sets $A = \{x_1, \ldots, x_m\}$ and $B = \{y_1, \ldots, y_n\}$ of Ob_S define

$A \vdash_{B(S)} B$ iff $(\forall a \in At_S)(f_S(x_1, a) \cap \ldots \cap f_S(x_m, a) \subseteq f_S(y_1, a) \cup \ldots \cup f_S(y_n, a))$,

$A \succ_{B(S)} B$ iff $(\exists a \in At_S)(f_S(x_1, a) \cap \ldots \cap f_S(x_m, a) \subseteq f_S(y_1, a) \cup \ldots \cup f_S(y_n, a))$.

LEMMA 1. 3.

(i) *Let S be an A-system. Then the system $B(S)$ defined as above is a B-system.*

(ii) *Let $S = A(W, V)$ be set-theoretical A-system over the pair (W, V) and $A, B \subseteq \mathbf{P}_{fin}(W)$. Then:*

$$A \vdash_{B(S)} B \text{ iff } (\forall a \in V)(\forall C \in a)(A \subseteq C \rightarrow B \cap C \neq \emptyset).$$
$$A \succ_{B(S)} B \text{ iff } (\exists a \in V)(\forall C \in a)(A \subseteq C \rightarrow B \cap C \neq \emptyset).$$

PROOF – straightforward, following the relevant definitions. ■

Lemma 1. 3. shows that in some abstract sense A-systems constitute a correct "semantics" for B-systems. The next theorem shows that each B-system can be represented as a B-system over some A-system.

THEOREM 1. 4. (Representation theorem for B-systems)
Let S be a B-system. Then there exists an A-system $S' = A(S)$ such that $S = B(A(S))$.

PROOF. The system S', which we are going to define will be a set theoretical A-system $A(W, V)$ over a pair (W, V). Since we want that the equality $S = B(A(W, V))$ to hold, we have to put $W = Sen_S$. It remains to show how to define the set V (recall that $V \subseteq \mathbf{P}(\mathbf{P}(W))$). We will do this by finding a characteristic property of the elements of V, which will separate it from the set $\mathbf{P}(\mathbf{P}(W))$. Suppose for the moment that the set V is defined. Then by the equality $S = B(A(W, V))$ and lemma 4. 2. (ii) we will have

$$A \vdash_S B \text{ iff } (\forall a \in V)(\forall C \in a)(A \subseteq C \rightarrow B \cap C \neq \emptyset),$$
$$A \succ_S B \text{ iff } (\exists a \in V)(\forall C \in a)(A \subseteq C \rightarrow B \cap C \neq \emptyset).$$

If a is an arbitrary element of V, then it will satisfy the following two conditions for any two sets $A, B \in \mathbf{P}_{fin}(W)$:

(*) $A \vdash_S B \rightarrow (\forall C \in a)(A \subseteq C \rightarrow B \cap C \neq \emptyset)$,

(**) $A \succ_S B \rightarrow (\exists C \in a)(A \subseteq C \text{and } B \cap C = \emptyset)$.

The required characteristic property for a will be to satisfy (*) and (**).

So we define: a set of subsets of Sen_S is called a good set in S if it satisfies (*) and (**). Note that condition (*) says that the elements of a good set are prime filters in $(Sen_S, \vdash_S)$.

Now we can start the proof of the theorem.

Put $W = Sen_S$ and V be the set of all good sets in S. Since we have $V \subseteq P(P(W))$ put S' to be the set-theoretical A-system $A(W, V)$ over the pair (W, V) and denote S' by $A(S)$. $A(S)$ will be called the canonical A-system over S.

To prove the theorem we need some lemmas.

LEMMA 1. 5. *Let $A, B, X, Y \in \mathbf{P}_{fin}(Sen_S)$. Then:*

(i) *If $X \not\vdash_S Y$, then there exists $C \in PrFil(S)$, denoted by $C(X \not\vdash_S Y)$, such that $X \subseteq C$ and $Y \cap C = \emptyset$.*

(ii) *If $A \not\succ_S B$ then there exists $C \in PrFil(S)$, denoted by $C(A \not\succ_S B)$, such that $A \subseteq C$ and $B \cap C = \emptyset$.*

(iii) *If $X \succ_S Y$ and $A \not\succ_S B$ then there exists $C \in PrFil(S)$ denoted by $C(X \succ_S Y, A \not\succ_S B)$ such that:*

1) *either $X \not\subseteq C$ or $Y \cap C \neq \emptyset$,*

2) *$A \subseteq C$ and $B \cap C = \emptyset$.*

PROOF.

(i) is exactly the Separation lemma for C-systems.

(ii) Let $A \not\succ_S B$. Then by (Incl) $A \not\vdash_S B$ and by (i) there exists a $C \in PrFil(S)$ such that $A \subseteq C$ and $B \cap C = \emptyset$. Put $C(A \not\succ_S B) = C$.

(iii) Suppose $X \succ_S Y$ and $A \not\succ_S B$. Then by lemma 1. 2 either $\exists x \in X$ such that $A \not\vdash_S x, B$ or $\exists y \in Y$ such that $A, y \not\vdash_S B$.

Case 1: $\exists x \in X$ $A \not\vdash_S x, B$. Then by (i) there exists $C_1 \in PrFil(S)$ such that $A \subseteq X$ and $(\{x\} \cup B) \cap C_1 = \emptyset$, hence $x \notin C_1$, $B \cap C_1 = \emptyset$ and $X \not\subseteq C_1$. This yields the conditions 1) and 2) of the assertion. Put in this case $C(X \succ_S Y, A \not\succ_S B) = C_1$.

Case 2: $\exists y \in Y\ A, y \not\vdash_S B$. Then by (i) there exists $C_2 \in PrFil(S)$ such that $A \cup \{y\} \subseteq C_2$ and $B \cap C_2 = \emptyset$. Then $A \subseteq C_2$, $y \in C_2$ and cosequently $Y \cap C_2 \neq \emptyset$. This yields the conditions 1) and 2) of the assertion. In this case put $C(X \succ_S Y, A \not\succ_S B) = C_2$.

■

LEMMA 1. 6. *For any* $X, Y \in P_{fin}(W)$ *the following holds*:

(i) $X \vdash_S Y$ iff $(\forall a \in V)(\forall C \in a)(X \subseteq C \rightarrow Y \cap C \neq \emptyset)$,

(ii) $X \succ_S Y$ iff $(\exists a \in V)(\forall C \in a)(X \subseteq C \rightarrow Y \cap C \neq \emptyset)$.

PROOF.

(i) $(\rightarrow)$ Suppose $X \vdash_S Y$, $a \in V$. Then a satisfies (*) and consequently $(\forall C \in a)(X \subseteq C \rightarrow Y \cap C \neq \emptyset)$.

$(\leftarrow)$ Suppose $X \not\vdash_S Y$. We shall define a good set $a \in V$ such that $(\exists C \in a)(X \subseteq C$ and $Y \cap C = \emptyset)$.

Put $a = \{C(X \not\vdash_S Y)\} \cup \{C(A \not\succ_S B)/A, B \in \mathbf{P}_{fin}(W)$ and $A \not\succ_S B\}$ and let $C = C(X \not\vdash_S Y)$. We have $C \in a$ and by lemma 1. 5(i) that $X \in C$ and $Y \cap C = \emptyset$. It remains to show that a is a good set. By lemma 1. 5 all elements of a are prime filters of S so (*) is fulfilled. To prove (**) suppose $A \not\succ_S B$. Then by the construction of a $C = C(A \not\succ_S B) \in a$ and by lemma 1. 5(ii) $A \subseteq C$ and $B \cap C \neq \emptyset$. So (**) is fulfilled and hence $a \in V$.

(ii) $(\rightarrow)$ Suppose $X \succ_S Y$. We shall construct $a \in V$ such that $(\forall C \in a)(X \subseteq C \rightarrow Y \cap C \neq \emptyset)$. Put $a = \{C(X \succ_S Y, A \not\succ_S B)/A, B \in \mathbf{P}_{fin}(W)$ and $A \not\succ_S B\}$. LEMMA 1. 5(iii) guarantees that $a \in V$ and that it satisfies the assertion.

$(\leftarrow)$ Suppose $X \not\succ_S Y$ and $a \in V$. Then by (**) $\exists C \in a$ such that $X \subseteq C$ and $Y \cap C = \emptyset$.

This completes the proof of the lemma. ■

Now the proof of theorem 1. 4 follows immediately from lemma 1. 5. and lemma 1. 6. ■

In some abstract sense theorem 1. 4 may by considered as "completeness" theorem for B-system with respect to their "semantics" in the class of A-systems.

2. Some informational relations in Attribute systems

Let S be an A-system and $x, y \in Ob_S$. We introduce the following six informational relations in S:

- Weak positive similarity

  $x\Sigma_S y$ iff $(\exists a \in At_S) f_S(x,a) \cap f_S(y,a) \neq \emptyset$,

- Weak negative similarity

  $xN_S y$ iff $(\exists a \in At_S) \bar{f}_S(x,a) \cap \bar{f}_S(y,a) \neq \emptyset$,

- Weak informational inclusion

  $x <_S y$ iff $(\exists a \in At_S) f_S(x,a) \subseteq f_S(y,a)$,

- Strong positive similarity

  $x\sigma_S y$ iff $(\forall a \in At_S) f_S(x,a) \cap f_S(y,a) \neq \emptyset$,

- Strong negative similarity

  $x\nu_S y$ iff $(\forall a \in At_S) \bar{f}_S(x,a) \cap \bar{f}_S(y,a) \neq \emptyset$,

- Strong informational inclusion

  $x \leq_S y (\forall a \in At) f_S(x,a) \subseteq \bar{f}_S(y,a)$.

The intuitive meaning of this relations is clear from their definitions.

LEMMA 2. 1.
(i) *Let $S = A(W,V)$ be a set-theoretical A-system over the pair (W,V). Then for any $x, y \in W$ the following is true*:

- $x\Sigma_S y$ iff $(\exists a \in V)(\exists A \in a)(x \in A \& y \in A)$,
- $xN_S y$ iff $(\exists a \in V)(\exists A \in a)(x \notin A \& y \notin A)$,
- $x <_S y$ iff $(\exists a \in V)(\forall A \in a)(x \in A \rightarrow y \in A)$,
- $x\sigma_S y$ iff $(\forall a \in V)(\exists A \in a)(x \in A \& y \in A)$,
- $x\nu_S y$ iff $(\forall a \in V)(\exists A \in a)(x \notin A \& y \notin A)$,
- $x \leq_S y$ iff $(\forall a \in V)(\forall A \in a)(x \in A \rightarrow y \in A)$.

PROOF – straightforward. ■

LEMMA 2. 2.

(i) *Let S be an A-system and $S' = B(S)$ by the B-system over S. Then for any $x, y \in Ob_S$ the following is true*:

- $x\Sigma_S y$ iff $x, y \not\vdash_{S'} \emptyset$
- $xN_S y$ iff $\emptyset \not\vdash_{S'} x, y$,
- $x <_S y$ iff $x \succ\!\!-_{S'} \emptyset$,
- $x\sigma_S y$ iff $x, y \not\succ\!\!-_{S'} \emptyset$,
- $x\nu_S y$ iff $\emptyset \not\succ\!\!-_{S'} x, y$,
- $x \leq_S y$ iff $x \vdash_{S'} y$.

(ii) *Let S' be a B-system and $S = A(S')$ be the canonical A-system over S'. Then for any $x, y \in Ob_S$ the following is true*:

- $x\Sigma_S y$ iff $x, y \not\vdash_{S'} \emptyset$
- $xN_S y$ iff $\emptyset \not\vdash_{S'} x, y$,
- $x <_S y$ iff $x \succ\!\!-_{S'} y$,
- $x\sigma_S y$ iff $x, y \not\succ\!\!-_{S'} \emptyset$,
- $x\nu_S y$ iff $\emptyset \not\succ\!\!-_{S'} x, y$,
- $x \leq_S y$ iff $x \vdash_{S'} y$.

PROOF – straightforward, following the relevant definitions. ■

Lemma 2. 2 attaches a new meaning of the relations $\Sigma, N, <, \sigma, \nu, \leq$ and suggests how to define them in B-systems. Namely we have the following definition.

Let S be a B-system. Then for $x, y \in Sen_S$ we define:

- $x\Sigma_S y$ iff $x, y \not\vdash_S \emptyset$
- $xN_S y$ iff $\emptyset \not\vdash_S x, y$,
- $x <_S y$ iff $x \succ\!\!-_S y$,

- $x\sigma_S y$ iff $x, y \not\succ\!\!-_S \emptyset$,
- $x\nu_S y$ iff $\emptyset \not\succ\!\!-_S x, y$,
- $x \leq_S y$ iff $x \vdash_S y$.

LEMMA 2. 3. *Let S be an A-system (B-system). Then the following conditions are true for any $x, y, z \in Ob_S$ ($x, y, z \in Sen_S$): (the subscript S is omitted)*

$S1.\ x \leq x$	$/x \vdash x,$	(Ref $\vdash$)
$S2.\ x \leq y \ \&\ y \leq z \rightarrow x \leq z,$	$/x \vdash y \ \&\ y \vdash z \rightarrow x \vdash z,$	(Cut $\vdash$)
$S3.\ x\Sigma y \rightarrow y\Sigma x,$	$/y, x \vdash \emptyset \rightarrow x, y \vdash \emptyset,$	(Permutation $\vdash$)
$S4.\ x\Sigma y \rightarrow x\Sigma x,$	$/x \vdash \emptyset \rightarrow x, y \vdash \emptyset,$	(Mono $\vdash$)
$S5.\ x\Sigma y \ \&\ y \leq z \rightarrow x\Sigma z,$	$/x, z \vdash \emptyset \ \&\ y \vdash z$ then $x, y \vdash \emptyset,$	(Cut $\vdash$)
$S6.\ x\Sigma x$ or $x \leq y$	$/x \vdash \emptyset \rightarrow x \vdash y,$	(Mono $\vdash$)
$S7.\ xNy \rightarrow yNx,$	$/\emptyset \vdash y, x \rightarrow \emptyset \vdash x, y,$	(Permutation $\vdash$)
$S8.\ xNy \rightarrow xNx,$	$/\emptyset \vdash x \rightarrow \emptyset \vdash x, y,$	(Mono $\vdash$)
$S9.\ x \leq y \ \&\ yNz \rightarrow xNz,$	$/x \vdash y \ \&\ \emptyset \vdash x, z \rightarrow \emptyset \vdash y, z$	(Cut $\vdash$)
$S10.\ yNy$ or $x \leq y,$	$/\emptyset \vdash y \rightarrow x \vdash y,$	(Mono $\vdash$)
$S11.\ x\Sigma z$ or xNz or $x \leq y,$	$/x, z \vdash \emptyset \ \&\ \emptyset \vdash z, x \rightarrow x \vdash y,$	(Cut $\vdash$)
$S12.\ x < x,$	$/x \succ\!\!- x$	(Ref$\succ\!\!-$)
$S13.\ x \leq y \ \&\ y < z \rightarrow x < z,$	$/x \vdash y \ \&\ y \succ\!\!- z \rightarrow x \succ\!\!- z$	(Cut$\succ\!\!-$)
$S14.\ x < y \ \&\ y \leq z \rightarrow x < z,$	$/x \succ\!\!- y \ \&\ y \vdash z \rightarrow x \succ\!\!- z$	(Cut$\succ\!\!-$)
$S15.\ x\sigma y \rightarrow y\sigma x,$	$/x, y \succ\!\!- \emptyset \succ\!\!- y, x \succ\!\!- \emptyset$	(Permutation$\succ\!\!-$)
$S16.\ x\sigma y \rightarrow x\sigma x,$	$/x \succ\!\!- \emptyset \rightarrow x, y \succ\!\!- \emptyset$	(Mono$\succ\!\!-$)
$S17.\ x\sigma y \ \&\ y \leq z \rightarrow x\sigma z,$	$/x, z \succ\!\!- \emptyset \ \&\ y \vdash z \rightarrow x, y \succ\!\!- \emptyset$	(Cut$\succ\!\!-$)
$S18.\ x\sigma x$ or $x < y,$	$/x \succ\!\!- \emptyset \rightarrow x \succ\!\!- y$	(Mono$\succ\!\!-$)
$S19.\ x\sigma y \ \&\ y < z \rightarrow x\Sigma z,$	$/x, z \vdash \emptyset \ \&\ y \succ\!\!- z \rightarrow x, y \succ\!\!- \emptyset$	(Cut$\succ\!\!-$)
$S20.\ x\sigma z$ or yNz or $x < y$	$/x, z \succ\!\!- \emptyset \ \&\ \emptyset \vdash y, z \rightarrow x \succ\!\!- y$	(Cut$\succ\!\!-$)
$S21.\ x\nu y \rightarrow y\nu x,$	$/\emptyset \succ\!\!- y, x \rightarrow \emptyset \succ\!\!- x, y$	(Permutation$\succ\!\!-$)
$S22.\ x\nu y \rightarrow x\nu x,$	$/\emptyset \succ\!\!- x \rightarrow \emptyset \succ\!\!- x, y$	(Mono$\succ\!\!-$)
$S23.\ x \leq y \ \&\ y\nu z \rightarrow x\nu z,$	$/\emptyset \succ\!\!- x, z \ \&\ x \vdash y \rightarrow \emptyset \succ\!\!- y, z$	(Cut$\succ\!\!-$)
$S24.\ y\nu y$ or $x < y,$	$/\emptyset \succ\!\!- y \rightarrow x \succ\!\!- y$	(Mono$\succ\!\!-$)
$S25.\ x < y \ \&\ y\nu z \rightarrow xNz,$	$/\emptyset \vdash x, z \ \&\ x \succ\!\!- y \rightarrow \emptyset \succ\!\!- y, z$	(Cut$\succ\!\!-$)
$S26.\ x\Sigma z$ or $y\nu z$ or $x < y$	$/x, z \vdash \emptyset \ \&\ \emptyset \succ\!\!- y, z \rightarrow x \succ\!\!- y$	(Cut$\succ\!\!-$)

PROOF – straightforward verification. ■

Lemma 2. 3. suggests the following definition, which is the first step in the abstract characterization of the relations $\leq, \Sigma, N, <, \sigma, \nu$.

Let $\underline{W} = (W, \leq, \Sigma, N, <, \sigma, \nu)$ be a relational system with $W \neq \emptyset$ and $\leq, \Sigma, N, <, \sigma, \nu$ be binary relations in W. We call $\underline{W}$ a bisimilarity structure if it satisfies the conditions S1-S26 from lemma 2. 3.

Now Lemma 2. 3. says that if S is an A-system (B-system) then the system $BiSim(S) = (W, \leq_S, \Sigma_S, N_S, <_S, \sigma_S, \nu_S)$ with $W = Ob_S$ ($W = Sens_S$) is a bi-similarity structure, called the bi-similarity structure over S.

The next lemma states some consequences from the axioms of a bi-similarity structure.

LEMMA 2. 4. *The following conditions hold in each bi-similarity structure*

(i) *Let* $S \in \{\Sigma, N, \sigma, \nu\}$. *Then* xSy *and* $u, v \in \{x, y\}$ *imply* uSv.

(ii) a. $xSy \ \& \ x \leq u \ \& \ y \leq v \rightarrow uSv, \ S \in \{\Sigma, \sigma\}$,
b. $xSy \ \& \ u \leq x \ \& \ v \leq y \rightarrow uSv, \ S \in \{N, \nu\}$,
c. $x\sigma y \ \& \ ((x \leq u \ \& \ y < v) \text{ or } (x < u \ \& \ y \leq v)) \rightarrow u\Sigma v$,
d. $x\nu y \ \& \ ((u \leq x \ \& \ v < y) \text{ or } (u < x \ \& \ v \leq y)) \rightarrow uNv$,
e. $x\sigma x \ \& \ x < y \rightarrow x\Sigma x$,
f. $y\nu y \ \& \ x < y \rightarrow xNx$.

(iii) a. $x\sigma y \rightarrow x\Sigma y$,
b. $x\nu y \rightarrow xNy$,
c. $x \leq y \rightarrow x < y$.

(iv) a. $x\bar{\Sigma}x \ \& \ x\bar{N}x \rightarrow \forall uv \ u \leq v, u < v, u\bar{\Sigma}v, u\bar{N}v, u\bar{\sigma}v, u\bar{\nu}v$,
b. $x\bar{\Sigma}x \ \& \ x\bar{N}y \rightarrow (y\bar{\Sigma}y \ \& \ y\bar{\nu}y) \ \& \ (y\sigma y \ \& \ y\bar{N}y)$.
c. $(x\bar{\Sigma}x \ \& \ x\bar{\nu}x)$ or $(x\bar{\sigma}x \ \& \ x\bar{N}x) \rightarrow \forall uv \ u < v, u\bar{\sigma}v, u\bar{\nu}v$

PROOF — exercise. ■

LEMMA 2. 5.

(i) *Let* S *be an A-system and* $B(S)$ *be the B-system over* S. *Then* $BiSim(S) = BiSim(B(S))$.

(ii) *Let* S *be a B-system and* $A(S)$ *be the canonical A-system over* S. *Then* $BiSim(S) = BiSim(A(S))$.

PROOF — direct consequence of lemma 2. 3 and lemma 2. 4. ■

THEOREM 2. 6. (Characterization theorem for bi-similarity structutres) *Let* $\underline{W} = (W, \leq, \Sigma, N, <, \sigma, \nu)$ *be a bi-similarity structure. Then:*

(i) *There exists a B-system* S *such that* $\underline{W} = BiSim(S)$.

(ii) *There exists an A-system S' such that $\underline{W} = BiSim(S')$.*

PROOF. (i) Since we want to have $\underline{W} = BiSim(S)$ then we have to put $Sens_S = W$. For the relations $\vdash$ and $\succ\!\!-$ we take the following definition. Namely for $X, Y \in \mathbf{P}_{fin}(W)$ define:

$$
\begin{array}{lll}
X \vdash Y iff & (I) & \exists x \in X\ \exists y \in Y\ x \leq y, or \\
& (II) & \exists x, y \in X\ x\bar{\Sigma}y, or \\
& (III) & \exists x, y \in Y\ x\bar{N}y, or \\
& (IV) & \exists a \in W\ a\bar{\Sigma}a\ \&\ a\bar{N}a. \\
X \succ\!\!- Y iff & (J) & \exists x \in X\ \exists y \in Y\ x < y, or \\
& (JJ) & \exists x, y \in X\ x\bar{\sigma}y, or \\
& (JJJ) & \exists x, y \in Y\ x\bar{\nu}y, or \\
& (JV) & \exists x\ ((x\bar{\Sigma}x\ \&\ x\bar{\nu}x) or (x\bar{\sigma}x\ \&\ x\bar{N}x)).
\end{array}
$$

The proof of (i) will follow from the following two assertions.

Assertion 1.

$(W, \vdash, \succ\!\!-)$ is a B-system.

Assertion 2.

For any $x, y \in W$ the following holds:

(a) $u \leq v$ iff $u \vdash v$,

(b) $u\Sigma v$ iff $u, v \not\vdash \emptyset$,

(c) uNv iff $\emptyset \not\vdash u, v$,

(d) $u < v$ iff $u \succ\!\!- v$,

(e) $u\sigma v$ iff $u, v \not\succ\!\!- \emptyset$,

(f) $u\nu v$ iff $\emptyset \not\succ\!\!- u, v$.

Proof of assertion 1 – long and routine checking the validity of the axioms of B-system by applying the conditions S1-S26 and lemma 2. 4. As an example we shall verify only the axiom (Cut). For the remaining axioms the reader can proceed in the same way.

For (Cut) suppose

(1) $X, a \vdash Y$,

(2) $X \vdash a, Y$

and proceed to show

(3) $X \vdash Y$.

For (1) and (2) we have to consider several cases following the definition of $\vdash$. Then we shall combine each case for (1) with each case for (2).

(1I) $\exists x \in X \cup \{a\}\ \exists y \in Y\ x \leq y$. If $x \in X$ then by (I) we get (3), so we consider in this case that

$$x = a:\ a \leq y,\ y \in Y.$$

(1II) $\exists x, y \in X \cup \{a\}\ x\bar{\Sigma}y$. If $x, y \in X$ then by (II) we get (3). So we consider here three cases:

(i) $x = a : a\bar{\Sigma}y, y \in X$,

(ii) $y = a : x\bar{\Sigma}a, x \in X$.

(iii) $x = y = a : a\bar{\Sigma}a$.

(1III) $\exists x, y \in Y\ x\bar{N}y$. Then directly by (III) we obtain (3). So this case will not be combined with the other cases for (2).

(1IV) $\exists a\ a\bar{\Sigma}a\ \&\ a\bar{N}a$. By (IV) we get (3) so this case also will not be combined with the other cases for (2).

(2I) $\exists u \in X\ \exists v \in \{a\} \cup Y\ u \leq v$. If $v \in Y$ then by (I) we get (3) so in this case we consider

$$v = a : u \leq a \text{ and } u \in X.$$

(2II) $\exists u, v \in X\ u\bar{\Sigma}v$. Then by (II) we obtain (3). This case will not be combined with the other cases for (1).

(2III) $\exists u, v \in \{a\} \cup Y\ u\bar{N}v$. If $u, v \in V$ then we obtain (3) by (III), so we will consider here three cases:

(j) $u = a : a\bar{N}v, v \in Y$,

(jj) $v = a : u\bar{N}a, u \in Y$,

(jjj) $u = v = a : a\bar{N}a$.

(2IV) $\exists a \in W\ a\bar{\Sigma}a\ \&\ a\bar{N}a$. Then by (IV) we get (3), so this case will not be combined with the other cases of (1).

Now we start to combine the possible cases for (1) and (2).

Case (1I) (2I): $a \leq y, y \in Y, u \leq a, u \in X$. By S1. we get $u \leq v$ and by (I) we get (3).

Case (1I) (2III). Here we have three sub-cases:

(j) $a \leq y, y \in Y, a\bar{N}v, v \in Y$. By S9' we obtain $y\overline{N}v$ and by (III) we obtain (3).

(jj) $a \leq y, y \in Y, u\bar{N}a, u \in Y$ - similar to (j).

(jjj) $a \leq y, y \in Y, a\bar{N}a$. By S9 we obtain $y\overline{N}y$ and by (III) we obtain (3).

Case (1II) (2I) We have to consider three cases.

(i) $a\bar{\Sigma}y, y \in X, u \leq a, u \in X$. Then by S5 we get $u\bar{\Sigma}y$ and by (II) we obtain (3).

(ii) $x\Sigma a, x \in X, u \leq a, u \in X$. Proceed as in (i).

(iii) $a\bar{\Sigma}a, u \leq a, u\Sigma a, u \in X$. By S5 we obtain $u\bar{\Sigma}u$ and by (II) we obtain (3).

Case (1II) (2III). We have to combine the cases (i)-(iii) with the cases (j)-(jjj).

(i) (j) $a\bar{\Sigma}y, x \in X, a\bar{N}v, v \in Y$. Then by S11 we get $y \leq v$ and by (I) we obtain (3). The cases (i) (jj), (ii) (j) and (ii) (jj) can be treated similarly.

(iii) (jjj) $a\bar{\Sigma}a, a\bar{N}a$. By (IV) we obtain (3).

■

Proof of assertion 2. As an example we shall verify (a).

(a) ($\rightarrow$) Suppose $u \leq v$. Then by (I) we have $u \vdash v$.

(a) ($\leftarrow$) Suppose $u \vdash v$, i. e. $\{u\} \vdash \{v\}$. We have to show $u \leq v$. For that purpose we have to consider the possible cases of the definition of $\vdash$.

Case (I): $\exists x \in \{u\}\ \exists y \in \{v\}\ x \leq y$. Then $x = u$, $y = v$ and hence $u \leq v$.

Case (II): $\exists x, y \in \{u\}\ x\bar{\Sigma}y$. Then $x = y = u$ and $u\bar{\Sigma}u$. By S6 we obtain $u \leq v$.

Case (III): $\exists x, y \in \{v\}\ x\bar{N}y$. Then $x = y = v$ and $v\bar{N}v$. By S10 we obtain $u \leq v$.

Case (IV): $\exists a\ a\bar{\Sigma}a\ \&\ a\bar{N}a$. By LEMMA 2. 4. (iv) we obtain $u \leq v$. ∎

This ends the proof of (i).

(ii) Put $S' = A(S)$. Then S' is an A-system. By lemma 2. 5 $BiSim(S) = BiSim(A(s))$. By (i) we have $\underline{W} = BiSim(S)$, hence $\underline{W} = BiSim(S')$. This ends the proof of the theorem. ∎

3. A modal logic INF for some informational relations

In this part we introduce a modal logic INF for the informational relations in A-systems, considered in the previous section. The aim of this logic is to provide a formal account for reasoning about objects in an A-system, including different modalities corresponding to some similarity relations and other kinds of informational relations. We shall discuss also the possibility to give a "query" meaning of the considered modal language. The first modal systems of this type have been introduced by Orlowska and Pawlak in [11] and [12]. Later this line of investigations was continued by Orlowska [4], [5], [6], [9], Vakarelov [22], [23], [25], [26] and [27]. The novelty of INF is that it contains the full variety of positive and negative similarity relations, which can not be characterized by the methods applied in the above mentioned papers.

The main results in this section is a completeness theorem for INF with respect to its standard semantics. The proof of the completeness theorem is essentially based on the characterization theorem for the informational relations, proved in section 2. Another result is the finite model property for INF, which yields its decidability.

Syntax of INF

The language of INF contains the following primitive symbols:

- VAR - an infinite set, whose elements are called propositional variables,
- $\wedge, \vee, \neg$ - the classical Boolean connectives,

- $[\leq], [\geq], [\Sigma], [N], [<], [>], [\sigma], [\nu], [U]$ - modal operations,
- ( ) - parentheses.

The notion of formula is the usual one, namely:

- all propositional variables are formulas,
- if A and B are formulas then $\neg A, (A \wedge B)$ and $(A \vee B)$ are formulas,
- if A is a formula then $[R]A$ is a formula, $R = \{\leq, \geq, \Sigma, N, <, >, \sigma, \nu, U\}$.
- Abbreviations: $A \Rightarrow B = \neg A \vee B, 1 = \neg A \vee A, 0 = \neg 1, \langle R \rangle = \neg [R] \neg$

Semantics of INF

We interpret the language of INF in a bi-similarity structures as in the usual Kripke semantics. Let $\underline{W}$ be a bi-similarity structure and for $x, y \in W$ we put $x \geq y$ iff $y \leq x$ and $x > y$ iff $y < x$. A function $v : VAR \rightarrow \mathbf{P}(W)$ is called a valuation if it assigns to each variable A a subset $V(A) \subseteq W$. Then the pair $M = (\underline{W}, v)$ is called a model over $\underline{W}$. The satisfiability relation $x \Vdash_v A$ (the formula A is true in a point $x \in W$ at the valuation v) is defined inductively:

$x \Vdash_v A$ iff $x \in v(A)$ for $a \in VAR$,

$x \Vdash_v \neg A$ iff $x \nVdash_v A$,

$x \Vdash_v A \wedge B$ iff $x \Vdash_v A$ and $x \Vdash_v B$,

$x \Vdash_v A \vee B$ iff $x \Vdash_v A$ or $x \Vdash_v B$,

$x \Vdash_v [R]A$ iff $(\forall y \in W)(xRy \rightarrow y \Vdash_v A)$ for $R \in \{\leq, \geq, \Sigma, N, <, >, \sigma, \nu\}$.

$x \Vdash_v [U]A$ iff $(\forall y \in W) y \Vdash_v A$.

Using the definition of $\langle R \rangle$ we obtain the following interpretation of $\langle R \rangle A$ and $\langle U \rangle A$:

$x \Vdash_v \langle R \rangle A$ iff $(\exists y \in W)(xRy$ and $y \Vdash_v A)$,

$x \Vdash \langle U \rangle A$ iff $(\exists y \in A)(y \Vdash A)$.

We say that a formula A is true in the model M if for any $x \in W$ we have $x \Vdash_v A$; A is true in a bi-similarity structure $\underline{W}$ if A is true in any model over $\underline{W}$.

Let S be a P-system, $\underline{W}$ be the similarity structure over S and $M = (\underline{W}, v)$ be a model over $\underline{W}$. For any formula A we put $v(A) = \{x \in W / x \Vdash_v A\}$. The set $v(A)$ may have different meanings. One is that it is the set of all objects from $W = Ob_S$, for which A is true (*at* v). Another meaning is that $v(A)$ may be considered also as a query to S: "give the set of all objects $x \in Ob_S$, for which A is true". This meaning leads to consider interpreted propositional variables in a given model as a simple queries and formulas as compound queries. Them modal formulas will be "modal queries". Let us consider the following example. Suppose in the above model M that A is a propositional variable such that $v(A) = \{x_0\}$. Then for $v(\langle\Sigma\rangle A)$ we can compute: $v(\langle\Sigma\rangle A) = \{x \in Ob_0 / (\exists y \in Ob_S)(x\Sigma y \text{ and } y \in \{x_0\}) = \{x \in Ob_S / x\Sigma x_0\}$. This is the following query to S: "give all objects of S which are positively weakly similar to x_0".

Axiomatization of INF

Axiom schemes:

(Bool) All Boolean tautologies,

(K) $[R](A \Rightarrow B) \Rightarrow [R]B), R \in \{\leq, \geq, \Sigma, N, <, >, \sigma, \nu, U\}$

$A0.$ $\langle\leq\rangle[\geq]A \Rightarrow A, \langle\geq\rangle[\leq]A \Rightarrow A, \langle<\rangle[>]A \Rightarrow A,$
$\langle<\rangle[>]A \Rightarrow A, [U]A \Rightarrow A, \langle U\rangle[U]A \Rightarrow A, [U]A \Rightarrow [U][U]A,$
$[U]A \Rightarrow [R]A, R \in \{\leq, \geq, \Sigma, N, <, >, \sigma, \nu, U\}$

$A1.$ $[\leq]A \Rightarrow A,$

$A2.$ $[\leq]A \Rightarrow [\leq][\leq]A,$

$A3.$ $\langle\Sigma\rangle[\Sigma]A \Rightarrow A,$

$A4.$ $\langle\Sigma\rangle 1 \Rightarrow ([\Sigma]A \Rightarrow A),$

$A5.$ $[\Sigma]A \Rightarrow [\Sigma][\leq]A,$

$A6.$ $[\leq]A \Rightarrow ([U]B \vee ([\Sigma]B \Rightarrow B)),$

$A7.$ $\langle N\rangle[N]A \Rightarrow A,$

$A8.$ $\langle N\rangle 1 \Rightarrow ([N]A \Rightarrow A),$

$A9.$ $[N]A \Rightarrow [N][\geq]A,$

$A10.$ $[\geq]A \Rightarrow ([U]B \vee ([N]B \Rightarrow B)),$

$A11.$ $[\leq]A \wedge [\Sigma]B \Rightarrow ([U]B \vee [U]([N]B \Rightarrow A)),$

$A12.$ $[<]A \Rightarrow A,$

$A13.$ $[<]A \Rightarrow [\leq][<]A,$

$A14.$ $[<]A \Rightarrow [<][\leq],$

$A15.$ $\langle\sigma\rangle[\sigma]A \Rightarrow A,$

A16. $\langle\sigma\rangle 1 \Rightarrow [\sigma]A \Rightarrow A)$,
A17. $[\sigma]A \Rightarrow [\sigma][\leq]A$,
A18. $[<]A \Rightarrow ([U]B \vee ([\sigma]B \Rightarrow B))$,
A19. $[\sigma]A \Rightarrow [\sigma][<]A$,
A20. $[<]A \wedge [\sigma]B \Rightarrow ([U]B \vee ([N]B \Rightarrow A))$,
A21. $\langle\nu\rangle[\nu]A \Rightarrow A$,
A22. $\langle\nu\rangle 1 \Rightarrow ([\nu][\geq]A$,
A23. $[\nu]A \Rightarrow [\nu][\geq]A$,
A24. $[>]A \Rightarrow ([U]B \vee ([\nu]B \Rightarrow B))$,
A25. $[N]A \Rightarrow [\nu][>]A$,
A26. $[<]A \wedge [\Sigma]B \Rightarrow ([U]B \vee [U]([\nu]B \Rightarrow A))$.

Rules of inference: modus ponens $(MP)A, A \Rightarrow B/B$,
necessitation $A/[R]A, R \in \{\leq, \geq, \Sigma, N, <, >, \sigma, \nu, U\}$.

The logic INF is the smallest set of formulas, containing all axiom schemes and closed under the rules of inference.

Let us note that the axioms A1-A26 are modal translations of the conditions S1-S26.

THEOREM 3. 1. (Completeness theorem for INF)
For any formula A of INF the following conditions are equivalent:

(i) *A is a theorem of INF,*

(ii) *A is true in all bi-similarity structures,*

(iii) *A is true in all bi-similarity structures over A-systems.*

PROOF.

(i) $\rightarrow$ (ii) in a standard way by showing the validity of all axioms and that the rules preserve validity.

(ii) $\leftrightarrow$ (iii) — by theorem 2. 6.

(ii) $\rightarrow$ (i) The proof can be done by the standard canonical-model-construction (for the relevant definitions and facts see [HC 84] or [Seg 71]).

Let W be the set of all maximal consistent sets of INF. Define for $x, y \in W$ and $R \in \{\leq, \geq, \Sigma, N, <, >, \sigma, \nu, U\}$, $[R]x = \{A \in FOR/[R]A \in x\}, xRy$ iff $[R]x \subseteq y$. It is easy to show by A0 that U is an equivalence relation containing all $R \in \{\leq, \geq, \Sigma, N, <, >, \sigma, \nu\}$ and that $\geq$ and $>$ are the converse relations of $\leq$ and $<$ respectively. For $a \in W$ let $W_a = \{x/aUx\}$ and R_b be the restriction of R in the set W_a, $R \in \{\leq, \geq, \Sigma, N, <, >, \sigma, \nu, U\}$. Since U is an equivalence relation, U_a is the universal relation in W_a. Then, using the axioms of INF, one can prove in a standard way the following

LEMMA 3. 2. *For any* $a \in W$ *the system* $\underline{W}_a = (W_a, \leq_a, \geq_a, \Sigma_a, N_a, <_a, >_a, \sigma_a, \nu_a)$ *is a bi-similarity structure.*

Now, suppose that A is not a theorem of INF. Then there exists a maximal consistent set $a \in W$ such that $A \notin a$. Take the canonical valuation $v(p) = \{x \in W_a/p \in x\}, p \in VAR$. Then in a standard way one can prove by induction on the construction of B that for any $x \in W_a : x \Vdash_v B$ iff $B \in x$. From here we get $a \nVdash_v A$, so A is not true in the similarity structure W_a, which ends the proof of the theorem.

Finite model property of INF

Now we shall prove by means of the filtration method (see [18]) that INF possesses finite model property in a sense that each non-theorem can be falsified in a finite model. First we shall formulate some basic definitions and facts about filtration, adapted to the logic INF.

Let Γ be a finite set of formulas, closed under subformulas and $M = ((W, \leq, \geq, \Sigma, N, <, >, \sigma, \nu), v)$ be a model over some bi-similarity structure $\underline{W}$. Define an equivalence relation $\sim$ in W in the following way:

$$x \sim y \text{ iff } (\forall A \in \Gamma)\ (x \Vdash_v A \leftrightarrow y \Vdash_v A).$$

Let for $x \in W$ $|x|\{y \in W/x \sim y\}$, $|W| = \{|x|/x \in W\}$ and

$$v'(p) = \{|x|/x \in v(p)\} \text{ for } p \in VAR.$$

We say that the model $M' = ((|W|, \leq', \Sigma', N'), v')$ is a filtration of M through Γ if M' is a model over similarity structure and the following conditions are satisfied for any $x, y \in W$ and $R \in \{\leq, \geq, \Sigma, N, <, >, \sigma, \nu, U\}$.

(FR1) If xRy then $|x|R'|y|$,

(FR2) If $|x|R'|y|$ then $(\forall[R]A \in \Gamma)(x \Vdash_v A \rightarrow y \Vdash_v A)$.

LEMMA 3. 3. (Filtration lemma)

(i) *The following is true for any formula* $A \in \Gamma$ *and* $x \in W$: $x \Vdash_v A$ *iff* $|x| \Vdash'_v A$.

(ii) *The set* $|W|$ *has at most* 2^n *elements, where* n *is the number of the elements of* Γ.

The proof of (i) is the same as in the standard modal logic (see [18]) and can be done by induction on the complexity of A. Conditions (FR1) and (FR2) are used when A is in the form $[R]B$.

For (ii) let f be a function from $|W|$ to the set $P(\Gamma)$ of all subsets of Γ defined as follows: $f(|x|) = \{B \in \Gamma / x \Vdash_v B\}$. It is easy to see that f is 1-1-function from $|W|$ into $\mathbf{P}(\Gamma)$. Since $\mathbf{P}(\Gamma)$ has 2^n elements then $|W|$ has no more than 2^n elements. ■

THEOREM 3. 4. (Filtration theorem for INF)
For any model $M = ((W, \leq, \geq, \Sigma, N, <, >, \sigma, \nu), v)$ over a bi-similarity structure $\underline{W}$ and formula A' there exist a finite set Γ of formulas, containing A' and closed under subformulas and a filtration $M' = ((|W|, \leq', \Sigma', N', \sigma', \nu'), v')$ of M through Γ such that $Card\Gamma \leq 8.n + 4$, where n is the number of subformulas of A'.

PROOF. Let the model M and the formula A' be given. Let Γ be the smallest set of formulas containing $A', \langle\Sigma\rangle, \langle N\rangle 1, \langle\sigma\rangle 1, \langle\nu\rangle 1$, closed under sub formulas and satisfying the following closure condition

(γ) For any formula A, if one of the formulas $[\Sigma]A, [N]A, [\leq]A, [\geq]A, [\sigma]A, [\nu]A, [<]A, [>]A$ is in Γ, then the others are also in Γ.

Obviously Γ is a finite set of formulas containing no more than $8n + 4$, where n is the number of subformulas of A'. Define $|W|$ and v' as in the definition of filtration. For $|x|, |y| \in |W|$ define:

$$
\begin{array}{lll}
(1) & |x| \leq' |y| \text{ iff } (\forall[\leq]A \in \Gamma) & (x \Vdash_v [\leq]A \rightarrow y \Vdash_v [\leq]A)\& \\
& & (y \Vdash_v [\geq]A \rightarrow x \Vdash_v [\geq]A)\& \\
& & (y \Vdash_v [\Sigma]A \rightarrow x \Vdash_v [\Sigma]A)\& \\
& & (x \Vdash_v [N]A \rightarrow y \Vdash_v [N]A)\& \\
& & (x \Vdash_v [<]A \rightarrow y \Vdash_v [<]A)\& \\
& & (y \Vdash_v [>]A \rightarrow x \Vdash_v [>]A)\& \\
& & (y \Vdash_v [\sigma]A \rightarrow x \Vdash_v [\sigma]A)\& \\
& & (x \Vdash_v [\nu]A \rightarrow y \Vdash_v [\nu]A)\& \\
& & (x \Vdash_v \langle\Sigma\rangle 1 \rightarrow y \Vdash_v \langle\Sigma\rangle 1)\& \\
& & (y \Vdash_v \langle N\rangle 1 \rightarrow x \Vdash_v \langle N\rangle 1)\& \\
& & (x \Vdash_v \langle\sigma\rangle 1 \rightarrow y \Vdash_v \langle\sigma\rangle 1)\& \\
& & (y \Vdash_v \langle\nu\rangle 1 \rightarrow y \Vdash_v \langle\nu\rangle 1), \\
(2) & |x| \geq' |y| \text{ iff } |y| \leq' |x|, & \\
(3) & |x|\Sigma'|y| \text{ iff } (\forall[\Sigma]A \in \Gamma) & (x \Vdash_v [\Sigma]A \rightarrow y \Vdash_v [\leq]A)\& \\
& & (y \Vdash_v [\Sigma]A \rightarrow x \Vdash_v [\leq]A)\& \\
& & x \Vdash_v \langle\Sigma\rangle 1 \;\&\; y \Vdash_v \langle\Sigma\rangle 1,
\end{array}
$$

$$
\begin{array}{lll}
(4) & |x|N'|y| \text{ iff } (\forall [N]A \in \Gamma) & (x \Vdash_v [N]A \rightarrow y \Vdash_v [\geq]A)\& \\
 & & (y \Vdash_v [N]A \rightarrow x \Vdash_v [\geq]A)\& \\
 & & x \Vdash_v \langle N\rangle 1 \;\&\; y \Vdash_v \langle N\rangle 1, \\
(5) & |x| <' |y| \text{ iff } (\forall [<]A \in \Gamma) & (x \Vdash_v [<]A \rightarrow y \Vdash_v [\leq]A)\& \\
 & & (y \Vdash_v [>]A \rightarrow y \Vdash_v [\geq]A)\& \\
 & & (y \Vdash_v [\Sigma]A \rightarrow x \Vdash_v [\sigma]A)\& \\
 & & (x \Vdash_v [N]A \rightarrow y \Vdash_v [\nu]A)\& \\
 & & (x \Vdash_v \langle\sigma\rangle 1 \rightarrow y \Vdash_v \langle\Sigma\rangle 1)\& \\
 & & (y \Vdash_v \langle\nu\rangle 1 \rightarrow x \Vdash_v < N\rangle 1)\& \\
(6) & |x| >' |y| \text{ iff } |y| <' |x|, & \\
(7) & |x|\sigma'|y| \text{ iff } (\forall [\sigma]A \in \Gamma) & (x \Vdash_v [\sigma]A \rightarrow y \Vdash_v [\leq]A)\& \\
 & & (y \Vdash_v [\sigma]A \rightarrow x \Vdash_v [\leq]A)\& \\
 & & (x \Vdash_v [\Sigma]A \rightarrow y \Vdash_v [<]A)\& \\
 & & (y \Vdash_v [\Sigma]A \rightarrow x \Vdash_v [<]A)\& \\
 & & x \Vdash_v \langle\sigma\rangle 1 \;\&\; y \Vdash_v \langle\sigma\rangle 1, \\
(8) & |x|\nu'|y| \text{ iff } (\forall [\nu]A \in \Gamma) & (x \Vdash_v [\nu]A \rightarrow y \Vdash_v [\geq]A)\& \\
 & & (y \Vdash_v [\nu]A \rightarrow x \Vdash_v [\geq]A)\& \\
 & & (x \Vdash_v [N]A \rightarrow y \Vdash_v [>]A)\& \\
 & & (y \Vdash_v [N]A \rightarrow x \Vdash_v [>]A)\& \\
 & & x \Vdash_v \langle\nu\rangle 1 \;\&\; x \Vdash_v \langle\nu\rangle 1,
\end{array}
$$

The required model is $M' = ((|W|, \leq', \Sigma', N', <', \sigma', \nu'), v')$. The proof that the conditions of filtration are satisfied and that M' is a model over a bi-similarity structure is long and routine and therefore is left to the reader.

THEOREM 3. 5. (Finite model property for INF)

(i) *For any formula A of INF, if A is not theorem of INF then there exists a model $M = (\underline{W}, v)$, such that $CardW \leq 2^{8n+4}$ in which A is not true.*

(ii) *INF is decidable.*

PROOF.

(i) Suppose that A is not a theorem of INF and let the number of subformulas of A is n. Then A is falsified in some model $M = (\underline{W}, v)$. So there exists x_0 such that $x_0 \not\Vdash_v A$. Let $M' = ((|W|, \leq', \Sigma', N', <', \sigma', \nu'), v')$ be the filtration of M existing by lemma 3. 4. and determined by A. By the filtration lemma the cardinality of $|W|$ is $\leq 2^{8.n+4}$, so A is falsified in model with cardinality $\leq 2^{8.n+4}$.

(ii) The decidability of INF follows from (i). ∎

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