A LOGIC OF INDISCERNIBILITY RELATIONS

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1. Introduction

In the paper we attempt to define a logic to reason about entities which are defined up to indiscernibility relations. Indiscernibility relations are assumed to be determined by properties of entities. The logic is a multimodal logic such that accessibility relations corresponding to modal operations are determined by sets of parameters.The special expressions are allowded representing set-theoretical combinations of the sets of parameters. Moreover, in the language of the logic special constants are used, interpreted as one-element sets of entities and one-element sets of properties of entities. Several conditions characterizing sets of entities, indiscernibility relations, and sets of properties of entities are shown to be expressible in the logic.

A treatment of indiscernibility presented in the present paper is inspired by rough set approach to knowledge representation (Pawlak (1981, 1982), Orlowska and Pawlak (1984), Orlowska (1983)) and by research on relative identity (Quine (1961), Geach (1976), Kraut (1980), Griffin (1977), Noonan (1980), Wiggins (1980)).

The focus in this paper is on definability of entities and sets of entities up to indiscernibility. We consider indiscernibility to be a binary relation in a set of entities, and moreover this relation is determined by some properties of entities. We assume that entities are indiscriminable from one another relative to the properties under consideration. We attempt to define a formal language in which various facts concerning definability can be expressed. In section 2 we define a semantical structure for the language. In sections 3 and 4 we recall the basic notions from the rough set theory. In sections 5 and 6 the language and the semantics of the language are defined. Section 7 provides examples of facts expressible in the language. In section 8 we consider weak indiscernibility relations determined by properties of entities.

2. Universe of discourse

We consider domains which can be characterized by specification of a non-empty set ENT whose elements are interpreted as entities, and a non-empty set PROP whose elements are interpreted as properties of entities. Properties of entities serve as expressive and discriminative resources in the domain. Entities may be discernible with respect to one subset of properties but indiscernible with respect to another. Whether or not entities are discernible from one another is thus a function of these properties, and hence indiscernibility should be relative with respect to a choosen subset of properties.

Given a set ENT of entities and a set PROP of properties, we consider a family $\{ind(P)\}_{P \subseteq PROP}$ of binary relations in set ENT. For a subset P of properties relation ind(P) is interpreted as an indiscernibility with respect to properties from set P. We assume that for any $P,Q \subseteq$ PROP the following conditions are satisfied:

(U1) ind(P) is reflexive, symmetric, and transitive
(U2) $ind(P \cup Q) = ind(P) \cap ind(Q)$
(U3) If $P \subseteq Q$ then $ind(Q) \subseteq ind(P)$
(U4) $ind(\emptyset) = ENT \times ENT$

Condition (U1) says that indiscernibility relations are equivalence relations. Indiscernibility relations determined by information systems satisfy these conditions. Condition (U2) shows how discrimination power of the union of sets of properties depends on discrimination power of the components of the union; namely, a pair of entities is discriminable relative to properties from a set $P \cup Q$ iff it is discriminable relative to both properties from P and Q. Condition (U3) says that discrimination power of a set of properties is, in general, better than that of its subsets, that is bigger sets of properties enable us to discern more entities. Condition (U4) says that if a set of properties is empty, then we are not able to discern any entities.

Any system of the form

$$U = (ENT, PROP, \{ind(P)\}_{P \subseteq PROP})$$

satisfying conditions (U1), (U2), (U3), (U4) will be referred to as a universe of discourse.

It is easy to see that the algebra

$$(\{ind(P)\}_{P \subseteq PROP}, \cap)$$

is a lower semilattice, where ind(PROP) is the zero element.

Example 2.1

Let PROP be a family of subsets of a nonempty set ENT, and for a set $P \subseteq PROP$ let ind(P) be defined as follows:
(ex1) $(e,e') \in ind(P)$ iff for all $p \in P$ $e \in p$ iff $e' \in p$.

Example 2.2

Let ENT be a family of subsets of a nonempty set PROP and for any $P \subseteq PROP$ let ind(P) be defined as follows:
(ex2) $(e,e') \in ind(P)$ iff for all $p \in P$ $p \in e$ iff $p \in e'$.

Example 2.3

Let ENT and PROP be nonempty families of subsets of a certain nonempty set, and let us define two families of indiscernibility relations:
(ex3) $(e,e') \in ind(P)$ iff for all $p \in P$ $e \subseteq p$ iff $e' \subseteq p$
(ex4) $(e,e') \in ind(P)$ iff for all $p \in P$ $p \subseteq e$ iff $p \subseteq e'$.

Indiscernibility relations defined in these examples are relativised versions of identities considered in Weingartner (1974). Relations (ex1) and (ex3) are relative intensional identities and relations (ex2) and (ex4) are relative extensional identities.

In the following we present an example of indiscernibility relations determined by an information system (Pawlak (1981)). In information systems properties of entities are expressed in terms of attributes and values of attributes. For example property to be green is formulated as the pair (colour, green). By an information system we mean a system

$$(OB, AT, \{VAL_a\}_{a \in AT}, f)$$

where OB is a nonempty set whose elements are called objects (e. g. flower); AT is a nonempty set whose elements are called attributes (e.g. colour); for each attribute a VAL_a is a set of values of the attribute a (e.g. green); $f: OB \times AT \rightarrow VAL = \bigcup VAL_a$ is an information function such that $f(o,a) \in VAL_a$ for any $o \in OB$ and $a \in AT$.

Each information system determines a family of indiscernibility relations in the set OB. For a set $P \subseteq AT$ we define relation ind(P) as follows:

$(o,o') \in ind(P)$ iff for all $a \in P$ $f(o,a) = f(o',a)$.

Example 2.4

In this example we consider an information system introduced in Novotny (1981). Let OB={p,q,r,x}, AT={p,q,r}, f(o,a)={a} if o=a, and f(o,a)=OB-{a} if o≠a. The system can be represented in the form of the following table:

	p	q	r
p	{p}	{p,r,x}	{p,q,x}
q	{q,r,x}	{q}	{p,q,x}
r	{q,r,x}	{p,r,x}	{r}
x	{q,r,x}	{p,r,x}	{p,q,x}

In the following we list equivalence classes determined by some of the indiscernibility relations generated by the system.

ind(p):	{p}	{q,r,x}		
ind(q):	{q}	{p,r,x}		
ind(r):	{r}	{p,q,x}		
ind(p,q):	{p}	{q}	{r,x}	
ind(p,q,r):	{p}	{q}	{r}	{x}

The system may be considered as self-referential in a sense, since each attribute enables us to distinguish itself from the remaining objects.

Example 2.5

Let OB={1,2,3} and let AT be the family of all the subsets of set OB. Let the information function be given by the following table:

	∅	{1}	{2}	{1,2}	{3}	{1,3}	{2,3}	{1,2,3}
1	F	T	F	T	F	T	F	T
2	F	F	T	T	F	F	T	T
3	F	F	F	F	T	T	T	T

It is easy to see that f(o,a)=T iff $o \in a$, and f(o,a)=F iff $o \notin a$. Moreover, for each attribute a the equivalence classes of relation ind(a) are a and OB-a.

3 Relative definability of sets of entities

Given a universe U, a relation ind(P), and an entity e, we define equivalence class $sim_P(e)$ of e determined by ind(P):

$$sim_P(e)=\{e' \in ENT: (e,e') \in ind(P)\}$$

Proposition 3.1

(a) $ind(P) \subseteq ind(Q)$ implies $sim_P(e) \subseteq sim_Q(e)$ for all $e \in ENT$

(b) $sim_{P \cup Q}(e)=sim_P(e) \cap sim_Q(e)$

Indiscernibility of entities influences definability of sets of entities in terms of properties from PROP. Given a set P of properties, in general sets of entities cannot be defined uniquely in terms of properties from P,they can be defined up to indiscernibility ind(P). Following the rough set theory, for a subset E of ENT and an indicsernibility ind(P) we define a lower approximation $\underline{ind}(P)E$ and an upper approximation $\overline{ind}(P)E$ of set E with respect to P as follows:

$\underline{ind}(P)E$ is the union of those equivalence classes determined by ind(P) which are included in E

$\overline{ind}(P)E$ is the union of those equivalence classes determined by ind(P) which have an element in common with E.

A detailed discussion of properties of approximations can be found in Pawlak (1982). We say that:

A set E is P-definable in universe U iff its lower and upper approximations with respect to P coincide or E equals the empty set.

For any set E of entities we define P-positive, P-negative,and P-borderline elements of E:

e is a P-positive element of E iff $e \in \underline{ind}(P)E$

e is a P-negative element of E iff $e \in ENT-\overline{ind}(P)E$

e is a P-borderline element of E iff $e \in \overline{ind}(P)E-\underline{ind}(P)E$.

P-positive elements of E are those entities which definitely, relative to properties from set P, belong to E. P-negative elements definitely, relative to P, do not belong to E. P-borderline elements form a doubtful region, they possibly belong to E, but we cannot decide it for certain in virtue of properties from P. Let us observe that the following conditions are satisfied:

Proposition 3.2

(a) $ind(P) \subseteq ind(Q)$ implies $\underline{ind}(Q)E \subseteq \underline{ind}(P)E$ and $\overline{ind}(P)E \subseteq \overline{ind}(Q)E$ for any $E \subseteq ENT$

(b) If $ind(P) \subseteq ind(Q)$ then Q-definability implies P-definability.

Condition (a) says that properties which provide better discernibility of entities than some other properties give greater lower approximations and smaller upper approximations of sets of entities. It follows that elements of sets of entities can be recognized with greater precision, more elements can be properly classified as positive or negative elements, and a set of borderline elements is smaller than for those other properties.

Following Pawlak (1984) we can define rough P-definability of sets of entities. We say that:

A set E is roughly P-definable in universe U iff $\underline{ind}(P)E \neq O$ and $\overline{ind}(P)E \neq ENT$ or $E=ENT$ or $E=\emptyset$.

Thus for roughly definable sets membership question can be decided approximately. However, if the lower approximation ind(P)E is empty, then there are no P-positive elements of E. Moreover, if the upper approximation ind(P)E equals set ENT, then there are no negative elements of E. We say that:

A set E is internally P-nondefinable in U iff $\underline{ind}(P)E=\emptyset$

A set E is externally P-nondefinable in U iff $\overline{ind}(P)E=ENT$

A set E is totally P-nondefinable in U iff E is internally P-nondefinable in U and externally P-nondefinable in U.

In terms of approximations we define relative equalities and inclusions. Namely, we define relations $\underline{eq}(P)$ and $\overline{eq}(P)$ of bottom and top equality, respectively, and relations $\underline{in}(P)$ and $\overline{in}(P)$ of bottom and top inclusion:

$E\ \underline{eq}(P)\ D$ iff $\underline{ind}(P)E = \underline{ind}(P)D$

$E\ \overline{eq}(P)\ D$ iff $\overline{ind}(P)E = \overline{ind}(P)D$

$E\ \underline{in}(P)\ D$ iff $\underline{ind}(P)E \subseteq \underline{ind}(P)D$

$E\ \overline{in}(P)\ D$ iff $\overline{ind}(P)E \subseteq \overline{ind}(P)D$

In section 7 we show how the properties of sets of entities listed in this section can be expressed in the language of logic of indiscernibility relations.

Example 3.1

Let ENT={1,2,3,4,5,6} and PROP={p1,p2,p3}, where p1=to be divisible by 2, p2=to be divisible by 3, and p3=to be divisible by 6. The indiscernibility relations generated by the subsets of set PROP determine the following equivalence classes:

ind(p1): {1,3,5} objects which are not divisible by 2
{2,4,6} objects which are divisible by 2

ind(p2): {1,2,4,5} {3,6}

ind(p3): {1,2,3,4,5} {6}

ind(p1,p2):
{1,5} objects which are divisible neither by 2 nor by 3
{2,4} objects which are divisible by 2 but not by 3
{3} objects which are not divisible by 2 and are divisible by 3
{6} objects which are divisible both by 2 and by 3

ind(p1,p3): {1,3,5} {2,4} {6}

ind(p2,p3): {1,2,4,5} {3} {6}

ind(p1,p2,p3): {1,5} {2,4} {3} {6}

Let E={1,2,3}. We have $\underline{ind}(\{p2,p3\})E=\{3\}$; $\overline{ind}(\{p1,p2\}) = \{1,2,3,4,5\}$; E is totally {p1}-nondefinable; E is internally {p3}-nondefinable. Observe that $ind(\{p1,p2\}) \subseteq ind(\{p2,p3\})$ and hence if a set is {p2,p3}-definable then it is {p1,p2}-definable. However the converse is not true, e.g. set {1,5} is {p1,p3}-definable, but not {p2,p3}-definable.

Example 3.2

Consider OB and PROP from example 3.1 and sets E={1,2,3,5}, and D={1,4}. We have $E\overline{eq}(\{p1,p3\})D$.

For sets K={1,2,3,4} and M={1,2,3,6} we have $K\underline{in}(\{p2,p3\})M$.

4 Independence of properties

In the present section the notions introduced in Pawlak (1984) in connection with dependencies of attributes in information systems will be applied to subsets of set PROP of a universe of discourse.

Given a universe $U=(ENT, PROP, \{ind(P)\}_{P\subseteq PROP})$, we say that

A set $P\subseteq PROP$ is independent in U iff for any $Q\subsetneq P$ we have $ind(P)\subsetneq ind(Q)$

A set P is dependent in U iff there is a $Q\subsetneq P$ such that $ind(P)=ind(Q)$

A set Q is superfluous in P iff $ind(P-Q)=ind(P)$.

It follows from these definitions that if we eliminate some properties from an independent set P, then discrimination power of the remaining properties is less than discrimination power of set P. Similarly, if a set P is dependent, then some of its elements are superfluous in PROP, and hence we can drop these superfluous properties without violating discrimination power. A detailed discussion of these notions in connection with information systems can be found in Pawlak and Rauszer (1984).

We also consider dependence of a set of properties on some other set of properties:

A set Q is dependent on a set P iff $ind(P)\subseteq ind(Q)$.

Let us observe that condition $ind(P)\subseteq ind(Q)$ says that if a pair of entities is discernible relative to properties from Q then it is also discernible relative to properties from P.

Conditions related to independence of properties will be shown to be expressible in the logic of indiscernibility relations.

Example 4.1

Consider a set of entities which are characterized in terms of their length, and assume that there are given results of measurement of length up to 1m and up to 1cm. Clearly, the property "length up to 1m" is dependent on the property "length up to 1cm", since if two entities can be distinguished with respect to their length measured up to 1m then they can also be distinguished with respect to length provided by more detailed measurement.

Example 4.2

Consider ENT and PROP from example 3.1. Observe that PROP is dependent, since ind({p1,p2})=ind({p1,p2,p3}). The property "to be divisible by 6" is superfluous in PROP.

5 Formalized language

Given a universe U, we define a formal language for U. Formulas of the language are intended to represent sets of entities. First, we define an auxiliary set EXPR of expressions representing subsets of set PROP. Elements of EXPR are formed from symbols taken from the following pairwise disjoint sets:

VARP a set of variables representing sets of properties

CONP a set of constants representing all the one-element subsets of PROP; a constant corresponding to set {p} will be denoted by **p**

{1} the set consisting of the constant representing set PROP

{-, ∪, ∩} the set of set-theoretical operations of complement, union, and intersection, respectively

{(,)} brackets.

Set EXPR is the least set satisfying the following conditions

$VARP \subseteq EXPR$, $CONP \subseteq EXPR$

$\mathbf{1} \in EXPR$

$A,B \in EXPR$ implies $-A$, $A \cup B$, $A \cap B \in EXPR$.

Formulas of the language are constructed from symbols taken from the following pairwise disjoint sets:

VARE a set of variables representing sets of entities

CONE a set of constants representing all the one-element subsets of ENT; a constant corresponding to set{e} will be denoted by **e**

{¬, ∨, ∧, →, ↔} the set of propositional operations of negation, disjunction, conjunction, implication, and equivalence, respectively

{$\underline{ind}$, $\overline{ind}$} the set of symbols of lower and upper approximation, respectively

{(,)} brackets.

Set FOR of formulas is the least set satisfying the following conditions:

$VARE \subseteq FOR$, $CONE \subseteq FOR$

$F,G \in FOR$ implies $\neg F$, $F \vee G$, $F \wedge G$, $F \rightarrow G$, $F \leftrightarrow G \in FOR$

$A \in EXPR$, $F \in FOR$ imply $\underline{ind}(A)F$, $\overline{ind}(A)F \in FOR$.

6 Semantics of the language

Semantics of the given language is defined by means of notions of model and satisfiability of the formulas in a model. By a model we mean a system

$M = (U,v)$, where

$U = (ENT, PROP, \{ind(P)\}_{P\ PROP})$ is a universe

v is a valuation function such that

$v(F) \subseteq ENT$ for $F \in VARE$

$v(A) \subseteq PROP$ for $A \in VARP$

$v(\mathbf{e}) = \{e\}$ for $\mathbf{e} \in CONE$, $e \in ENT$

$v(\mathbf{p}) = \{p\}$ for $\mathbf{p} \in CONP$, $p \in PROP$

$v(\mathbf{1}) = PROP$

$v(A \cup B) = v(A) \cup v(B)$

$v(A \cap B) = v(A) \cap v(B)$

$v(-A) = PROP - v(A)$ for any $A,B \in EXPR$.

Given a model M, we define satisfiability of the formulas by entities. We say that an entity e satisfies a formula F in model M (M,e sat F) iff the following conditions are satisfied:

for $F \in VARE$ M,e sat F iff $e \in v(F)$

for $F \in CONE$ M,e sat F iff $v(F)=\{e\}$

M,e sat $\neg F$ iff not M,e sat F

M,e sat $F \vee G$ iff M,e sat F or M,e sat G

M,e sat $F \wedge G$ iff M,e sat F and M,e sat G

M,e sat $F \rightarrow G$ iff M,e sat $\neg F \vee G$

M,e sat $F \leftrightarrow G$ iff M,e sat $(F \rightarrow G) \wedge (G \rightarrow F)$

M,e sat $\underline{ind}(A)F$ iff for all e' if $(e,e') \in ind(v(A))$ then M,e' sat F

M,e sat $\overline{ind}(A)F$ iff there is an e' such that $(e,e') \in ind(v(A))$ and M,e' sat F.

To every formula F we assign a set $ext_M F$ (extension of F in model M) of those entities which satisfy F in model M:

$ext_M F = \{e \in ENT: M,e \text{ sat } F\}$

Extensions of compound formulas depend on extensions of their components according to the following rules:

Proposition 6.1

(a) For $F \in VARE$ $ext_M F = v(F)$

(b) $ext_M \mathbf{e} = \{e\}$

(c) $ext_M \neg F = -ext\ F$
(d) $ext_M (F \vee G) = ext_M F \cup ext_M G$
(e) $ext_M (F \wedge G) = ext_M F \cap ext_M G$
(f) $ext_M (F \rightarrow G) = -ext_M F \cup ext_M G$
(g) $ext_M (F \leftrightarrow G) = ext_M F \cap ext_M G \cup -ext_M F \cap -ext_M G$
(h) $ext_M \underline{ind}(A)F = \underline{ind}(v(A)) ext_M F$
(i) $ext_M \overline{ind}(A)F = \overline{ind}(v(A)) ext_M F$

Thus the classical propositional operations correspond to the set-theoretical operations, and $\underline{ind}(A)$, $\overline{ind}(A)$ correspond to operations of lower and upper approximation, respectively. A formula F is said to be true in a model M ($\models_M F$) iff $ext_M F = ENT$.

The language presented in this section is a slight modification of the language defined in Orlowska (1983). The idea to introduce propositional constants representing one-element sets of entities is due to Vakarelov (1984).

7 Expressiveness of the language

In the present section we list some facts expressible in the given language. First, we present properties of sets of entities.

Proposition 7.1
(a) $\models_M F \rightarrow \underline{ind}(A)F$ iff $ext_M F$ is $v(A)$-definable
(b) $\models_M \neg \underline{ind}(A)F$ iff $ext_M F$ is internally $v(A)$-nondefinable
(c) $\models_M \overline{ind}(A)F$ iff $ext_M F$ is externally $v(A)$-nondefinable
(d) $\models_M \neg(\overline{ind}(A)F \rightarrow \underline{ind}(A)F)$ iff $ext_M F$ is totally $v(A)$-nondefinable

Proposition 7.2
(a) $\models_M \underline{ind}(A)F \leftrightarrow \underline{ind}(A)G$ iff $ext_M F\ \underline{eq}(v(A))\ ext_M G$
(b) $\models_M \underline{ind}(A)F \rightarrow \underline{ind}(A)G$ iff $ext_M F\ \underline{in}(v(A))\ ext_M G$

The formula in condition (a) expresses the bottom equality and the formula in (b) expresses the bottom inclusion of sets of entities. In a similar way we can express the top equality and the top inclusion.

In the following we present examples of properties of indiscernibility relations expressible in the language.

Proposition 7.3
(a) For any formula F $\models_M \underline{ind}(A)F \rightarrow F$ iff $ind(v(A))$ is reflexive
(b) For any formula F $\models_M F \rightarrow \underline{ind}(A)\overline{ind}(A)F$ iff $ind(v(A))$ is symmetric
(c) For any formula F $\models_M \underline{ind}(A)F \rightarrow \underline{ind}(A)\underline{ind}(A)F$ iff $ind(v(A))$ is transitive

Proof of condition (a): Let us assume that there is $e \in ENT$ such that $(e,e) \notin ind(v(A))$. Let us consider a formula of the form $\neg e$. We will show that formula $\underline{ind}(A)\neg e \rightarrow \neg e$ is not true in M. More exactly, we will show that e does not satisfy this formula in M. Obviously, e does not satisfy $\neg e$. We will show that e satisfies $\underline{ind}(A)\neg e$. Suppose conversely, then there exists e' such that $(e,e') \in ind(v(A))$ and not M,e' sat $\neg e$. Hence M,e' sat e and we conclude that e'=e. It follows that $(e,e) \in ind(v(A))$, a contradiction. In a standard way we can easily show that if relation $ind(v(A))$ is reflexive then the respective scheme represents formulas which are true in the model under consideration. In a similar way we can prove conditions (b) and (c).

Let us observe that this theorem provides a characterization of relations in one model, this characterization is stronger than that provided by ordinary modal logics T, B, and S5.

Proposition 7.4
(a) For any formula F $\models_M \overline{ind}(A)F \rightarrow \underline{ind}(A)F$ iff equivalence classes determined by ind(v(A) are one-element sets.

(b) For any formulas F, G $\models_M \overline{ind}(A)(F\wedge G)\wedge\overline{ind}(A)(F\wedge\neg G)$ iff each equivalence class of ind(v(A)) has at least two elements
(c) For any formulas F,G $\models_M \overline{ind}(A)(F\wedge G)\wedge\overline{ind}(A)(F\wedge\neg G)\to\underline{ind}(A)F$ iff each equivalence class of ind(v(A)) has exactly two elements
(d) For any formula G $\models_M (G\to\underline{ind}(A)F)\to((\underline{ind}(A)G\to F\wedge\neg F)\vee(\underline{ind}(A)G\leftrightarrow\underline{ind}(A)F))$ iff $ext_M F$ is an equivalence class of ind(v(A)) or the empty set.

In a similar way we can define schemes of formulas expressing the fact that equivalence classes of an indiscernibility relation have at least or exactly n elements for any n>1.

Proposition 7.5
The following conditions are equivalent:
(a) For any formula F $\models_M \overline{ind}(A)F\to\overline{ind}(B)F$
(b) $ind(v(A)) \subseteq ind(v(B))$
(c) v(B) is dependent on v(A).

Proof: We will show that (a) implies (b). Suppose that there exists a pair (e,d) of entities such that $(e,d)\in ind(v(A))$ and $(e,d)\notin ind(v(B))$. We will show that e does not satisfy formula $\overline{ind}(A)\mathbf{d}\to\overline{ind}(B)\mathbf{d}$. Clearly e satisfies $\overline{ind}(A)\mathbf{d}$. Suppose that M,e sat $\overline{ind}(B)\mathbf{d}$. Hence there is e' such that $(e,e')\in ind(v(B))$ and M,e' sat **d**. It follows that e'=d and hence $(e,d)\in ind(v(B))$, a contradiction. Also inclusion of relations implies that the formula in question is true and hence (b) implies (a). By the definition of dependencies of sets of properties conditions (b) and (c) are equivalent.

Proposition 7.6
(a) For all formulas F $\models_M \overline{ind}(A-B)F\leftrightarrow\overline{ind}(A)F$ iff set v(B) is superfluous in set v(A)
(b) For all F∈FOR and for all **p**∈CONP
$\models_M \overline{ind}(\mathbf{1})F\to\overline{ind}(\mathbf{1}-\mathbf{p})F$ and
$\models_M \neg(\overline{ind}(\mathbf{1})F\leftrightarrow\overline{ind}(\mathbf{1}-\mathbf{p})F)$
iff set PROP is independent.

Proof of condition (b): For any constant **p** the first formula is true in a model M iff $ind(PROP)\subseteq ind(PROP-\{p\})$. The second formula says that this inclusion is proper. Thus by (U3) any proper subset of PROP generates a greater indiscernibility relation than set PROP, and hence PROP is independent. Conversely, if PROP is independent then the given formulas are true.

In the following we list examples of true formulas.

Proposition 7.7
For any model M the following formulas are true:
(a) $\neg\underline{ind}(A)F\leftrightarrow\overline{ind}(A)\neg F$
(b) $\underline{ind}(A)F\vee\underline{ind}(B)F\to\underline{ind}(A\cup B)F$
(c) $\underline{ind}(A\cap B)\to\underline{ind}(A)F\wedge\underline{ind}(B)F$
(d) $\underline{ind}(A)(F\to G)\to(\underline{ind}(A)F\to\underline{ind}(A)G)$
(e) $\underline{ind}(A)(F\wedge G)\leftrightarrow\underline{ind}(A)F\wedge\underline{ind}(A)G$
(f) $\underline{ind}(A)F\vee\underline{ind}(A)G\to\underline{ind}(A)(F\vee G)$
(g) $\overline{ind}(A)e\wedge\overline{ind}(B)e\leftrightarrow\overline{ind}(A\cup B)e$
(h) $(\overline{ind}(C)e\to\overline{ind}(A)e)\wedge(\overline{ind}(C)e\to\overline{ind}(B)e)\to$
$\to(\overline{ind}(C)e\to\overline{ind}(A\cup B)e)$.

Proof of (g): By (b) the right hand side formula implies the left hand side formula. Assume that for an arbitrary d∈ENT we have M,d sat $\overline{ind}(A)e$ and M,d sat $\overline{ind}(B)e$. Hence there are d',d'' such that $(d,d')\in ind(A)$, $(d,d'')\in ind(B)$, and d'=d''=e. We conclude that $(d,e)\in ind(A)\cap ind(B)=ind(A\cup B)$, so M,d sat $\overline{ind}(A\cup B)e$.

8. Weak indiscernibility

In this section we consider indiscernibility relations which satisfy conditions (U2), (U3), (U4) and

(U1') ind(P) is reflexive and symmetric.

In this case sets $sim_P(e)=\{e' \in ENT:(e,e') \in ind(P)\}$ are similarity classes of relation ind(P). They provide a covering of set ENT, but they are not necessarily pairwise disjoint. The approximations of sets of entities are defined as follows:

$\underline{ind}(P)E$ is the union of those similarity classes of ind(P) which are included in E

$\overline{ind}(P)E$ is the union of those similarity classes of ind(P) which have an element in common with E.

Several properties of these approximations can be found in Zakowski (1983).

In the language of a logic of nontransitive indiscernibility relations satisfiability of modal formulas should be defined as follows:

M,e sat $\underline{ind}(A)F$ iff there is an e' such that $e \in sim_{v(A)}(e')$ and for all d if $d \in sim_{v(A)}(e')$ then M,d sat F

M,e sat $\overline{ind}(A)F$ iff there is an e' such that $e \in sim_{v(A)}(e')$ and there is $d \in sim_{v(A)}(e')$ such that M,d sat F.

For these operations conditions 6.1(h),(i) hold and hence the operations correspond to lower and upper approximation, respectively.

Example 8.1

Consider a nondeterministic information system (Orlowska and Pawlak (1984)) in which sets of values of attributes are assigned to objects, that is an information function f: $OB \times AT \rightarrow VAL$, where $VAL = \bigcup VAL_a$, satisfies the condition: for each $o \in OB$ and $a \in AT$ $f(o,a) \subseteq VAL_a$. We define a family ind(P) for $P \subseteq AT$:

$(o,o') \in ind(P)$ iff for all $a \in P$ $f(o,a) \cap f(o,a') \neq \emptyset$.

Clearly, relations ind(P) are not transitive.

Example 8.2

Assume that we are given four entities e1,e2,e3,e4 such that e2 and e4 are green, and e1,e2,e4 are small. The indiscernibility relation determined by the property "to be green or to be small" is not transitive, since the pairs (e3,e1) and (e1,e2) belong to this relation, but (e3,e2) does not.

The proposals made in the present paper for modifying the syntax and semantics of the ordinary modal logics enable us to express several interesting facts related to indiscernibility. We can easily define a language for a class of models, assuming that in any model entity constants and property constants are interpreted as arbitrary one-element subsets of the respective sets from the model. The problem of complete axiomatization of such languages is open. Some results for a simpler language without expressions representing sets of properties can be obtained in a way similar to that developed for the combinatory dynamic logic (Solomon (1984)).

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